Orthonormal bases for product measures

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1 Measure and integration theory

Let $\mathcal{B}$ be the Borel $\sigma$-algebra of $\mathbb{R}$, and let $\mathcal{B}^{-}$ be the Borel $\sigma$-algebra of $[-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$: the elements of $\mathcal{B}^{-}$ are those subsets of $\mathbb{R}$ of the form $B, B \cup \{-\infty\}, B \cup \{\infty\}, B \cup \{-\infty, \infty\}$, with $B \in \mathcal{B}$.

Let $(X, \mathcal{A}, \mu)$ be a measure space. It is a fact that if $f_n$ is a sequence of $\mathcal{A} \to \mathcal{B}$ measurable functions then $\sup_n f_n$ and $\inf_n f_n$ are $\mathcal{A} \to \mathcal{B}$ measurable, and thus if $f_n$ is a sequence of $\mathcal{A} \to \mathcal{B}$ measurable functions that converge pointwise to a function $f : X \to \mathbb{R}$, then $f$ is $\mathcal{A} \to \mathcal{B}$ measurable.\footnote{Heinz Bauer, Measure and Integration Theory, p. 52, Corollary 9.7.} If $f_1, \ldots, f_n$ are $\mathcal{A} \to \mathcal{B}$ measurable, then so are $f_1 \lor \cdots \lor f_n$ and $f_1 \land \cdots \land f_n$, and a function $f : X \to \mathbb{R}$ is $\mathcal{A} \to \mathcal{B}$ measurable if and only if both $f^+ = f \lor 0$ and $f^- = -(f \land 0)$ are $\mathcal{A} \to \mathcal{B}$ measurable. In particular, if $f$ is $\mathcal{A} \to \mathcal{B}$ measurable then so is $|f| = f^+ + f^-$.

A simple function is a function $f : X \to \mathbb{R}$ that is $\mathcal{A} \to \mathcal{B}$ measurable and whose range is finite. Let $E = E(\mathcal{A})$ be the collection of nonnegative simple functions. It is straightforward to prove that

$$u, v \in E, \alpha \geq 0 \implies \alpha u, u + v, u \lor v, u \land v \in E.$$  

Define $I_\mu : E \to [0, \infty]$ by

$$I_\mu u = \sum_{i=1}^{n} a_i \mu(A_i),$$

where $u$ has range $\{a_1, \ldots, a_n\}$ and $A_i = u^{-1}(a_i)$. One proves that $I_\mu : E \to [0, \infty]$ is positive homogeneous, additive, and order preserving.\footnote{Heinz Bauer, Measure and Integration Theory, pp. 55–56, §10.}

It is a fact\footnote{Heinz Bauer, Measure and Integration Theory, p. 57, Theorem 11.1.} that if $u_n$ is a nondecreasing sequence in $E$ and $u \in E$ then

$$u \leq \sup_n u_n \implies I_\mu u \leq \sup_n I_\mu u_n.$$
It follows that if \( u_n \) and \( v_n \) are sequences in \( E \) then

\[
\sup_n u_n = \sup_n v_n \Rightarrow \sup_n I_\mu u_n = \sup_n I_\mu v_n.
\] (1)

Define \( E^* = E^*(\mathcal{A}) \) to be the set of all functions \( f : X \to [0, \infty] \) for which there is a nondecreasing sequence \( u_n \) in \( E \) satisfying \( \sup_n u_n = f \), in other words, there is a sequence \( u_n \) in \( E \) satisfying \( u_n \uparrow f \). From (1), for \( f \in E^* \) and sequences \( u_n, v_n \in E \) with \( \sup_n u_n = f \) and \( \sup_n v_n = f \), it holds that \( \sup_n I_\mu u_n = \sup_n I_\mu v_n \). Also, if \( u \in E \) then \( u_n = u \) is a nondecreasing sequence in \( E \) with \( u = \sup_n u_n \), so \( u \in E^* \). Then it makes sense to extend \( I_\mu \) from \( E \to [0, \infty] \) to \( E^* \to [0, \infty] \) by defining \( I_\mu f = \sup_n I_\mu u_n \). One proves\(^4\) that

\[
f, g \in E^*, \alpha \geq 0 \Rightarrow \alpha f, f + g, f \cdot g, f \vee g, f \wedge g \in E^*
\]

and that \( I_\mu : E^* \to [0, \infty] \) is positive homogeneous, additive, and order preserving.

The monotone convergence theorem\(^5\) states that if \( f_n \) is a sequence in \( E^* \) then \( \sup_n f_n \in E^* \) and

\[
I_\mu \left( \sup_n f_n \right) = \sup_n I_\mu f_n.
\]

We now prove a characterization of \( E^* \).\(^6\)

**Theorem 1.** \( E^* \) is equal to the set of functions \( X \to [0, \infty] \) that are \( \mathcal{A} \to \mathcal{B} \) measurable.

**Proof.** If \( f \in E^* \), then there is a sequence \( u_n \) in \( E \) with \( u_n \uparrow f \). Because each \( u_n \) is measurable \( \mathcal{A} \to \mathcal{B} \), so is \( f \).

Now suppose that \( f : X \to [0, \infty] \) is \( \mathcal{A} \to \mathcal{B} \) measurable. For \( n \geq 1 \) and \( 0 \leq i \leq n2^n - 1 \) let

\[
A_{i,n} = \{ f \geq 2^{-n} \} \cap \{ f < (i+1)2^{-n} \} = \{ 2^{-n} \leq f < (i+1)2^{-n} \},
\]

and for \( i = n2^n \) let

\[
A_{i,n} = \{ f \geq n \}.
\]

Because \( f \) is \( \mathcal{A} \to \mathcal{B} \) measurable, the sets \( A_{i,n} \) belong to \( \mathcal{A} \). For each \( n \), the sets \( A_{0,n}, \ldots, A_{n2^n-1,n}, A_{n2^n,n} \) are pairwise disjoint and their union is equal to \( X \). It is apparent that

\[
A_{i,n} = A_{2i,n+1} \cup A_{2i+1,n+1}, \quad 0 \leq i \leq n2^n - 1. \tag{2}
\]

Define

\[
u_n = \sum_{i=0}^{n2^n} 2^{-n} 1_{A_{i,n}},
\]


which belongs to $E$. For $x \in X$, either $f(x) = \infty$ or $0 \leq f(x) < \infty$. In the first case, $u_n(x) = n$ for all $n \geq 1$. In the second case, $u_n(x) \leq f(x) < u_n(x) + 2^{-n}$ for all $n > f(x)$. Therefore $u_n(x) \uparrow f(x)$ as $n \to \infty$, and because this is true for each $x \in X$, this means $u_n \uparrow f$ and so $f \in E^*$.

So far we have defined $I_\mu : E^* \to [0, \infty]$. Suppose that $f : X \to \mathbb{R}$ is $\mathcal{A} \to \mathcal{B}$ measurable. Then $f^+, f^- : X \to [0, \infty]$ are $\mathcal{A} \to \mathcal{B}$ measurable so by Theorem 1, $f^+, f^- \in E^*$. Then $I_\mu f^+, I_\mu f^- \in [0, \infty]$. We say that a function $f : X \to \mathbb{R}$ is $\mu$-integrable if it is $\mathcal{A} \to \mathcal{B}$ measurable and $I_\mu f^+ < \infty$ and $I_\mu f^- < \infty$. One checks that a function $f : X \to \mathbb{R}$ is $\mu$-integrable if and only if it is $\mathcal{A} \to \mathcal{B}$ measurable and $I_\mu |f| < \infty$. If $f : X \to \mathbb{R}$ is $\mu$-integrable, we now define $I_\mu f \in \mathbb{R}$ by

$$I_\mu f = I_\mu f^+ - I_\mu f^-.$$

For example, if $\mu(X) < \infty$ and $S$ is a subset of $X$ that does not belong to $\mathcal{A}$, define $f : X \to \mathbb{R}$ by $f = 1_S - 1_{X \setminus S}$. Then $f^+ = 1_S$ and $f^- = 1_{X \setminus S}$, and thus $f$ is not $\mathcal{A} \to \mathcal{B}$ measurable, so it is not $\mu$-integrable. But $|f| = 1$ belongs to $E$, and $I_\mu |f| = \mu(X) < \infty$ by hypothesis, showing that $|f|$ is $\mu$-integrable while $f$ is not.

One proves that if $f, g : X \to \mathbb{R}$ are $\mu$-integrable and $\alpha \in \mathbb{R}$ then $\alpha f$ is $\mu$-integrable and

$$I_\mu(\alpha f) = \alpha I_\mu f,$$

if $f + g$ is defined on all $X$ then $f + g$ is $\mu$-integrable and

$$I_\mu(f + g) = I_\mu f + I_\mu g,$$

and $f \vee g, f \wedge g$ are $\mu$-integrable.\(^7\) Furthermore, $I_\mu$ is order preserving.

Let $f : X \to \mathbb{C}$ be a function and write $f = u + iv$. One proves that $f$ is Borel measurable (i.e. $\mathcal{A} \to \mathcal{B}_\mathbb{C}$ measurable), if and only if $u$ and $v$ are measurable $\mathcal{A} \to \mathcal{B}$. We define $f$ to be $\mu$-integrable if both $u$ and $v$ are $\mu$-integrable, and define

$$I_\mu f = I_\mu u + iI_\mu v.$$

2 \hspace{1cm} \mathcal{L}^2

Let $(X, \mathcal{A}, \mu)$ be a measure space and for $1 \leq p < \infty$ let $\mathcal{L}^p(\mu)$ be the collection of Borel measurable functions $f : X \to \mathbb{C}$ such that $|f|^p$ is $\mu$-integrable. For complex $a, b$, because $x \mapsto x^p$ is convex we have by Jensen’s inequality

$$\left|\frac{a + b}{2}\right|^p \leq \left(\frac{1}{2}|a| + \frac{1}{2}|b|\right)^p \leq \frac{1}{2}|a|^p + \frac{1}{2}|b|^p = \frac{1}{2}(|a|^p + |b|^p),$$

so $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$. Thus if $f, g \in \mathcal{L}^p(\mu)$ then

$$|f + g|^p \leq 2^{p-1}(|f|^p + |g|^p),$$

\(^7\)Heinz Bauer, Measure and Integration Theory, p. 65, Theorem 12.3.
which implies that $\mathcal{L}^p(\mu)$ is a linear space.

For Borel measurable $f : X \to \mathbb{C}$ define

$$\|f\|_{L^p} = \left( \int_X |f|^p d\mu \right)^{1/p}.$$  

For $f, g \in \mathcal{L}^p(\mu)$, by Hölder’s inequality, with $\frac{1}{p} + \frac{1}{p'} = 1$ (for which $p' = \frac{p}{p-1}$),

\[
\|f + g\|_{L^p}^p \leq \int_X |f + g|^{p-1} d\mu + \int_X |g| |f|^{p-1} d\mu \leq \|f\|_{L^p} \|f + g\|_{L^{p'}} + \|g\|_{L^p} \|f + g\|_{L^{p'}} = \|f\|_{L^p} \|f + g\|_{L^{p'}} + \|g\|_{L^p} \|f + g\|_{L^{p'}},
\]

which implies that $\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$, and hence $\|\cdot\|_{L^p}$ is a seminorm on $\mathcal{L}^p(\mu)$.

Let $\mathcal{M}^p(\mu)$ be the set of those $f \in \mathcal{L}^p(\mu)$ such that $\|f\|_{L^p} = 0$. $\mathcal{M}^p(\mu)$ is a linear subspace of $\mathcal{L}^p(\mu)$, and we define

$$L^p(\mu) = \mathcal{L}^p(\mu)/\mathcal{M}^p(\mu) = \{f + \mathcal{M}^p(\mu) : f \in \mathcal{L}^p(\mu)\}.$$  

$L^p(\mu)$ is a normed linear space with the norm $\|\cdot\|_{L^p}$.

It is a fact that if $V$ is a normed linear space then $V$ is complete if and only if each absolutely convergent series in $V$ converges in $V$. Suppose that $f_k$ is a sequence in $\mathcal{L}^p(\mu)$ with $\sum_{k=1}^{\infty} \|f_k\|_{L^p} < \infty$. For $n \geq 1$ let $g_n(x) = (\sum_{k=1}^{n} |f_k(x)|)^p$ and define $g : X \to [0, \infty]$ by

$$g(x) = \left( \sum_{k=1}^{\infty} |f_k(x)| \right)^p = \lim_{n \to \infty} g_n(x),$$

which is $\mathcal{A} \to \mathcal{B}$ measurable, being the pointwise limit of a sequence of functions each of which is $\mathcal{A} \to \mathcal{B}$ measurable. Because $g_1 \leq g_2 \leq \cdots$, by the monotone convergence theorem,

$$\int_X gd\mu = \lim_{n \to \infty} \int_X g_n d\mu.$$  

But

$$\left( \int_X g_n d\mu \right)^{1/p} = \left\| \sum_{k=1}^{n} |f_k| \right\|_{L^p} \leq \sum_{k=1}^{n} \|f_k\|_{L^p} \leq \sum_{k=1}^{\infty} \|f_k\|_{L^p},$$

which implies that $\int_X gd\mu < \infty$, meaning that $g : X \to [0, \infty]$ is integrable. The fact that $g$ is integrable implies $\mu(E) = 0$, where $E = \{x \in X : g(x) = \infty\} \in \mathcal{A}$. For $x \in X \setminus E$, $\sum_{k=1}^{\infty} |f_k(x)| < \infty$ and because $\mathbb{C}$ is complete this implies that $\sum_{k=1}^{\infty} f_k(x) \in \mathbb{C}$, and so it makes sense to define $f : X \to \mathbb{C}$ by

$$f(x) = 1_{X \setminus E}(x) \sum_{k=1}^{\infty} f_k(x),$$

where 1}_{X \setminus E}(x) is the indicator function of $X \setminus E$. 

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which is Borel measurable. Furthermore, \(|f|^p \leq g\), and because \(g\) is integrable this implies that \(f \in L^p(\mu)\). For \(x \in X \setminus E\),

\[
\lim_{n \to \infty} \left| \sum_{k=1}^{n} f_k(x) - f(x) \right|^p = 0
\]

and

\[
\left| \sum_{k=1}^{n} f_k(x) - f(x) \right|^p \leq g(x),
\]

so by the dominated convergence theorem,\(^8\)

\[
\lim_{n \to \infty} \int_X \left| \sum_{k=1}^{n} f_k(x) - f(x) \right|^p d\mu = 0.
\]

Because \(x \mapsto x^{1/p}\) is continuous this implies

\[
\lim_{n \to \infty} \left\| \sum_{k=1}^{n} f_k - f \right\|_{L^p} = 0.
\]

Hence, if \(f_k\) is a sequence in \(L^p(\mu)\) such that \(\sum_{k=1}^{\infty} \|f_k\|_{L^p} < \infty\) then there is some \(f \in L^p(\mu)\) such that \(\sum_{k=1}^{n} f_k \to f\) in the norm \(\|\cdot\|_{L^p}\). This implies that \(L^p(\mu)\) is a Banach space.

We say that the \(\sigma\)-algebra \(\mathcal{A}\) is countably generated if there is a countable subset \(\mathcal{C}\) of \(\mathcal{A}\) such that \(\mathcal{A} = \sigma(\mathcal{C})\) and we say that a topological space is separable if there exists a countable dense subset of it. It can be proved that if \(\mathcal{A}\) is countably generated and \(\mu\) is \(\sigma\)-finite, then for \(1 \leq p < \infty\) there is a countable collection of simple functions that is dense in \(L^p(\mu)\), showing that \(L^p(\mu)\) is separable.\(^9\)

**Theorem 2.** Let \((X, \mathcal{A}, \mu)\) be a measure space and let \(1 \leq p < \infty\). \(L^p(\mu)\) with the norm \(\|\cdot\|_{L^p}\) is a Banach space, and if \(\mathcal{A}\) is countably generated and \(\mu\) is \(\sigma\)-finite then \(L^p(\mu)\) is separable.

For \(f, g \in L^2(\mu)\), let

\[
\langle f, g \rangle_{L^2(\mu)} = \int_X f \cdot \overline{g} d\mu.
\]

This is an inner product on \(L^2(\mu)\), and thus \(L^2(\mu)\) is a Hilbert space.

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\(^9\)Donald L. Cohn, *Measure Theory*, second ed., p. 102, Proposition 3.4.5.
3 Product measures

Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be measure spaces and let $\mathcal{A}_1 \otimes \mathcal{A}_2$ be the product $\sigma$-algebra. For $Q \subset X_1 \times X_2$, write

$$Q_{x_1} = \{x_2 \in X_2 : (x_1, x_2) \in Q\}, \quad Q_{x_2} = \{x_1 \in X_1 : (x_1, x_2) \in Q\}.$$

One proves that if $\mu_1$ and $\mu_2$ are $\sigma$-finite, then for each $Q \in \mathcal{A}_1 \otimes \mathcal{A}_2$ the function $x_1 \mapsto \mu_2(Q_{x_1})$ is $\mathcal{A}_1 \rightarrow \mathcal{B}$ measurable and the function $x_2 \mapsto \mu_1(Q_{x_2})$ is $\mathcal{A}_2 \rightarrow \mathcal{B}$ measurable. If $\mu_1$ and $\mu_2$ are $\sigma$-finite, one proves that there is a unique measure $\mu : \mathcal{A}_1 \otimes \mathcal{A}_2 \rightarrow [0, \infty]$ that satisfies

$$\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2), \quad A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2.$$

The measure $\mu$ satisfies

$$\mu(Q) = \int_{X_1} \mu_2(Q_{x_1}) d\mu_1(x_1) = \int_{X_2} \mu_1(Q_{x_2}) d\mu_2(x_2)$$

for $Q \in \mathcal{A}_1 \otimes \mathcal{A}_2$, and is itself $\sigma$-finite. We write $\mu = \mu_1 \otimes \mu_2$, and call $\mu$ the product measure of $\mu_1$ and $\mu_2$.

Let $X'$ be a set and let $f : X_1 \times X_2 \rightarrow X'$ be a function. For $x_1 \in X_1$, define $f_{x_1} : X_2 \rightarrow X'$ by

$$f_{x_1}(x_2) = f(x_1, x_2), \quad x_2 \in X_2$$

and for $x_2 \in X_2$, define $f_{x_2} : X_1 \rightarrow X'$ by

$$f_{x_2}(x_1) = f(x_1, x_2), \quad x_1 \in X_1.$$

For $Q \subset X_1 \times X_2$,

$$(1_Q)_{x_1} = 1_{Q_{x_1}}, \quad (1_Q)_{x_2} = 1_{Q_{x_2}}.$$  

It is straightforward to prove that if $(X', \mathcal{A}')$ is a measurable space and $f : (X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2) \rightarrow (X', \mathcal{A}')$ is measurable, then for each $x_1 \in X_1$ the function $f_{x_1} : X_2 \rightarrow X'$ is measurable $\mathcal{A}_2 \rightarrow \mathcal{A}'$ and for each $x_2 \in X_2$ the function $f_{x_2} : X_1 \rightarrow X'$ is measurable $\mathcal{A}_1 \rightarrow \mathcal{A}'$.

**Tonelli’s theorem**\(^{13}\) states that if $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ are $\sigma$-finite measure spaces and $f : X_1 \times X_2 \rightarrow [0, \infty]$ is $\mathcal{A}_1 \otimes \mathcal{A}_2 \rightarrow \mathcal{B}$ measurable, then the functions

$$x_2 \mapsto \int_{X_1} f_{x_2} d\mu_1, \quad x_1 \mapsto \int_{X_2} f_{x_1} d\mu_2$$

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\(^{11}\)Heinz Bauer, *Measure and Integration Theory*, p. 136, Theorem 23.3.


are \( \mathcal{A}_2 \rightarrow \mathcal{R} \) measurable and \( \mathcal{A}_1 \rightarrow \mathcal{R} \) measurable respectively, and

\[
\int_{X_1 \times X_2} f \, d(\mu_1 \otimes \mu_2)
= \int_{X_2} \left( \int_{X_1} f_{x_2} \, d\mu_1 \right) \, d\mu_2(x_2) 
= \int_{X_1} \left( \int_{X_2} f_{x_1} \, d\mu_2 \right) \, d\mu_1(x_1). 
\]

**Fubini’s theorem**\(^\text{14}\) states that if \((X_1, \mathcal{A}_1, \mu_1)\) and \((X_2, \mathcal{A}_2, \mu_2)\) are \(\sigma\)-finite measure spaces and \(f : X_1 \times X_2 \rightarrow \mathbb{R}\) is \(\mu_1 \otimes \mu_2\)-integrable then there is some \(A_1 \in \mathcal{A}_1\) with \(\mu_1(A_1) = 0\) such that for \(x_1 \in X_1 \setminus A_1\) the function \(f_{x_1} : X_2 \rightarrow \mathbb{R}\) is \(\mu_2\)-integrable, and there is some \(A_2 \in \mathcal{A}_2\) with \(\mu_2(A_2) = 0\) such that for \(x_2 \in X_2 \setminus A_2\) the function \(f_{x_2} : X_1 \rightarrow \mathbb{R}\) is \(\mu_1\)-integrable. Furthermore, define \(F_1 : X_1 \rightarrow \mathbb{R}\) by \(F_1(x_1) = \int_{X_2} f_{x_1} \, d\mu_2\) for \(x_1 \in X_1 \setminus A_1\) and \(F_1(x_1) = 0\) for \(x_1 \in A_1\), and define \(F_2 : X_2 \rightarrow \mathbb{R}\) by \(F_2(x_2) = \int_{X_1} f_{x_2} \, d\mu_1\) for \(x_2 \in X_2 \setminus A_2\) and \(F_2(x_2) = 0\) for \(x_2 \in A_2\). The functions \(F_1\) and \(F_2\) are \(\mu_1\)-integrable and \(\mu_2\)-integrable respectively, and

\[
\int_{X_1 \times X_2} f \, d(\mu_1 \otimes \mu_2) = \int_{X_1} F_1 \, d\mu_1 = \int_{X_2} F_2 \, d\mu_2. 
\]

Suppose that \((X_1, \mathcal{A}_1, \mu_1)\) and \((X_2, \mathcal{A}_2, \mu_2)\) are \(\sigma\)-finite measure spaces. For \(e : X_1 \rightarrow \mathbb{C}\) and \(f : X_2 \rightarrow \mathbb{C}\), define \(e \otimes f : X_1 \times X_2 \rightarrow \mathbb{C}\) by

\[
(e \otimes f)(x_1, x_2) = e(x_1) f(x_2),
\]

which is Borel measurable \(X_1 \times X_2 \rightarrow \mathbb{C}\) if \(e\) and \(f\) are Borel measurable. If \(e \in \mathcal{L}^2(\mu_1)\) and \(f \in \mathcal{L}^2(\mu_2)\), then by Tonelli’s theorem \(e \otimes f : X_1 \times X_2 \rightarrow \mathbb{C}\) belongs to \(\mathcal{L}^2(\mu_1 \otimes \mu_2)\). For \(e, e' \in \mathcal{L}^2(\mu_1)\) and \(f, f' \in \mathcal{L}^2(\mu_2)\), by Fubini’s theorem,

\[
\langle e \otimes f, e' \otimes f' \rangle_{\mathcal{L}^2(\mu_1 \otimes \mu_2)}
= \int_{X_1 \times X_2} e(x_1) f(x_2) \overline{e'(x_1) f'(x_2)} \, d(\mu_1 \otimes \mu_2)(x_1, x_2)
= \int_{X_1} \left( \int_{X_2} e(x_1) \overline{e'(x_1)} \, d\mu_1(x_1) \right) f(x_2) \overline{f'(x_2)} \, d\mu_2(x_2)
= \langle e, e' \rangle_{\mathcal{L}^2(\mu_1)} \cdot \langle f, f' \rangle_{\mathcal{L}^2(\mu_2)}.
\]

Therefore, if \(E \subset \mathcal{L}^2(\mu_1)\) is an orthonormal set in \(L^2(\mu_1)\) and \(F \subset \mathcal{L}^2(\mu_2)\) is an orthonormal set in \(L^2(\mu_2)\), then \(\{ e \otimes f : e \in E, f \in F \} \subset \mathcal{L}^2(\mu_1 \otimes \mu_2)\) is an orthonormal set in \(L^2(\mu_1 \otimes \mu_2)\).

**Theorem 3.** Let \((X_1, \mathcal{A}_1, \mu_1)\) and \((X_2, \mathcal{A}_2, \mu_2)\) be \(\sigma\)-finite measure spaces and suppose that \(L^2(\mu_1)\) and \(L^2(\mu_2)\) are separable. If \(E \subset \mathcal{L}^2(\mu_1)\) is an orthonormal

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\(^{14}\)Heinz Bauer, *Measure and Integration Theory*, p. 139, Corollary 23.7.
basis for $L^2(\mu_1)$ and $F \subset L^2(\mu_2)$ is an orthonormal basis for $L^2(\mu_2)$, then
\[ \Phi = \{ e \otimes f : e \in E, f \in F \} \subset L^2(\mu_1 \otimes \mu_2) \] is an orthonormal basis for $L^2(\mu_1 \otimes \mu_2)$.

Proof. To show that $\Phi$ is an orthonormal basis for $L^2(\mu_1 \otimes \mu_2)$ it suffices to prove that if $h \in L^2(\mu_1 \otimes \mu_2)$ belongs to the orthogonal complement of $\Phi$ then $h \in \mathcal{N}^2(\mu_1 \otimes \mu_2)$. Thus, suppose that $h \in L^2(\mu_1 \otimes \mu_2)$ and that $\langle h, e \otimes f \rangle_{L^2(\mu_1 \otimes \mu_2)} = 0$ for all $e \in E, f \in F$. Using Fubini’s theorem,
\[
\int_{X_1} e(x_1) \left( \int_{X_2} h_{x_1}(x_2) f(x_2) d\mu_2(x_2) \right) d\mu_1(x_1) = 0.
\]
Because this is true for all $e \in E$ and $E$ is dense in $L^2(\mu_1)$, it follows that there is some $A_f \in \mathcal{H}_1$ with $\mu_1(A_f) = 0$ such that $\int_{X_2} h_{x_1} f d\mu_2 = 0$ for $x_1 \not\in A_f$. Let $A_1 = \bigcup_{f \in F} A_f$, for which $\mu_1(A_1) = 0$. If $x_1 \not\in A_1$ then $\int_{X_2} h_{x_1} f d\mu_2 = 0$ for all $f \in F$, and because $F$ is dense in $L^2(\mu_2)$ this implies that $h_{x_1} = 0 \mu_2$-almost everywhere. Then
\[
\int_{X_1 \times X_2} |h|^2 d(\mu_1 \otimes \mu_2) = \int_{X_1} \left( \int_{X_2} |h_{x_1}|^2 d\mu_2 \right) d\mu_1(x_1)
= \int_{X_1 \setminus A_1} \left( \int_{X_2} |h_{x_1}|^2 d\mu_2 \right) d\mu_1(x_1)
= 0,
\]
which implies that $h = 0 \mu_1 \otimes \mu_2$-almost everywhere. \qed