Rademacher functions

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1 Binary expansions

Define $S : \{0,1\}^{\mathbb{N}} \to [0,1]$ by

$$S(\sigma) = \sum_{k=1}^{\infty} \frac{\sigma_k}{2^k}, \qquad \sigma \in \{0,1\}^{\mathbb{N}}.$$

For example, for $\sigma_1 = 0$ and $\sigma_2 = 1, \sigma_3 = 1, \ldots,$

$$S(\sigma) = \frac{0}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{2};$$

for $\sigma_1 = 1$ and $\sigma_2 = 0$, $\sigma_3 = 0, \ldots,$

$$S(\sigma) = \frac{1}{2} + \frac{0}{4} + \frac{0}{8} + \cdots = \frac{1}{2}.$$

Let $\sigma \in \{0,1\}^{\mathbb{N}}$. If there is some $n \in \mathbb{N}$ such that $\sigma_n = 0$ and $\sigma_k = 1$ for all $k \geq n+1$, then defining

$$\tau_k = \begin{cases} \sigma_k & k \leq n-1 \\ 1 & k = n \\ 0 & k \geq n, \end{cases}$$

we have

$$S(\sigma) = \sum_{k=1}^{n-1} \frac{\sigma_k}{2^k} + \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \sum_{k=1}^{n-1} \frac{\sigma_k}{2^k} + \frac{1}{2^n} = S(\tau).$$

One proves that if either (i) there is some $n \in \mathbb{N}$ such that $\sigma_n = 0$ and $\sigma_k = 1$ for all $k \geq n+1$ or (ii) there is some $n \in \mathbb{N}$ such that $\sigma_n = 1$ and $\sigma_k = 0$ for all $k \geq n+1$, then $S^{-1}(S(\sigma))$ contains exactly two elements, and that otherwise $S^{-1}(S(\sigma))$ contains exactly one element.

In words, except for the sequence whose terms are only 0 or the sequence whose terms are only 1, $S^{-1}(S(\sigma))$ contains exactly two elements when σ is eventually 0 or eventually 1, and $S^{-1}(S(\sigma))$ contains exactly one element otherwise.

We define $\epsilon : [0,1] \to \{0,1\}^{\mathbb{N}}$ by taking $\epsilon(t)$ to be the unique element of $S^{-1}(t)$ if $S^{-1}(t)$ contains exactly one element, and to be the element of $S^{-1}(t)$ that is eventually 0 if $S^{-1}(t)$ contains exactly two elements. For $k \in \mathbb{N}$ we define $\epsilon_k : [0,1] \to \{0,1\}$ by

$$\epsilon_k(t) = \epsilon(t)_k, \qquad t \in [0,1].$$

Then, for all $t \in [0,1]$,

$$t = S(\epsilon(t)) = \sum_{k=1}^{\infty} \frac{\epsilon_k(t)}{2^k}, \tag{1}$$

which we call **the binary expansion of** t.

2 Rademacher functions

For $k \in \mathbb{N}$, the k**th Rademacher function** $r_k : [0,1] \to \{-1,+1\}$ is defined by

$$r_k(t) = 1 - 2\epsilon_k(t), \qquad t \in [0,1].$$

We can rewrite the binary expansion of $t \in [0,1]$ in (1) as

$$\sum_{k=1}^{\infty} \frac{r_k(t)}{2^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2^k} - 2 \cdot \frac{\epsilon_k(t)}{2^k} \right) = 1 - 2\sum_{k=1}^{\infty} \frac{\epsilon_k(t)}{2^k} = 1 - 2t. \tag{2}$$

Define $r : \mathbb{R} \to \{-1,+1\}$ by

$$r(x) = (-1)^{[x]},$$

where $[x]$ denotes the greatest integer $\leq x$. Thus, for $0 \leq x < 1$ we have $r(x) = 1$, for $1 \leq x < 2$ we have $r(x) = -1$, and r has period 2.

Lemma 1. *For any* $n \in \mathbb{N}$,

$$r_n(t) = (-1)^{[2^n t]} = r(2^n t), \qquad t \in [0,1]$$

In the following theorem we use the Rademacher functions to prove an identity for trigonometric functions.[1]

Theorem 2. *For any nonzero real* x,

$$\prod_{k=1}^{\infty} \cos \frac{x}{2^k} = \frac{\sin x}{x}.$$

[1] Mark Kac, *Statistical Independence in Probability, Analysis and Number Theory*, p. 4, §3.

Proof. Let $n \in \mathbb{N}$ and let $c_1, \ldots, c_n \in \mathbb{R}$. The function

$$\sum_{k=1}^{n} c_k r_k$$

is constant on each of the intervals

$$\left[\frac{s}{2^n}, \frac{s+1}{2^n}\right), \qquad 0 \leq s \leq 2^n - 1. \tag{3}$$

There is a bijection between $\Delta_n = \{-1, +1\}^n$ and the collection of intervals (3). Without explicitly describing this bijection, we have

$$\begin{aligned} \int_0^1 \exp\left(i \sum_{k=1}^{n} c_k r_k(t)\right) dt &= \sum_{s=0}^{2^n-1} \int_{s \cdot 2^{-n}}^{(s+1)\cdot 2^{-n}} \exp\left(i \sum_{k=1}^{n} c_k r_k(t)\right) dt \\ &= \sum_{\delta \in \Delta_n} \frac{1}{2^n} \exp\left(i \sum_{k=1}^{n} \delta_k c_k\right) \\ &= \sum_{\delta \in \Delta_n} \prod_{k=1}^{n} \frac{e^{i\delta_k c_k}}{2} \\ &= \prod_{k=1}^{n} \frac{e^{ic_k} + e^{-ic_k}}{2}, \end{aligned}$$

giving

$$\int_0^1 \exp\left(i \sum_{k=1}^{n} c_k r_k(t)\right) dt = \prod_{k=1}^{n} \cos c_k. \tag{4}$$

We have

$$\int_0^1 e^{ix(1-2t)} dt = e^{ix} \frac{e^{-2ixt}}{-2ix}\bigg|_0^1 = e^{ix}\left(\frac{e^{-2ix}}{-2ix} + \frac{1}{2ix}\right) = \frac{\sin x}{x}. \tag{5}$$

Using (2) we check that the sequence of functions $\sum_{k=1}^{n} \frac{r_k(t)}{2^k}$ converges uniformly on $[0,1]$ to $1 - 2t$, and hence using (5) we get

$$\int_0^1 \exp\left(ix \sum_{k=1}^{n} \frac{r_k(t)}{2^k}\right) dt \to \int_0^1 e^{ix(1-2t)} dt = \frac{\sin x}{x}$$

as $n \to \infty$. Combining this with (4), which we apply with $c_k = \frac{x}{2^k}$, we get

$$\prod_{k=1}^{n} \cos \frac{x}{2^k} \to \frac{\sin x}{x}$$

as $n \to \infty$, proving the claim. □

We now give an explicit formula for the measure of those t for which exactly l of $r_1(t), \dots, r_n(t)$ are equal to 1.[2] We denote by μ Lebesgue measure on $\mathbb{R}$. We can interpret the following formula as stating the probability that out of n tosses of a coin, exactly l of the outcomes are heads.

Theorem 3. *For $n \in \mathbb{N}$ and $0 \le l \le n$,*

$$\mu\{t \in [0,1] : r_1(t) + \cdots + r_n(t) = 2l - n\} = \frac{1}{2^n}\binom{n}{l}.$$

Proof. Define $\phi : [0,1] \to \mathbb{R}$ by

$$\phi(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\left(-(2l-n)+\sum_{k=1}^n r_k(t)\right)} dx$$

But for $m \in \mathbb{Z}$,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{imx} dx = \delta_{m,0} = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0, \end{cases} \tag{6}$$

hence

$$\phi(t) = \begin{cases} 1 & \sum_{k=1}^n r_k(t) = 2l - n \\ 0 & \sum_{k=1}^n r_k(t) \neq 2l - n. \end{cases}$$

Therefore

$$\begin{aligned} \mu\left\{t \in [0,1] : \sum_{k=1}^n r_k(t) = 2l - n\right\} &= \int_0^1 \phi(t) dt \\ &= \int_0^1 \frac{1}{2\pi}\int_0^{2\pi} e^{ix\left(-(2l-n)+\sum_{k=1}^n r_k(t)\right)} dx dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-ix(2l-n)} \int_0^1 e^{ix\sum_{k=1}^n r_k(t)} dt dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-ix(2l-n)} \cos^n x dx; \end{aligned}$$

the last equality uses (4) with $c_1 = x, \dots, c_n = x$. Furthermore, writing

$$\cos^n x = 2^{-n}(e^{ix} + e^{-ix}) = 2^{-n}\sum_{k=0}^n \binom{n}{k} e^{ix(2k-n)},$$

[2] Mark Kac, *Statistical Independence in Probability, Analysis and Number Theory*, pp. 8–9.

we calculate using (6) that

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} e^{-ix(2l-n)}\cos^n x dx &= 2^{-n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2\pi}\int_0^1 e^{-ix(2l-n)}e^{ix(2k-n)}dx \\
&= 2^{-n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2\pi}\int_0^1 e^{ix(2k-2l)}dx \\
&= 2^{-n}\sum_{k=0}^{n}\binom{n}{k}\delta_{2k-2l,0} \\
&= 2^{-n}\sum_{k=0}^{n}\binom{n}{k}\delta_{k,l} \\
&= 2^{-n}\binom{n}{l},
\end{aligned}$$

proving the claim. □

We now prove that the expected value of a product of distinct Rademacher functions is equal to the product of their expected values.[3]

Theorem 4. *If $k_1, \dots, k_n$ are positive integers and $k_1 < \cdots < k_n$, then*

$$\int_0^1 r_{k_1}(t)\cdots r_{k_n}(t)dt = 0.$$

Proof. Write $J = \int_0^1 r_{k_1}(t)\cdots r_{k_n}(t)dt$ and define

$$\phi(x) = \prod_{s=2}^{n} r(2^{k_s-k_1}x), \qquad x \in \mathbb{R},$$

which satisfies

$$\phi(x+1) = \prod_{s=2}^{n} r(2^{k_s-k_1}x + 2^{k_s-k_1}) = \prod_{s=2}^{n} r(2^{k_s-k_1}x) = \phi(x).$$

[3]Masayoshi Hata, *Problems and Solutions in Real Analysis*, p. 185, Solution 13.2.

Hence, as ϕ has period 1 and r has period 2,

$$\begin{aligned}
J &= \int_0^1 r_{k_1}(t)\phi(2^{k_1}t)dt \\
&= \int_0^1 r(2^{k_1}t)\phi(2^{k_1}t)dt \\
&= \frac{1}{2^{k_1}}\int_0^{2^{k_1}} r(x)\phi(x)dx \\
&= \frac{1}{2^{k_1}}\sum_{j=0}^{2^{k_1-1}-1}\int_{2j}^{2j+2} r(x)\phi(x)dx \\
&= \frac{1}{2^{k_1}}\sum_{j=0}^{2^{k_1-1}-1}\int_0^2 r(x)\phi(x)dx \\
&= \frac{1}{2}\int_0^2 r(x)\phi(x)dx.
\end{aligned}$$

But, as ϕ has period 1,

$$\int_0^2 r(x)\phi(x)dx = \int_0^1 \phi(x)dx - \int_1^2 \phi(x)dx = \int_0^1 \phi(x)dx - \int_0^1 \phi(x)dx = 0,$$

hence $J = 0$, proving the claim. □

For each $n \in \mathbb{N}$, if f is a function defined on the integers we define

$$I_n(f) = \int_0^1 f\left(\sum_{k=1}^n r_k(t)\right) dt.$$

Lemma 5. *For any $n \in \mathbb{N}$,*

$$I_n(x^2) = n, \qquad I_n(x^4) = 3n^2 - 2n.$$

Proof. Using Theorem 4 we get

$$\begin{aligned}
I_n(x^2) &= \int_0^1 \left(\sum_{k=1}^n r_k(t)\right)^2 dt \\
&= \int_0^1 \sum_{k=1}^n r_k(t)^2 + \sum_{j\neq k} r_j(t)r_k(t)dt \\
&= \int_0^1 \sum_{k=1}^n r_k(t)^2 dt \\
&= n.
\end{aligned}$$

Using Theorem 4 we get, since $r_j(t)^4 = r_j(t)^2 = 1$ and $r_j(t)^3 = r_j(t)$,

$$\begin{aligned}
I_n(x^4) &= \int_0^1 \left(\sum_{k=1}^n r_k(t)\right)^4 dt \\
&= \int_0^1 \sum_{k=1}^n r_k(t)^4 + \binom{4}{3} \sum_{j=1}^n \sum_{k\neq j} r_j(t)^3 r_k(t) + \binom{4}{2} \sum_{j=1}^n \sum_{k\neq j} r_j(t)^2 r_k(t)^2 \\
&\quad + \binom{4}{2} \sum_{j=1}^n \sum_{j,\,k,\,l \text{ all distinct}} r_j(t)^2 r_k(t) r_l(t) \\
&\quad + \sum_{j,\,k,\,l,\,m \text{ all distinct}} r_j(t) r_k(t) r_l(t) r_m(t) dt \\
&= n + \binom{4}{2} n(n-1).
\end{aligned}$$

□

Our proof of the next identity follows Hata.[4]

Lemma 6. *For any* $n \in \mathbb{N}$,

$$I_n(|x|) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos^n x}{x^2} dx.$$

Proof. For $n \in \mathbb{N}$ and $c_1, \ldots, c_n \in \mathbb{R}$,

$$\int_0^1 \exp\left(i \sum_{k=1}^n c_k r_k(t)\right) dt = \int_0^1 \cos\left(\sum_{k=1}^n c_k r_k(t)\right) dt + i \int_0^1 \sin\left(\sum_{k=1}^n c_k r_k(t)\right) dt,$$

and since (4) tells us that the left-hand side of the above is real, it follows that we can write (4) as

$$\int_0^1 \cos\left(\sum_{k=1}^n c_k r_k(t)\right) dt = \prod_{k=1}^n \cos c_k. \tag{7}$$

Suppose that ξ is a positive real number. Using $t = x\xi$ and doing integration by parts,

$$\begin{aligned}
\int_0^\infty \frac{1 - \cos x\xi}{x^2} dx &= \xi \int_0^\infty \frac{1 - \cos t}{t^2} dt \\
&= \xi \frac{1 - \cos t}{-t}\Bigg|_0^\infty + \xi \int_0^\infty \frac{\sin t}{t} dt \\
&= \xi \int_0^\infty \frac{\sin t}{t} dt \\
&= \xi \frac{\pi}{2}.
\end{aligned}$$

[4] Masayoshi Hata, *Problems and Solutions in Real Analysis*, p. 188, Solution 13.6.

It is thus apparent that for any real ξ,

$$\int_0^\infty \frac{1-\cos x\xi}{x^2}dx = |\xi|\frac{\pi}{2}.$$

For any $n \in \mathbb{N}$, applying the above with $\xi = \sum_{k=1}^n r_k(t)$ we get

$$\begin{aligned} I_n(|x|) &= \frac{2}{\pi}\int_0^1 \left|\sum_{k=1}^n r_k(t)\right| \frac{\pi}{2}dt \\ &= \frac{2}{\pi}\int_0^1 \int_0^\infty \frac{1-\cos\left(x\sum_{k=1}^n r_k(t)\right)}{x^2}dxdt \\ &= \frac{2}{\pi}\int_0^\infty \frac{1}{x^2}\int_0^1 1-\cos\left(x\sum_{k=1}^n r_k(t)\right) dtdx \\ &= \frac{2}{\pi}\int_0^\infty \frac{1}{x^2}\left(1-I_n(\cos x\cdot)\right)dx. \end{aligned}$$

Applying (7) with $c_k = x$ for each k, this is equal to

$$\frac{2}{\pi}\int_0^\infty \frac{1}{x^2}\left(1-\prod_{k=1}^n \cos x\right)dx = \frac{2}{\pi}\int_0^\infty \frac{1-\cos^n x}{x^2}dx,$$

completing the proof. □

We use the above formula for $I_n(|x|)$ to obtain an asymptotic formula for $I_n(|x|)$.[5]

Theorem 7.

$$I_n(|x|) \sim \sqrt{\frac{2}{\pi}}\sqrt{n}.$$

Proof. By Lemma 6,

$$I_n(|x|) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos^n x}{x^2}dx.$$

For $0 \le \epsilon < 1$, define $\phi_\epsilon : \left[0, \frac{\pi}{2}\right) \to \mathbb{R}$ by

$$\phi_\epsilon(x) = \frac{x^2}{2(1-\epsilon)} + \log\cos x.$$

We also define

$$\alpha_\epsilon = \arccos\sqrt{1-\epsilon}, \qquad \beta_\epsilon = \int_{\alpha_\epsilon}^\infty \frac{1-\cos^n x}{x^2}dx,$$

[5] Mark Kac, *Statistical Independence in Probability, Analysis and Number Theory*, p. 12, Masayoshi Hata, *Problems and Solutions in Real Analysis*, p. 188, Solution 13.6.

and for $\sigma > 0$,

$$K_{\epsilon,\sigma} = \int_0^{\alpha_\epsilon} \frac{1 - \exp\left(-\frac{nx^2}{\sigma}\right)}{x^2} dx.$$

Let $0 < \epsilon < 1$. Until the end of the proof, at which point we take $\epsilon \to 0$, we shall keep ϵ fixed. For $0 < x < \alpha_\epsilon$ we have, using $\arccos \sqrt{1-\epsilon} \le \sqrt{\epsilon}$,

$$\phi_0(x) = \frac{x^2}{2} + \log \cos x < \frac{\epsilon}{2} + \log \sqrt{1-\epsilon} = \frac{\epsilon}{2} + \frac{1}{2}\log(1-\epsilon) < 0,$$

hence

$$\cos x < \exp\left(-\frac{x^2}{2}\right).$$

On the other hand,

$$\phi_\epsilon'(x) = \frac{x}{1-\epsilon} - \tan x, \qquad \phi_\epsilon''(x) = \frac{1}{1-\epsilon} - \sec^2 x,$$

so $\phi_\epsilon(0) = \phi_\epsilon'(0) = 0$ and $\phi_\epsilon''(t) > 0$ for all $0 \le t < \alpha_\epsilon$, giving

$$\phi_\epsilon(x) > 0,$$

and hence

$$\exp\left(-\frac{x^2}{2(1-\epsilon)}\right) < \cos x.$$

Collecting what we have established so far, for $0 < x < \alpha_\epsilon$ we have

$$\exp\left(-\frac{x^2}{2(1-\epsilon)}\right) < \cos x < \exp\left(-\frac{x^2}{2}\right).$$

This shows that

$$K_{\epsilon,2(1-\epsilon)} = \int_0^{\alpha_\epsilon} \frac{1 - \exp\left(-\frac{nx^2}{2(1-\epsilon)}\right)}{x^2} dx \ge \int_0^{\alpha_\epsilon} \frac{1-\cos^n x}{x^2} dx,$$

and therefore

$$K_{\epsilon,2(1-\epsilon)} + \beta_\epsilon \ge \frac{\pi}{2} I_n(|x|).$$

On the other hand,

$$K_{\epsilon,2} = \int_0^{\alpha_\epsilon} \frac{1 - \exp\left(-\frac{nx^2}{2}\right)}{x^2} dx \le \int_0^{\alpha_\epsilon} \frac{1-\cos^n x}{x^2} dx,$$

so

$$K_{\epsilon,2} + \beta_\epsilon \le \frac{\pi}{2} I_n(|x|).$$

Now summarizing what we have obtained, we have

$$K_{\epsilon,2} + \beta_\epsilon \le \frac{\pi}{2} I_n(|x|) \le K_{\epsilon,2(1-\epsilon)} + \beta_\epsilon. \tag{8}$$

For $\sigma > 0$, doing the change of variable $t = \sqrt{\frac{n}{\sigma}}x$,

$$K_{\epsilon,\sigma} = \int_0^{\alpha_\epsilon} \frac{1 - \exp\left(-\frac{nx^2}{\sigma}\right)}{x^2} dx = \sqrt{\frac{n}{\sigma}} \int_0^{\sqrt{\frac{n}{\sigma}}\alpha_\epsilon} \frac{1 - e^{-t^2}}{t^2} dt.$$

As $n \to \infty$, the right-hand side of this is asymptotic to

$$\sqrt{\frac{n}{\sigma}} \int_0^\infty \frac{1 - e^{-t^2}}{t^2} dt = \sqrt{\frac{n}{\sigma}} \sqrt{\pi}.$$

Dividing (8) by $\sqrt{n}$ and taking the limsup then gives

$$\limsup_{n\to\infty} \frac{\pi}{2} \frac{I_n(|x|)}{\sqrt{n}} \leq \sqrt{\frac{\pi}{2(1-\epsilon)}},$$

or

$$\limsup_{n\to\infty} \frac{I_n(|x|)}{\sqrt{n}} \leq \sqrt{\frac{2}{\pi(1-\epsilon)}};$$

indeed β_ϵ depends on n, but $\beta_\epsilon < \frac{2}{\alpha_\epsilon}$, which does not depend on n. Taking $\epsilon \to 0$ yields

$$\limsup_{n\to\infty} \frac{I_n(|x|)}{\sqrt{n}} \leq \sqrt{\frac{2}{\pi}}.$$

On the other hand, taking the liminf of (8) divided by $\sqrt{n}$ gives

$$\liminf_{n\to\infty} \frac{\pi}{2} \frac{I_n(|x|)}{\sqrt{n}} \geq \sqrt{\frac{\pi}{2}},$$

or

$$\liminf_{n\to\infty} \frac{I_n(|x|)}{\sqrt{n}} \geq \sqrt{\frac{2}{\pi}}.$$

Combining the limsup and the liminf inequalities proves the claim. □

Lemma 8. *For any $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$,*

$$I_n(e^{\xi|x|}) < I_n(e^{\xi x}) + I_n(e^{-\xi x}) = 2(\cosh \xi)^n.$$

We will use the following theorem to establish an estimate similar to but weaker than the **law of the iterated logarithm**.[6]

Theorem 9. *For any $\epsilon > 0$, for almost all $t \in [0, 1]$,*

$$\sum_{n=1}^\infty \frac{1}{n^{2+\epsilon}} \exp\left(\sqrt{\frac{2\log n}{n}} \left|\sum_{k=1}^n r_k(t)\right|\right) dt < \infty.$$

[6] Masayoshi Hata, *Problems and Solutions in Real Analysis*, p. 189, Solution 13.7.

Proof. Define $f_n : [0,1] \to (0,\infty)$ by

$$f_n(t) = \frac{1}{n^{2+\epsilon}} \exp\left(\sqrt{\frac{2\log n}{n}} \left| \sum_{k=1}^{n} r_k(t) \right| \right).$$

Applying Lemma 8 with $\xi = \sqrt{\frac{2\log n}{n}}$,

$$\int_0^1 f_n(t)dt \le \frac{1}{n^{2+\epsilon}} \cdot 2 \cdot \left(\cosh \sqrt{\frac{2\log n}{n}} \right)^n.$$

It is not obvious, but we take as given the asymptotic expansion

$$\left(\cosh \sqrt{\frac{2\log n}{n}} \right)^n = n - \frac{1}{3}(\log n)^2 + \frac{\frac{8}{45}(\log n)^3 + \frac{1}{18}(\log n)^4}{n} + O(n^{-3/2}),$$

and using this,

$$\frac{1}{n^{2+\epsilon}} \cdot 2 \cdot \left(\cosh \sqrt{\frac{2\log n}{n}} \right)^n = \frac{2}{n^{1+\epsilon}} + O\left(\frac{(\log n)^2}{n^{2+\epsilon}} \right) = \frac{2}{n^{1+\epsilon}} + O(n^{-2}).$$

Thus

$$\sum_{n=1}^{\infty} \int_0^1 f_n(t)dt = \sum_{n=1}^{\infty} \left(\frac{2}{n^{1+\epsilon}} + O(n^{-2}) \right) < \infty.$$

Because each f_n is nonnegative, using this with the **monotone convergence theorem** gives the claim. □

Theorem 10. *For almost all* $t \in [0,1]$,

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^n r_k(t)|}{\sqrt{n \log n}} \le \sqrt{2}.$$

Proof. Let $\epsilon > 0$. By Theorem 9, for almost all $t \in [0,1]$ there is some n_t such that $n \ge n_t$ implies that $f_n(t) < 1$, where we are talking about the functions f_n defined in the proof of that theorem; certainly the terms of a convergent series are eventually less than 1. That is, for almost all $t \in [0,1]$ there is some n_t such that $n \ge n_t$ implies that (taking logarithms),

$$(-2-\epsilon)\log n + \sqrt{\frac{2\log n}{n}} \left| \sum_{k=1}^n r_k(t) \right| < 0,$$

and rearranging,

$$\frac{|\sum_{k=1}^n r_k(t)|}{\sqrt{n\log n}} < \sqrt{2} + \frac{\epsilon}{\sqrt{2}} = \sqrt{2} + \epsilon'.$$

For each $s \in \mathbb{N}$, let E_s be those $t \in [0,1]$ such that

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^n r_k(t)|}{\sqrt{n\log n}} > \sqrt{2} + \frac{1}{s}.$$

For each s, taking $0 < \epsilon' < \frac{1}{s}$ we get that almost all $t \in [0,1]$ do not belong to E_s. That is, for each s, the set E_s has measure 0. Therefore

$$E = \bigcup_{s=1}^{\infty} E_s$$

has measure 0. That is, for almost all $t \in [0,1]$, for all $s \in \mathbb{N}$ we have $t \notin E_s$, i.e.

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^n r_k(t)|}{\sqrt{n \log n}} \leq \sqrt{2} + \frac{1}{s},$$

and this holding for all $s \in \mathbb{N}$ yields

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^n r_k(t)|}{\sqrt{n \log n}} \leq \sqrt{2},$$

completing the proof. □

3 Hypercubes

Let m_n be Lebesgue measure on $\mathbb{R}^n$, and let $Q_n = [0,1]^n$.[7]

Theorem 11. *If $f \in C([0,1])$, then*

$$\lim_{n\to\infty} \int_{Q_n} f\left(\frac{x_1 + \cdots + x_n}{n}\right) dm_n(x) = f\left(\frac{1}{2}\right).$$

Proof. Define $X_n : Q_n \to \mathbb{R}$ by

$$X_n = \frac{x_1 + \cdots + x_n}{n}, \qquad x \in Q_n.$$

We have

$$\int_{Q_n} X_n dm_n(x) = \frac{1}{n} \sum_{k=1}^n \int_0^1 x_k \cdot 1 dx_k = \frac{1}{n} \sum_{k=1}^n \frac{1}{2} = \frac{1}{2},$$

[7] Masayoshi Hata, *Problems and Solutions in Real Analysis*, p. 161, Solution 11.1.

and we define

$$
\begin{aligned}
V_n &= \int_{Q_n} \left(X_n - \frac{1}{2} \right)^2 dm_n(x) \\
&= \int_{Q_n} \sum_{k=1}^n \frac{x_k^2}{n^2} + \sum_{j \neq k} \frac{x_j x_k}{n^2} - X_n + \frac{1}{4} dm_n(x) \\
&= \frac{1}{n^2} \sum_{k=1}^n \int_0^1 x_k^2 dx_k + \frac{1}{n^2} \sum_{j=1}^n \sum_{k \neq j} \int_0^1 x_j dx_j \int_0^1 x_k dx_k - \frac{1}{2} + \frac{1}{4} \\
&= \frac{1}{n^2} \sum_{k=1}^n \frac{1}{3} + \frac{1}{n^2} \sum_{j=1}^n \sum_{k \neq j} \frac{1}{4} - \frac{1}{4} \\
&= \frac{1}{3n} + \frac{n-1}{4n} - \frac{1}{4} \\
&= \frac{n^{-1}}{12}.
\end{aligned}
$$

Suppose that c_n is a sequence of positive real numbers tending to 0, and define $J_n = J_n(c)$ to be those $x \in Q_n$ such that

$$
\left| X_n(x) - \frac{1}{2} \right| \geq c_n.
$$

Then

$$
\begin{aligned}
V_n &= \int_{Q_n} \left(X_n - \frac{1}{2} \right)^2 dm_n(x) \\
&\geq \int_{J_n} \left(X_n - \frac{1}{2} \right)^2 dm_n(x) \\
&\geq \int_{J_n} c_n^2 dm_n(x) \\
&= c_n^2 m_n(J_n),
\end{aligned}
$$

so

$$
m_n(J_n) \leq \frac{V_n}{c_n^2} = \frac{n^{-1}}{12 c_n^2}.
$$

Take $c_n = n^{-1/3}$, giving

$$
m_n(J_n) \leq \frac{n^{-1/3}}{12}.
$$

Let $\epsilon > 0$. Because f is continuous, there is some $\delta > 0$ such that $|t - \frac{1}{2}| < \delta$ implies that $|f(t) - f(\frac{1}{2})| < \epsilon$; furthermore, we take δ such that

$$
\frac{\|f\|_\infty \delta}{6} < \epsilon.
$$

Set $N > \delta^{-3}$. For $n \geq N$ and $x \in Q_n \setminus J_n$,

$$\left| X_n(x) - \frac{1}{2} \right| < c_n = n^{-1/3} \leq N^{-1/3} < \delta,$$

and so

$$\left| (f(X_n(x)) - f\left(\frac{1}{2}\right) \right| < \epsilon.$$

This gives us

$$\begin{aligned}
\left| \int_{Q_n} f(X_n(x)) dm_n(x) - f\left(\frac{1}{2}\right) \right| &= \left| \int_{Q_n} f(X_n(x)) - f\left(\frac{1}{2}\right) dm_n(x) \right| \\
&\leq \int_{J_n} \left| f(X_n(x)) - f\left(\frac{1}{2}\right) \right| dm_n(x) \\
&\quad + \int_{Q_n \setminus J_n} \left| f(X_n(x)) - f\left(\frac{1}{2}\right) \right| dm_n(x) \\
&\leq \int_{J_n} 2 \|f\|_\infty \, dm_n(x) + \int_{Q_n \setminus J_n} \epsilon dm_n(x) \\
&\leq 2 \|f\|_\infty \, m_n(J_n) + \epsilon \\
&\leq 2 \|f\|_\infty \frac{n^{-1/3}}{12} + \epsilon \\
&< \frac{\|f\|_\infty \delta}{6} + \epsilon \\
&< 2\epsilon,
\end{aligned}$$

which proves the claim. □