Jointly measurable and progressively measurable stochastic processes

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1 Jointly measurable stochastic processes

Let $E = \mathbb{R}^d$ with Borel $\mathscr{E}$, let $I = \mathbb{R}_{\geq 0}$, which is a topological space with the subspace topology inherited from $\mathbb{R}$, and let $(\Omega, \mathscr{F}, P)$ be a probability space. For a stochastic process $(X_t)_{t \in I}$ with state space E, we say that X is **jointly measurable** if the map $(t, \omega) \mapsto X_t(\omega)$ is measurable $\mathscr{B}_I \otimes \mathscr{F} \to \mathscr{E}$.

For $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ is called **left-continuous** if for each $t \in I$,

$$X_s(\omega) \to X_t(\omega), \qquad s \uparrow t.$$

We prove that if the paths of a stochastic process are left-continuous then the stochastic process is jointly measurable.[1]

Theorem 1. *If X is a stochastic process with state space E and all the paths of X are left-continuous, then X is jointly measurable.*

Proof. For $n \geq 1$, $t \in I$, and $\omega \in \Omega$, let

$$X_t^n(\omega) = \sum_{k=0}^{\infty} 1_{[k2^{-n},(k+1)2^{-n})}(t) X_{k2^{-n}}(\omega).$$

Each X^n is measurable $\mathscr{B}_I \otimes \mathscr{F} \to \mathscr{E}$: for $B \in \mathscr{E}$,

$$\{(t,\omega) \in I \times \Omega : X_t^n(\omega) \in B\} = \bigcup_{k=0}^{\infty} [k2^{-n}, (k+1)2^{-n}) \times \{X_{k2^{-n}} \in B\}.$$

Let $t \in I$. For each n there is a unique k_n for which $t \in [k_n 2^{-n}, (k_n+1)2^{-n})$, and thus $X_t^n(\omega) = X_{k_n 2^{-n}}(\omega)$. Furthermore, $k_n 2^{-n} \uparrow t$, and because $s \mapsto X_s(\omega)$ is left-continuous, $X_{k_n 2^{-n}}(\omega) \to X_t(\omega)$. That is, $X^n \to X$ pointwise on $I \times \Omega$, and

[1] cf. Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 153, Lemma 4.51.

because each X_n is measurable $\mathscr{B}_I \otimes \mathscr{F} \to \mathscr{E}$ this implies that X is measurable $\mathscr{B}_I \otimes \mathscr{F} \to \mathscr{E}$.[2] Namely, the stochastic process $(X_t)_{t\in I}$ is jointly measurable, proving the claim. $\square$

2 Adapted stochastic processes

Let $\mathscr{F}_I = (\mathscr{F}_t)_{t\in I}$ be a filtration of $\mathscr{F}$. A stochastic process X is said to be **adapted to the filtration** $\mathscr{F}_I$ if for each $t \in I$ the map

$$\omega \mapsto X_t(\omega), \qquad \Omega \to E,$$

is measurable $\mathscr{F}_t \to \mathscr{E}$, in other words, for each $t \in I$,

$$\sigma(X_t) \subset \mathscr{F}_t.$$

For a stochastic process $(X_t)_{t\in I}$, the **natural filtration of** X is

$$\sigma(X_s : s \leq t).$$

It is immediate that this is a filtration and that X is adapted to it.

3 Progressively measurable stochastic processes

Let $\mathscr{F}_I = (\mathscr{F}_t)_{t\in I}$ be a filtration of $\mathscr{F}$. A function $X : I \times \Omega \to E$ is called **progressively measurable with respect to the filtration** $\mathscr{F}_I$ if for each $t \in I$, the map

$$(s,\omega) \mapsto X(s,\omega), \qquad [0,t] \times \Omega \to E,$$

is measurable $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t \to \mathscr{E}$. We denote by $\mathscr{M}^0(\mathscr{F}_I)$ the set of functions $I \times \Omega \to E$ that are progressively measurable with respect to the filtration $\mathscr{F}_I$. We shall speak about a stochastic process $(X_t)_{t\in I}$ being progressively measurable, by which we mean that the map $(t,\omega) \mapsto X_t(\omega)$ is progressively measurable.

We denote by $\mathrm{Prog}(\mathscr{F}_I)$ the collection of those subsets A of $I \times \Omega$ such that for each $t \in I$,

$$([0,t] \times \Omega) \cap A \in \mathscr{B}_{[0,t]} \otimes \mathscr{F}_t.$$

We prove in the following that this is a σ-subalgebra of $\mathscr{B}_I \otimes \mathscr{F}$ and that it is the coarsest σ-algebra with which all progressively measurable functions are measurable.

Theorem 2. *Let $\mathscr{F}_I = (\mathscr{F}_t)_{t\in I}$ be a filtration of $\mathscr{F}$.*

1. $\mathrm{Prog}(\mathscr{F}_I)$ *is a σ-subalgebra of $\mathscr{B}_I \otimes \mathscr{F}$, and is the σ-algebra generated by the collection of functions $I \times \Omega \to E$ that are progressively measurable with respect to the filtration $\mathscr{F}_I$:*

$$\mathrm{Prog}(\mathscr{F}_I) = \sigma(\mathscr{M}^0(\mathscr{F}_I)).$$

[2] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 142, Lemma 4.29.

2. *If $X : I \times \Omega \to E$ is progressively measurable with respect to the filtration $\mathscr{F}_I$, then the stochastic process $(X_t)_{t\in I}$ is jointly measurable and is adapted to the filtration.*

Proof. If $A_1, A_2, \ldots \in \mathrm{Prog}(\mathscr{F}_I)$ and $t \in I$ then

$$([0,t]\times\Omega) \cap \bigcup_{n\geq 1} A_n = \bigcup_{n\geq 1} (([0,t]\times\Omega)\cap A_n),$$

which is a countable union of elements of the σ-algebra $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t$ and hence belongs to $\mathscr{B}_{[0,t]}\otimes\mathscr{F}_t$, showing that $\bigcup_{n\geq 1} A_n \in \mathrm{Prog}(\mathscr{F}_I)$. If $A_1, A_2 \in \mathrm{Prog}(\mathscr{F}_I)$ and $t \in I$ then

$$([0,t]\times\Omega)\cap(A_1\cap A_2) = (([0,t]\times\Omega)\cap A_1)\cap(([0,t]\times\Omega)\cap A_2),$$

which is an intersection of two elements of $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t$ and hence belongs to $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t$, showing that $A_1 \cap A_2 \in \mathrm{Prog}(\mathscr{F}_I)$. Thus $\mathrm{Prog}((\mathscr{F}_t)_{t\in I})$ is a σ-algebra.

If $X : I \times \Omega \to E$ is progressively measurable, $B \in \mathscr{E}$, and $t \in I$, then

$$([0,t]\times\Omega)\cap X^{-1}(B) = \{(s,\omega) \in [0,t]\times\Omega : X(s,\omega) \in B\}.$$

Because X is progressively measurable, this belongs to $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t$. This is true for all t, hence $X^{-1}(B) \in \mathrm{Prog}(\mathscr{F}_I)$, which means that X is measurable $\mathrm{Prog}(\mathscr{F}_I) \to \mathscr{E}$.

If $X : I \times \Omega \to E$ is measurable $\mathrm{Prog}(\mathscr{F}_I) \to \mathscr{E}$, $t \in I$, and $B \in \mathscr{E}$, then because $X^{-1}(B) \in \mathrm{Prog}(\mathscr{F}_I)$, we have $([0,t]\times\Omega)\cap X^{-1}(B) \in \mathscr{B}_{[0,t]}\otimes\mathscr{F}_t$. This is true for all $B \in \mathscr{E}$, which means that $(s,\omega) \mapsto X(s,\omega)$, $[0,t]\times\Omega \to E$, is measurable $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t$, and because this is true for all t, X is progressively measurable. Therefore a function $I \times \Omega \to E$ is progressively measurable if and only if it is measurable $\mathrm{Prog}(\mathscr{F}_I) \to \mathscr{E}$, which shows that $\mathrm{Prog}(\mathscr{F}_I)$ is the coarsest σ-algebra with which all progressively measurable functions are measurable.

If $X : I \times \Omega \to E$ is a progressively measurable function and $B \in \mathscr{E}$,

$$X^{-1}(B) = \bigcup_{k\geq 1} (([0,k]\times\Omega)\cap X^{-1}(B)).$$

Because X is progressively measurable,

$$([0,k]\times\Omega)\cap X^{-1}(B) \in \mathscr{B}_{[0,k]}\otimes\mathscr{F}_k \subset \mathscr{B}_I\otimes\mathscr{F},$$

thus $X^{-1}(B)$ is equal to a countable union of elements of $\mathscr{B}_I \otimes \mathscr{F}$ and so itself belongs to $\mathscr{B}_I \otimes \mathscr{F}$. Therefore X is measurable $\mathscr{B}_I \otimes \mathscr{F} \to \mathscr{E}$, namely X is jointly measurable.

Because $\mathrm{Prog}(\mathscr{F}_I)$ is the σ-algebra generated by the collection of progressively measurable functions and each progressively measurable function is measurable $\mathscr{B}_I \otimes \mathscr{F}$,

$$\mathrm{Prog}(\mathscr{F}_I) \subset \mathscr{B}_I \otimes \mathscr{F},$$

and so $\mathrm{Prog}(\mathscr{F}_I)$ is indeed a σ-subalgebra of $\mathscr{B}_I \otimes \mathscr{F}$.

Let $t \in I$. That X is progressively measurable means that

$$(s,\omega) \mapsto X(s,\omega), \qquad [0,t] \times \Omega$$

is measurable $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t \to \mathscr{E}$. This implies that for each $s \in [0,t]$ the map $\omega \mapsto X(s,\omega)$ is measurable $\mathscr{F}_t \to \mathscr{E}$.[3] (Generally, if a function is jointly measurable then it is separately measurable in each argument.) In particular, $\omega \mapsto X(t,\omega)$ is measurable $\mathscr{F}_t \to \mathscr{E}$, which means that the stochastic process $(X_t)_{t\in I}$ is adapted to the filtration, completing the proof. □

We now prove that if a stochastic process is adapted and left-continuous then it is progressively measurable.[4]

Theorem 3. *Let $(\mathscr{F}_t)_{t\in I}$ be a filtration of $\mathscr{F}$. If $(X_t)_{t\in I}$ is a stochastic process that is adapted to this filtration and all its paths are left-continuous, then X is progressively measurable with respect to this filtration.*

Proof. Write $X(t,\omega) = X_t(\omega)$. For $t \in I$, let Y be the restriction of X to $[0,t] \times \Omega$. We wish to prove that Y is measurable $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t \to \mathscr{E}$. For $n \geq 1$, define

$$Y_n(s,\omega) = \sum_{k=0}^{2^n-1} 1_{[kt2^{-n},(k+1)t2^{-n})}(s) Y(kt2^{-n},\omega) + 1_{\{t\}}(s) Y(t,\omega).$$

Because X is adapted to the filtration, each Y_n is measurable $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t \to \mathscr{E}$. Because X has left-continuous paths, for $(s,\omega) \in [0,t] \times \Omega$,

$$Y_n(s,\omega) \to Y(s,\omega).$$

Since Y is the pointwise limit of Y_n, it follows that Y is measurable $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t \to \mathscr{E}$, and so X is progressively measurable. □

4 Stopping times

Let $\mathscr{F}_I = (\mathscr{F}_t)_{t\in I}$ be a filtration of $\mathscr{F}$. A function $T : \Omega \to [0,\infty]$ is called a **stopping time with respect to the filtration $\mathscr{F}_I$** if

$$\{T \leq t\} \in \mathscr{F}_t, \qquad t \in I.$$

It is straightforward to prove that a stopping time is measurable $\mathscr{F} \to \mathscr{B}_{[0,\infty]}$. Let

$$\mathscr{F}_\infty = \sigma(\mathscr{F}_t : t \in I).$$

[3] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 152, Theorem 4.48.

[4] cf. Daniel W. Stroock, *Probability Theory: An Analytic View*, second ed., p. 267, Lemma 7.1.2.

We define

$$\mathscr{F}_T = \{A \in \mathscr{F}_\infty : \text{if } t \in I \text{ then } A \cap \{T \le t\} \in \mathscr{F}_t\}.$$

It is straightforward to check that T is measurable $\mathscr{F}_T \to \mathscr{B}_{[0,\infty]}$, and in particular $\{T < \infty\} \in \mathscr{F}_T$.

For a stochastic process $(X_t)_{t\in I}$ with state space E, we define $X_T : \Omega \to E$ by

$$X_T(\omega) = 1_{\{T<\infty\}}(\omega) X_{T(\omega)}(\omega).$$

We prove that if X is progressively measurable then X_T is measurable $\mathscr{F}_T \to \mathscr{E}$.[5]

Theorem 4. *If $\mathscr{F}_I = (\mathscr{F}_t)_{t\in I}$ is a filtration of $\mathscr{F}$, $(X_t)_{t\in I}$ is a stochastic process that is progressively measurable with respect to $\mathscr{F}_I$, and T is a stopping time with respect to $\mathscr{F}_I$, then X_T is measurable $\mathscr{F}_T \to \mathscr{E}$.*

Proof. For $t \in I$, using that T is a stopping time we check that $\omega \mapsto T(\omega) \wedge t$ is measurable $\mathscr{F}_t \to \mathscr{B}_{[0,t]}$, and then $\omega \mapsto (T(\omega) \wedge t, \omega)$, $\Omega \to [0,t] \times \Omega$, is measurable $\mathscr{F}_t \to \mathscr{B}_{[0,t]} \otimes \mathscr{F}_t$.[6] Because X is progressively measurable, $(s,\omega) \mapsto X_s(\omega)$ is measurable $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t \to \mathscr{E}$. Therefore the composition

$$\omega \mapsto X_{T(\omega)\wedge t}(\omega), \qquad \Omega \to E,$$

is measurable $\mathscr{F}_t \to \mathscr{E}$, and a fortiori it is measurable $\mathscr{F}_\infty \to \mathscr{E}$. We have

$$X_T(\omega) = \lim_{n\to\infty} 1_{\{T\le n\}}(\omega) X_{T(\omega)\wedge n}(\omega),$$

and because $\omega \mapsto 1_{\{T\le n\}}(\omega) X_{T(\omega)\wedge n}(\omega)$ is measurable $\mathscr{F}_\infty \to \mathscr{E}$, it follows that $\omega \mapsto X_T(\omega)$ is measurable $\mathscr{F}_\infty \to \mathscr{E}$. For $B \in \mathscr{E}$,

$$\{X_T \in B\} \cap \{T \le t\} = \{\omega \in \Omega : X_{T(\omega)\wedge t}(\omega) \in B\} \cap \{T \le t\} \in \mathscr{F}_t,$$

therefore $\{X_T \in B\} \in \mathscr{F}_T$. This means that X_T is measurable $\mathscr{F}_T \to \mathscr{E}$. □

For a stochastic process $(X_t)_{t\in I}$, a filtration $\mathscr{F}_I = (\mathscr{F}_t)_{t\in I}$, and a stopping time T with respect to the filtration, we define

$$X_t^T(\omega) = X_{T(\omega)\wedge t}(\omega),$$

and $(X_t^T)_{t\in I}$ is a stochastic process. We prove that if X is progressively measurable with respect to $\mathscr{F}_I$ then the stochastic proces X^T is progressively measurable with respect to $\mathscr{F}_I$.[7]

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[5] Sheng-wu He and Jia-gang Wang and Jia-An Yan, *Semimartingale Theory and Stochastic Calculus*, p. 86, Theorem 3.12.

[6] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 152, Lemma 4.49.

[7] Ioannis Karatzas and Steven Shreve, *Brownian Motion and Stochastic Calculus*, p. 9, Proposition 2.18.

Theorem 5. *If $(X_t)_{t\in I}$ is a stochastic process that is progressively measurable with respect to a filtration $\mathscr{F}_I = (\mathscr{F}_t)_{t\in I}$ and T is a stopping time with respect to $\mathscr{F}_I$, then X^T is progressively measurable with respect to $\mathscr{F}_I$.*

Proof. Let $t \in I$. Because T is a stopping time, for each $s \in [0,t]$ the map $\omega \mapsto T(\omega)\wedge s$ is measurable $\mathscr{F}_s \to \mathscr{B}_{[0,t]}$ and a fortiori is measurable $\mathscr{F}_t \to \mathscr{B}_{[0,t]}$. Therefore $(s,\omega) \mapsto T(\omega) \wedge s$ is measurable $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t \to \mathscr{B}_{[0,t]}$,[8] and $(s,\omega) \mapsto \omega$ is measurable $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t \to \mathscr{F}_t$. This implies that

$$(s,\omega) \mapsto (T(\omega) \wedge s, \omega), \qquad [0,t] \times \Omega \to [0,t] \times \Omega,$$

is measurable $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t \to \mathscr{B}_{[0,t]} \otimes \mathscr{F}_t$.[9] Because X is progressively measurable,

$$(s,\omega) \mapsto X_s(\omega), \qquad [0,t] \times \Omega \to E,$$

is measurable $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t \to \mathscr{E}$. Therefore the composition

$$(s,\omega) \mapsto X_{T(\omega)\wedge s}(\omega), \qquad [0,t] \times \Omega \to E,$$

is measurable $\mathscr{B}_{[0,t]} \otimes \mathscr{F}_t \to \mathscr{E}$, which shows that X^T is progressively measurable. □

[8] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 152, Theorem 4.48.

[9] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 152, Lemma 4.49.