Meager sets of periodic functions

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The following is often useful.¹

**Theorem 1.** If \((X, \mu)\) is a measure space, \(1 \leq p \leq \infty\), and \(f_n \in L^p(\mu)\) is a sequence that converges in \(L^p(\mu)\) to some \(f \in L^p(\mu)\), then there is a subsequence of \(f_n\) that converges pointwise almost everywhere to \(f\).

**Proof.** Assume that \(1 \leq p < \infty\). For each \(n\) there is some \(a_n\) such that
\[
\|f_{a_n} - f\|_p < 2^{1-n}.
\]

Then
\[
\sum_{n=1}^\infty \|f_{a_n} - f\|_p^p < \sum_{n=1}^\infty 2^{-np} = \frac{2^{-p}}{1-2^{-p}} < \infty.
\]

Let \(\epsilon > 0\). We have
\[
\left\{ x \in X : \limsup_{n \to \infty} |f_{a_n}(x) - f(x)| > \epsilon \right\} \subset \bigcap_{N=1}^\infty \bigcup_{n=N}^\infty \left\{ x \in X : |f_{a_n}(x) - f(x)| > \epsilon \right\}.
\]

For any \(N\), this gives, using Chebyshev’s inequality,
\[
\mu \left( \left\{ x \in X : \limsup_{n \to \infty} |f_{a_n}(x) - f(x)| > \epsilon \right\} \right)
\leq \sum_{n=N}^\infty \mu \left( \left\{ x \in X : |f_{a_n}(x) - f(x)| > \epsilon \right\} \right)
\leq \epsilon^{-p} \sum_{n=N}^\infty \|f_{a_n} - f\|_p^p.
\]

Because \(\sum_{n=1}^\infty \|f_{a_n} - f\|_p^p < \infty\), we have \(\sum_{n=N}^\infty \|f_{a_n} - f\|_p^p \to 0\) as \(N \to \infty\), which implies that
\[
\mu \left( \left\{ x \in X : \limsup_{n \to \infty} |f_{a_n}(x) - f(x)| > \epsilon \right\} \right) = 0.
\]

This is true for each \( \epsilon > 0 \), hence
\[
\mu \left( \left\{ x \in X : \limsup_{n \to \infty} |f_{a_n}(x) - f(x)| > \epsilon \right\} \right) = 0,
\]
which means that for almost all \( x \in X \),
\[
\lim_{n \to \infty} |f_{a_n}(x) - f(x)| = 0.
\]

Assume that \( p = \infty \). Let
\[
E_k = \{ x \in X : |f_k(x)| > \|f_k\|_\infty \}.
\]
The measure of each of these sets is 0, so for \( E = \bigcup_k E_k \) we have
\[
\mu(E) = 0.
\]
For \( x \not\in E \),
\[
|f(x) - f_k(x)| \leq \|f - f_k\|_\infty \to 0, \quad k \to \infty,
\]
showing that for almost all \( x \in X \), \( f_k(x) \to f(x) \).

The following results are in the pattern of \( A \) being a strict subset of \( X \) implying that \( A \) is meager in \( X \).

We first work out two proofs of the following theorem.

**Theorem 2.** For \( 1 < p \leq \infty \), \( L^p(T) \) is a meager subset of \( L^1(T) \).

**Proof.** For \( n \geq 1 \), let
\[
C_n = \left\{ f \in L^1(T) : \|f\|_p \leq n \right\}.
\]
Let \( n \geq 1 \). If a sequence \( f_k \in C_n \) converges in \( L^1(T) \) to some \( f \in L^1(T) \), then there is a subsequence \( f_{a_k} \) of \( f_k \) such that for almost all \( x \in T \), \( f_{a_k}(x) \to f(x) \), and so \( f_{a_k}(x)^p \to f(x)^p \). Applying the dominated convergence theorem gives
\[
\frac{1}{2\pi} \int_T |f(x)|^p dx = \lim_{k \to \infty} \frac{1}{2\pi} \int_T |f_{a_k}(x)|^p dx = \lim_{k \to \infty} \|f_{a_k}\|_p \leq n^p,
\]
hence \( \|f\|_p \leq n \), showing that \( f \in C_n \). Therefore, \( C_n \) is a closed subset of \( L^1(T) \). On the other hand, let \( f \in C_n \) and let \( g \in L^1(T) \setminus L^p(T) \). Then \( f + \frac{1}{k}g \to f \) in \( L^1(T) \), and for each \( k \) we have \( f + \frac{1}{k}g \not\in C_n \), as that would imply \( g \in L^p(T) \). This shows that \( f \) does not belong to the interior of \( C_n \). Because \( C_n \) is closed and has empty interior, it is nowhere dense. Therefore
\[
L^p(T) = \bigcup_{n=1}^{\infty} \left\{ f \in L^1(T) : \|f\|_p \leq n \right\}
\]
is meager in \( L^1(T) \).

**Proof.** The open mapping theorem tells us that if \( X \) is an \( F \)-space, \( Y \) is a topological vector space, \( \Lambda : X \to Y \) is continuous and linear, and \( \Lambda(X) \) is not meager in \( Y \), then \( \Lambda(X) = Y \), \( \Lambda \) is an open mapping, and \( Y \) is an \( F \)-space.

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Let \( j : L^p(\mathbb{T}) \rightarrow L^1(\mathbb{T}) \) be the inclusion map. For \( f \in L^p(\mathbb{T}) \),
\[
\| j(f) \|_1 = \| f \|_1 \leq \| f \|_p,
\]
showing that the inclusion map is continuous. On the other hand, \( j \) is not onto, so the open mapping theorem tells us that \( j(L^p(\mathbb{T})) = L^p(\mathbb{T}) \) is meager in \( L^1(\mathbb{T}) \).

Suppose that \( X \) is a topological vector space, that \( Y \) is an \( F \)-space, and that \( \Lambda_n \) is a sequence of continuous linear maps \( X \rightarrow Y \). Let \( L \) be the set of those \( x \in X \) such that
\[
\Lambda x = \lim_{n \rightarrow \infty} \Lambda_n x
\]
exists. It is a consequence of the uniform boundedness principle that if \( L \) is not meager in \( X \), then \( L = X \) and \( \Lambda : X \rightarrow Y \) is continuous.\(^3\)

For \( n \geq 1 \), define \( \Lambda_n : L^2(\mathbb{T}) \rightarrow \mathbb{C} \) by
\[
\Lambda_n f = \sum_{|k| \leq n} \hat{f}(k), \quad f \in L^1(\mathbb{T}).
\]
Define
\[
L = \left\{ f \in L^2(\mathbb{T}) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists} \right\}.
\]
The sequence \( t \mapsto \sum_{k=1}^n \frac{e^{ikt}}{k} \) is a Cauchy sequence in \( L^2(\mathbb{T}) \), hence converges to some \( f \in L^2(\mathbb{T}) \), which satisfies
\[
\hat{f}(k) = \begin{cases} 
\frac{1}{k} & k \geq 1 \\
0 & k \leq 0.
\end{cases}
\]
Then
\[
\Lambda_n f = \sum_{k=1}^n \frac{1}{k} \rightarrow \infty, \quad n \rightarrow \infty,
\]
meaning that \( f \in L^2(\mathbb{T}) \setminus L \). This shows that \( L \neq L^2(\mathbb{T}) \). Therefore, the above consequence of the uniform boundedness principle tells us that \( L \) is meager.