Meager sets of periodic functions

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The following is often useful.[1]

Theorem 1. *If* (X, μ) *is a measure space,* $1 \leq p \leq \infty$*, and* $f_n \in L^p(\mu)$ *is a sequence that converges in* $L^p(\mu)$ *to some* $f \in L^p(\mu)$*, then there is a subsequence of* f_n *that converges pointwise almost everywhere to* f*.*

Proof. Assume that $1 \leq p < \infty$. For each n there is some a_n such that

$$\left\| f_{a_n} - f \right\|_p < 2^{-n}.$$

Then

$$\sum_{n=1}^\infty \left\| f_{a_n} - f \right\|_p^p < \sum_{n=1}^\infty 2^{-np} = \frac{2^{-p}}{1 - 2^{-p}} < \infty.$$

Let $\epsilon > 0$. We have

$$\left\{ x \in X : \limsup_{n \to \infty} |f_{a_n}(x) - f(x)| > \epsilon \right\} \subset \bigcap_{N=1}^\infty \bigcup_{n=N}^\infty \left\{ x \in X : |f_{a_n}(x) - f(x)| > \epsilon \right\}.$$

For any N, this gives, using Chebyshev's inequality,

$$\begin{aligned} &\mu\left(\left\{ x \in X : \limsup_{n \to \infty} |f_{a_n}(x) - f(x)| > \epsilon \right\} \right) \\ \leq &\sum_{n=N}^\infty \mu\left(\left\{ x \in X : |f_{a_n}(x) - f(x)| > \epsilon \right\} \right) \\ \leq &\epsilon^{-p} \sum_{n=N}^\infty \left\| f_{a_n} - f \right\|_p^p. \end{aligned}$$

Because $\sum_{n=1}^\infty \left\| f_{a_n} - f \right\|_p^p < \infty$, we have $\sum_{n=N}^\infty \left\| f_{a_n} - f \right\|_p^p \to 0$ as $N \to \infty$, which implies that

$$\mu\left(\left\{ x \in X : \limsup_{n \to \infty} |f_{a_n}(x) - f(x)| > \epsilon \right\} \right) = 0.$$

[1] Walter Rudin, *Real and Complex Analysis*, third ed., p. 68, Theorem 3.12.

This is true for each $\epsilon > 0$, hence

$$\mu\left(\left\{x \in X : \limsup_{n\to\infty} |f_{a_n}(x) - f(x)| > 0\right\}\right) = 0,$$

which means that for almost all $x \in X$,

$$\lim_{n\to\infty} |f_{a_n}(x) - f(x)| = 0.$$

Assume that $p = \infty$. Let

$$E_k = \{x \in X : |f_k(x)| > \|f_k\|_\infty\}.$$

The measure of each of these sets is 0, so for

$$E = \bigcup_k E_k$$

we have $\mu(E) = 0$. For $x \notin E$,

$$|f(x) - f_k(x)| \leq \|f - f_k\|_\infty \to 0, \qquad k \to \infty,$$

showing that for almost all $x \in X$, $f_k(x) \to f(x)$. □

The following results are in the pattern of A being a strict subset of X implying that A is meager in X.

We first work out two proofs of the following theorem.

Theorem 2. *For $1 < p \leq \infty$, $L^p(\mathbb{T})$ is a meager subset of $L^1(\mathbb{T})$.*

Proof. For $n \geq 1$, let

$$C_n = \left\{f \in L^1(\mathbb{T}) : \|f\|_p \leq n\right\}.$$

Let $n \geq 1$. If a sequence $f_k \in C_n$ converges in $L^1(\mathbb{T})$ to some $f \in L^1(\mathbb{T})$, then there is a subsequence f_{a_k} of f_k such that for almost all $x \in \mathbb{T}$, $f_{a_k}(x) \to f(x)$, and so $f_{a_k}(x)^p \to f(x)^p$. Applying the dominated convergence theorem gives

$$\frac{1}{2\pi}\int_{\mathbb{T}} |f(x)|^p dx = \lim_{k\to\infty} \frac{1}{2\pi}\int_{\mathbb{T}} |f_{a_k}(x)|^p dx = \lim_{k\to\infty} \|f_{a_k}\|_p^p \leq n^p,$$

hence $\|f\|_p \leq n$, showing that $f \in C_n$. Therefore, C_n is a closed subset of $L^1(\mathbb{T})$ On the other hand, let $f \in C_n$ and let $g \in L^1(\mathbb{T}) \setminus L^p(\mathbb{T})$. Then $f + \frac{1}{k}g \to f$ in $L^1(\mathbb{T})$, and for each k we have $f + \frac{1}{k}g \notin C_n$, as that would imply $g \in L^p(\mathbb{T})$. This shows that f does not belong to the interior of C_n. Because C_n is closed and has empty interior, it is nowhere dense. Therefore

$$L^p(\mathbb{T}) = \bigcup_{n=1}^{\infty} \left\{f \in L^1(\mathbb{T}) : \|f\|_p \leq n\right\}$$

is meager in $L^1(\mathbb{T})$. □

Proof. The open mapping theorem tells us that if X is an F-space, Y is a topological vector space, $\Lambda : X \to Y$ is continuous and linear, and $\Lambda(X)$ is not meager in Y, then $\Lambda(X) = Y$, Λ is an open mapping, and Y is an F-space.[2]

[2] Walter Rudin, *Functional Analysis*, second ed., p. 48, Theorem 2.11.

Let $j : L^p(\mathbb{T}) \to L^1(\mathbb{T})$ be the inclusion map. For $f \in L^p(\mathbb{T})$,

$$\|j(f)\|_1 = \|f\|_1 \leq \|f\|_p ,$$

showing that the inclusion map is continuous. On the other hand, j is not onto, so the open mapping theorem tells us that $j(L^p(\mathbb{T})) = L^p(\mathbb{T})$ is meager in $L^1(\mathbb{T})$. $\square$

Suppose that X is a topological vector space, that Y is an F-space, and that Λ_n is a sequence of continuous linear maps $X \to Y$. Let L be the set of those $x \in X$ such that

$$\Lambda x = \lim_{n\to\infty} \Lambda_n x$$

exists. It is a consequence of the uniform boundedness principle that if L is not meager in X, then $L = X$ and $\Lambda : X \to Y$ is continuous.[3]

For $n \geq 1$, define $\Lambda_n : L^2(\mathbb{T}) \to \mathbb{C}$ by

$$\Lambda_n f = \sum_{|k|\leq n} \hat{f}(k), \qquad f \in L^1(\mathbb{T}).$$

Define

$$L = \left\{ f \in L^2(\mathbb{T}) : \lim_{n\to\infty} \Lambda_n f \text{ exists} \right\}.$$

The sequence $t \mapsto \sum_{k=1}^n \frac{e^{ikt}}{k}$ is a Cauchy sequence in $L^2(\mathbb{T})$, hence converges to some $f \in L^2(\mathbb{T})$, which satisfies

$$\hat{f}(k) = \begin{cases} \frac{1}{k} & k \geq 1 \\ 0 & k \leq 0. \end{cases}$$

Then

$$\Lambda_n f = \sum_{k=1}^n \frac{1}{k} \to \infty, \qquad n \to \infty,$$

meaning that $f \in L^2(\mathbb{T}) \setminus L$. This shows that $L \neq L^2(\mathbb{T})$. Therefore, the above consequence of the uniform boundedness principle tells us that L is meager.

[3]Walter Rudin, *Functional Analysis*, second ed., p. 45, Theorem 2.7.