The $p$-adic solenoid

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1 Definition

We shall be speaking about locally compact abelian groups, and unless we say otherwise, by morphism we mean a continuous group homomorphism.

For $p$ prime and $n \in \mathbb{Z}_{\geq 0}$, $p^n \mathbb{Z}$ is a closed subgroup of the locally compact abelian group $\mathbb{R}$, and the quotient $\mathbb{R}/p^n \mathbb{Z}$ is a compact abelian group. For $n \geq m$, let $\phi_{n,m}: \mathbb{R}/p^n \mathbb{Z} \to \mathbb{R}/p^m \mathbb{Z}$ be the projection map, which is a morphism. The compact abelian groups $\mathbb{R}/p^n \mathbb{Z}$ and the morphisms $\phi_{n,m}$ are an inverse system, and the inverse limit is a compact abelian group denoted $\mathbb{T}_p$, called the $p$-adic solenoid, with morphisms $\phi_n: \mathbb{T}_p \to \mathbb{R}/p^n \mathbb{Z}$. Because the maps $\phi_{n,m}: \mathbb{R}/p^n \mathbb{Z} \to \mathbb{R}/p^m \mathbb{Z}$ are surjective, the maps $\phi_n: \mathbb{T}_p \to \mathbb{R}/p^n \mathbb{Z}$ are surjective.

Let $\pi_n: \mathbb{R} \to \mathbb{R}/p^n \mathbb{Z}$ be the projection map, which is a morphism. The projection maps $\pi_n$ are compatible with the inverse system $\phi_{n,m}$, so there is a unique morphism $\pi: \mathbb{R} \to \mathbb{T}_p$ such that $\phi_n \circ \pi = \pi_n$ for all $n \in \mathbb{Z}_{\geq 0}$. If $x, y \in \mathbb{R}$ are distinct, then for sufficiently large $n$ we have $\pi_n(x) \neq \pi_n(y)$. If $\pi(x) = \pi(y)$ then $\pi_n(x) = \phi_n(\pi(x)) = \phi_n(\pi(y)) = \pi_n(y)$, a contradiction. Therefore $\pi: \mathbb{R} \to \mathbb{T}_p$ is injective. Furthermore, the maps $\pi_n: \mathbb{R} \to \mathbb{R}/p^n \mathbb{Z}$ being surjective implies that the image $\pi(\mathbb{R})$ is dense in $\mathbb{T}_p$.

2 Pontryagin dual

If $G$ is a locally compact abelian group, we denote by $G^*$ the collection of morphisms $G \to S^1$. We assign $G^*$ the coarsest topology such that for all $g \in G$, the map $\gamma \mapsto \gamma(x)$ is continuous $G^* \to S^1$, and with this topology, $G^*$ is a locally compact abelian group, called the Pontryagin dual of $G$.

If $\phi: G \to H$ is a morphism of locally compact abelian groups, then $\phi^*: H^* \to G^*$ defined by

$$
\phi^*(\theta)(g) = \theta(\phi(g)), \quad \theta \in H^*, \quad g \in G.
$$

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1 Alain M. Robert, *A Course in $p$-adic Analysis*, Chapter 1, §4, p. 29.
is a morphism. Say \( \phi \) is surjective, and \( \phi^*(\theta_1) = \phi^*(\theta_2) \) but that \( \theta_1 \neq \theta_2 \). Then there is some \( h \in H \) such that \( \theta_1(h) \neq \theta_2(h) \). Since \( \phi : G \to H \) is surjective, there is some \( g \in G \) such that \( \phi(g) = h \). But then

\[
\theta_1(h) = \theta_2(\phi(g)) = \phi^*(\theta_1)(g) = \phi^*(\theta_2)(g) = \theta_2(\phi(g)) = \theta_2(h),
\]

contradicting \( \theta_1(h) \neq \theta_2(h) \). Therefore, if \( \phi : G \to H \) is surjective then \( \phi^* : H^* \to G^* \) is injective.

Let

\[
\frac{1}{p^n} \mathbb{Z} = \left\{ \frac{j}{p^n} : j \in \mathbb{Z} \right\} \subset \mathbb{Q},
\]

which with the discrete topology is a discrete abelian group.

**Theorem 1.** For prime \( p \) and \( n \in \mathbb{Z}_{\geq 0} \), the map \( \Phi_n : \frac{1}{p^n} \mathbb{Z} \to (\mathbb{R}/p^n\mathbb{Z})^* \) defined by

\[
\Phi_n(a)(x + p^n\mathbb{Z}) = e^{2\pi i a x}, \quad a \in \frac{1}{p^n} \mathbb{Z}, \quad x + p^n\mathbb{Z} \in \mathbb{R}/p^n\mathbb{Z},
\]

is an isomorphism of topological groups.

**Proof.** Write \( a = \frac{j}{p^n}, j \in \mathbb{Z} \). If \( x + p^n\mathbb{Z} = y + p^n\mathbb{Z} \), then \( x - y \in p^n\mathbb{Z} \), so \( x - y = p^n k \) for some \( k \in \mathbb{Z} \). Then

\[
\Phi_n(a)(x + p^n\mathbb{Z}) = e^{2\pi i a x} = e^{2\pi i \frac{j}{p^n} (p^n k + y)} = e^{2\pi i k + 2\pi i \frac{j}{p^n} y} = e^{2\pi i a y} = \Phi_n(a)(y + p^n\mathbb{Z}),
\]

showing that \( \Phi_n \) is well-defined. Furthermore, one checks that indeed \( \Phi_n(a) \in (\mathbb{R}/p^n\mathbb{Z})^* \) for each \( a \in \frac{1}{p^n} \mathbb{Z} \).

It is apparent that \( \Phi_n(a + b) = \Phi_n(a) \cdot \Phi_n(b) \). \( \Phi \) is continuous because \( \frac{1}{p^n} \mathbb{Z} \) is discrete. If \( \Phi_n(a) = \Phi_n(b) \), this means that for all \( x + p^n\mathbb{Z} \in \mathbb{R}/p^n\mathbb{Z} \),

\[
e^{2\pi i a x} = e^{2\pi i b x},
\]

equivalently, that \( (a - b)x \in \mathbb{Z} \) for all \( x \in \mathbb{R} \), whence \( a = b \).

Thus \( \Phi_n \) is injective.

Let \( \gamma \in (\mathbb{R}/p^n\mathbb{Z})^* \). Define \( \Gamma : \mathbb{R} \to S^1 \) by \( \Gamma = \gamma \circ \pi_n \), so that \( \Gamma \in \mathbb{R}^* \). We take as given that because \( \Gamma \in \mathbb{R}^* \), there is some \( y \in \mathbb{R} \) such that \( \Gamma(x) = e^{2\pi iyx} \) for all \( x \in \mathbb{R} \). In particular, for \( x = p^n \), on the one hand

\[
\Gamma(p^n) = \gamma(\pi_n(p^n)) = \gamma(0 + p^n\mathbb{Z}) = 1,
\]

and on the other hand

\[
\Gamma(p^n) = e^{2\pi iy p^n},
\]

so \( yp^n \in \mathbb{Z} \), i.e. \( y \in \frac{1}{p^n} \mathbb{Z} \), and it follows that \( \gamma = \Phi_n(y) \). Therefore \( \Phi_n \) is surjective.

The **open mapping theorem for topological groups** states that if \( G, H \) are locally compact groups, \( f : G \to H \) is a surjective morphism, and \( G \) is \( \sigma \)-compact, then \( f \) is open. \( \mathbb{Z} \) is discrete and countable, hence is \( \sigma \)-compact, so \( \Phi_n \) is open. Therefore \( \Phi_n \) is an isomorphism of topological groups. \( \square \)
Because the morphisms \( \phi_{n,m} : \mathbb{R}/p^n\mathbb{Z} \to \mathbb{R}/p^m\mathbb{Z} \) are surjective, the morphisms \( \phi_{n,m}^* : (\mathbb{R}/p^m\mathbb{Z})^* \to (\mathbb{R}/p^n\mathbb{Z})^* \) are injective. For \( m \leq n \), define \( \iota_{m,n} : 1_{p^m\mathbb{Z}} \to 1_{p^n\mathbb{Z}} \) by \( \iota_{m,n}(j_{p^m}) = j_{p^m} = \frac{p^n - j}{p^n} \in 1_{p^n\mathbb{Z}} \); this is an injective morphism. One checks that the following diagram commutes.

\[
\begin{array}{ccc}
(\mathbb{R}/p^m\mathbb{Z})^* & \xrightarrow{\phi_{n,m}^*} & (\mathbb{R}/p^n\mathbb{Z})^* \\
\downarrow{\phi_*} & & \downarrow{\phi_*} \\
1_{p^m\mathbb{Z}} & \xrightarrow{\iota_{m,n}} & 1_{p^n\mathbb{Z}}
\end{array}
\]

The discrete groups \( 1_{p^m\mathbb{Z}} \) and the morphisms \( \iota_{m,n} \) are a direct system. The localization of \( \mathbb{Z} \) away from \( p \) is the abelian group

\[
\mathbb{Z}[1/p] = \left\{ \frac{j}{p^m} : j \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0} \right\} \subset \mathbb{Q}.
\]

We assign \( \mathbb{Z}[1/p] \) the discrete topology. One proves that \( \mathbb{Z}[1/p] \) with the maps \( \iota_{m} : 1_{p^m\mathbb{Z}} \to \mathbb{Z}[1/p] \) defined by

\[
\iota_{m}\left(\frac{j}{p^m}\right) = \frac{j}{p^m}
\]

is the direct limit of this direct system.\(^3\) The direct system \( \iota_{m,n} : 1_{p^m\mathbb{Z}} \to 1_{p^n\mathbb{Z}} \) is dual to the inverse system \( \phi_{n,m} : \mathbb{R}/p^n\mathbb{Z} \to \mathbb{R}/p^m\mathbb{Z} \). It follows that the Pontryagin dual of the limit of either system is isomorphic as a topological group to the limit of the other system. That is,

\[
\mathbb{T}_p^* \cong \mathbb{Z}[1/p], \quad (\mathbb{Z}[1/p])^* \cong \mathbb{T}_p,
\]

as topological groups.

### 3 \( p \)-adic integers

For \( n \geq m \), let \( \psi_{n,m} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z} \) be the projection map. With the discrete topology, \( \mathbb{Z}/p^n\mathbb{Z} \) is a compact abelian group, as it is finite. Then \( \psi_{n,m} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z} \) is an inverse system, and its inverse limit is a compact abelian group denoted \( \mathbb{Z}_p \), called the \( p \)-adic integers, with morphisms \( \psi_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z} \). Because the morphisms \( \psi_{n,m} \) are surjective, the morphisms \( \psi_n \) are surjective.

Let \( \lambda_n : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{R}/p^n\mathbb{Z} \) be the inclusion map. Then the morphisms \( \Lambda_n = \lambda_n \circ \psi_n : \mathbb{Z}_p \to \mathbb{R}/p^n\mathbb{Z} \) are compatible with the inverse system \( \phi_{n,m} : \mathbb{R}/p^n\mathbb{Z} \to \mathbb{R}/p^m\mathbb{Z} \).

\(^3\)A direct limit of discrete abelian groups is the direct limit of abelian groups. On direct limits of abelian groups, cf. Luis Ribes and Pavel Zalesskii, *Profinite Groups*, p. 15, Proposition 1.2.1.
The map $\mathbb{R}/p^n\mathbb{Z} \to \mathbb{R}/p^m\mathbb{Z}$, so there is a unique morphism $\Lambda : \mathbb{Z}_p \to T_p$ such that $\phi_n \circ \Lambda = \Lambda_n$ for all $n \in \mathbb{Z}_{\geq 0}$. Suppose that $x, y \in \mathbb{Z}_p$ are distinct and that $\Lambda(x) = \Lambda(y)$. It is a fact that there is some $n$ such that $\psi_n(x) \neq \psi_n(y)$. Because $\lambda_n$ is injective, this implies that $\Lambda_n(x) \neq \Lambda_n(y)$, and this contradicts that $\Lambda(x) = \Lambda(y)$. Therefore $\Lambda : \mathbb{Z}_p \to T_p$ is injective.

One proves that $\ker \phi_0 = \Lambda(\mathbb{Z}_p)$, so that $0 \to \mathbb{Z}_p \to T_p \to \mathbb{R}/\mathbb{Z} \to 0$ is a short exact sequence of topological groups.\(^4\)

It can be proved that for each $m \in \mathbb{Z}_{>0}$ such that $\gcd(m, p) = 1$, the $p$-adic solenoid $T_p$ has a unique cyclic subgroup of order $m$, and on the other hand that there is no element in $T_p$ whose order is a power of $p$, namely, $T_p$ has no $p$-torsion.\(^5\)

### 4 Further reading

Garrett has written several notes on the $p$-adic solenoid.\(^6\) The $p$-adic solenoid occurs in several places in the books of Hofmann and Morris.\(^7\) For properties of the $p$-adic solenoid involving homological algebra, see the below references.\(^8\)