Proof of the pentagonal number theorem

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Let \( A_0 = \prod_{k=1}^{\infty} (1 - z^k) \). We will use the identity

\[
\prod_{k=1}^{N} (1 - a_k) = 1 - a_1 - \sum_{k=2}^{N} a_k (1 - a_1) \cdots (1 - a_{k-1}),
\]

which is straightforward to prove by induction. We apply the identity with \( a_k = z^k \) and \( N = \infty \), which gives

\[
A_0 = 1 - z - \sum_{k=2}^{\infty} z^k (1 - z) \cdots (1 - z^{k-1})
\]
\[
= 1 - z - \sum_{k=0}^{\infty} z^{k+2} (1 - z) \cdots (1 - z^{k+1}).
\]

For \( n \geq 1 \) let \( A_n = \sum_{k=0}^{\infty} z^{nk} (1 - z^n) \cdots (1 - z^{n+k}) \). We have \( A_0 = 1 - z - z^2 A_1 \),
and for $n \geq 1$ we have

\[
A_n = 1 - z^n + \sum_{k=1}^{\infty} z^{nk}(1 - z^n) \cdots (1 - z^{n+k})
\]

\[
= 1 - z^n + \sum_{k=1}^{\infty} z^{nk}(1 - z^{n+1}) \cdots (1 - z^{n+k})
\]

\[
- \sum_{k=1}^{\infty} z^{n(k+1)}(1 - z^{n+1}) \cdots (1 - z^{n+k})
\]

\[
= 1 - z^n + z^n(1 - z^{n+1}) + \sum_{k=2}^{\infty} z^{nk}(1 - z^{n+1}) \cdots (1 - z^{n+k})
\]

\[
- \sum_{k=1}^{\infty} z^{n(k+1)}(1 - z^{n+1}) \cdots (1 - z^{n+k})
\]

\[
= 1 - z^{2n+1} + \sum_{k=0}^{\infty} z^{n(k+2)}(1 - z^{n+1}) \cdots (1 - z^{n+k+2})
\]

\[
- \sum_{k=0}^{\infty} z^{n(k+2)}(1 - z^{n+1}) \cdots (1 - z^{n+k+1})
\]

\[
= 1 - z^{2n+1} - \sum_{k=0}^{\infty} z^{n(k+2)+n+k+2}(1 - z^{n+1}) \cdots (1 - z^{n+k+1})
\]

\[
= 1 - z^{2n+1} - z^{3n+2} + \sum_{k=0}^{\infty} z^{(n+1)k}(1 - z^{n+1}) \cdots (1 - z^{n+k+1})
\]

\[
= 1 - z^{2n+1} - z^{3n+2} A_{n+1}.
\]

Therefore $A_n = 1 - z^{2n+1} - z^{3n+2} A_{n+1}$ for all $n \geq 0$.

We then check by induction that for all $M$

\[
A_0 = 1 - z + \sum_{n=1}^{M} (-1)^n \left( z^{n(3n+1)/2} - z^{(n+1)(3n+2)/2} \right)
\]

\[
+ (-1)^{M+1} z^{(M+1)(3M+2)/2} A_{M+1},
\]

and taking $M = \infty$ gives the pentagonal number theorem.