1 Reproducing kernels

We shall often speak about functions $F : X \times X \to \mathbb{R}$, where $X$ is a nonempty set. For $x \in X$, we define $F_x : X \to \mathbb{R}$ by $F_x(y) = F(x, y)$ and for $y \in X$ we define $F^y : X \to \mathbb{R}$ by $F^y(x) = F(x, y)$. $F$ is said to be symmetric if $F(x, y) = F(y, x)$ for all $x, y \in X$ and positive-definite if for any $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{R}$ it holds that

$$\sum_{1 \leq i, j \leq n} c_i c_j F(x_i, x_j) \geq 0.$$

**Lemma 1.** If $F : X \times X \to \mathbb{R}$ is symmetric and positive-definite then

$$F(x, y)^2 \leq F(x, x) F(y, y), \quad x, y \in X.$$

**Proof.** For $\alpha, \beta \in \mathbb{R}$ define

$$C(\alpha, \beta) = \alpha^2 F(x, x) + \alpha \beta F(x, y) + \beta \alpha F(y, x) + \beta^2 F(y, y)$$

$$= \alpha^2 F(x, x) + 2 \alpha \beta F(x, y) + \beta^2 F(y, y),$$

which is $\geq 0$. Let

$$P(\alpha) = C(\alpha, F(x, y))$$

$$= \alpha^2 F(x, x) + 2 \alpha F(x, y)^2 + F(x, y)^2 F(y, y),$$

which is $\geq 0$. In the case $F(x, x) = 0$, the fact that $P \geq 0$ implies that $F(x, y) = 0$. In the case $F(x, y) \neq 0$, $P(\alpha)$ is a quadratic polynomial and because $P \geq 0$ it follows that the discriminant of $P$ is $\leq 0$:

$$4 F(x, y)^4 - 4 \cdot F(x, x) \cdot F(x, y)^2 F(y, y) \leq 0.$$

That is, $F(x, y)^4 \leq F(x, y)^2 F(x, x) F(y, y)$, and this implies that $F(x, y)^2 \leq F(x, x) F(y, y)$. \qed

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A real reproducing kernel Hilbert space is a Hilbert space $H$ contained in $\mathbb{R}^X$, where $X$ is a nonempty set, such that for each $x \in X$ the map $\Lambda_x f = f(x)$ is continuous $H \to \mathbb{R}$. In this note we speak always about real Hilbert spaces.

Let $H \subset \mathbb{R}^X$ be a reproducing kernel Hilbert space. Because $H$ is a Hilbert space, the Riesz representation theorem states that $\Phi : H \to H^*$ defined by

$$(\Phi g)(f) = \langle f, g \rangle_H, \quad g, f \in H$$

is an isometric isomorphism. Because $H$ is a reproducing kernel Hilbert space, $\Lambda_x \in H^*$ for each $x \in X$ and we define $T_x = \Phi^{-1} \Lambda_x \in H$, which satisfies

$$f(x) = \Lambda_x(f) = \langle f, T_x \rangle_H, \quad f \in H.$$ 

In particular, because $T_x \in H$, for $y \in X$ it holds that

$$T_x(y) = \Lambda_y(T_x) = \langle T_x, T_y \rangle_H.$$ 

Define $K : X \times X \to \mathbb{R}$ by

$$K(x, y) = \langle T_x, T_y \rangle_H,$$

called the reproducing kernel of $H$. For $x, y \in X$,

$$T_x(y) = \langle T_x, T_y \rangle_H = K(x, y) = K(y, x),$$

which means that $T_x = K_x$.

A reproducing kernel is symmetric and positive-definite:

$$K(x, y) = \langle T_x, T_y \rangle_H = \langle T_y, T_x \rangle_H = K(y, x)$$

and

$$\sum_{1 \leq i, j \leq n} c_i c_j K(x_i, x_j) = \left\langle \sum_{1 \leq i \leq n} c_i T_{x_i}, \sum_{1 \leq j \leq n} c_j T_{x_j} \right\rangle_H$$

$$\geq 0.$$ 

**Lemma 2.** If $E$ is an orthonormal basis for a reproducing kernel Hilbert space $H \subset \mathbb{R}^X$ with reproducing kernel $K : X \times X \to \mathbb{R}$, then

$$K(x, y) = \sum_{e \in E} e(x)e(y), \quad x, y \in X.$$

**Proof.** Because $E$ is an orthonormal basis for $H$, Parseval’s identity tell us

$$\langle T_x, T_y \rangle_H = \sum_{e \in E} \langle T_x, e \rangle \langle T_y, e \rangle = \sum_{e \in E} \langle e, T_x \rangle \langle e, T_y \rangle = \sum_{e \in E} e(x)e(y).$$

\[\square\]
If $H \subset \mathbb{R}^X$ is a reproducing kernel Hilbert space with reproducing kernel $K : X \times X \rightarrow \mathbb{R}$ and $V$ is a closed linear subspace of $H$, then $V$ is itself a reproducing kernel Hilbert space, with some reproducing kernel $G : X \times X \rightarrow \mathbb{R}$. The following theorem expresses $G$ in terms of $K$.

**Theorem 3.** Let $H \subset \mathbb{R}^X$ be a reproducing kernel Hilbert space with reproducing kernel $K : X \times X \rightarrow \mathbb{R}$, let $V$ be a closed linear subspace of $H$ with reproducing kernel $G : X \times X \rightarrow \mathbb{R}$, and let $P_V : H \rightarrow V$ be the projection onto $V$. Then

$$G_x = P_V K_x, \quad x \in X.$$

**Proof.** $H = V \oplus V^\perp$, thus for $f \in H$ there are unique $g \in V, h \in V^\perp$ such that $f = g + h$, and $P_V f = g$. Then $f - P_V f \in V^\perp$. Therefore for $x \in X$, as $G_y \in V$ it holds that

$$\langle f, G_y \rangle_H = \langle f - P_V f + P_V f, G_y \rangle_H = \langle P_V f, G_y \rangle_H = \langle P_V f \rangle(y).$$

In particular, for $x, y \in X$ and $f = K_x$,

$$(P_V K_x)(y) = \langle K_x, G_y \rangle_H = \langle G_y, T_x \rangle_H = G_y(x) = G(y, x) = G(x, y) = G_x(y),$$

which means that $P_V K_x = G_x$, proving the claim. \hfill \Box

The Moore-Aronszajn theorem states that if $X$ is a nonempty set and $K : X \times X \rightarrow \mathbb{R}$ is a symmetric and positive-definite function, then there is a unique reproducing kernel Hilbert space $H \subset \mathbb{R}^X$ for which $K$ is the reproducing kernel.

We now prove that given a symmetric positive-definite kernel there is a unique reproducing Hilbert space for which it is the reproducing kernel.

## 2 Sobolev spaces on $[0, T]$

Let $f \in \mathbb{R}^{[0, T]}$. The following are equivalent.\(^5\)

1. $f$ is absolutely continuous.
2. $f$ is differentiable at almost all $t \in [0, T]$, $f' \in L^1$, and

$$f(t) = f(0) + \int_0^t f'(s)ds, \quad t \in [0, T].$$


\(^3\)http://individual.utoronto.ca/jordanbell/notes/pvm.pdf


\(^5\)Elias M. Stein and Rami Shakarchi, *Real Analysis*, p. 130, Theorem 3.11.
3. There is some \( g \in L^1 \) such that
\[
    f(t) = f(0) + \int_0^t g(s)ds, \quad t \in [0,T].
\]
In particular, if \( f \) is absolutely continuous and \( f' = 0 \) almost everywhere then
\[
    \int_0^T f'(s)ds = 0
\]
and so \( f(t) = f(0) \) for all \( t \in [0,T] \). That is, if \( f \) is absolutely continuous and \( f' = 0 \) almost everywhere then \( f \) is constant.

Let \( H \) be the set of those absolutely continuous functions \( f \in \mathbb{R}^{[0,T]} \) such that \( f(0) = 0 \) and \( f' \in L^2 \). For \( f, g \in H \) define
\[
    \langle f, g \rangle_H = \int_0^T f'(s)g'(s)ds.
\]
If \( \|f\|_H = 0 \) then \( \int_0^T f'(s)^2ds = 0 \), which implies that \( f' = 0 \) almost everywhere and hence that \( f \) is constant, and therefore \( f = 0 \). Thus \( \langle \cdot, \cdot \rangle_H \) is indeed an inner product on \( H \).

If \( f_n \) is a Cauchy sequence in \( H \) then \( f'_n \) is a Cauchy sequence in \( L^2 \) and hence converges to some \( g \in L^2 \). Then the function \( f \in \mathbb{R}^{[0,T]} \) defined by
\[
    f(t) = \int_0^t g(s)ds, \quad t \in [0,T],
\]
is absolutely continuous, \( f(0) = 0 \), and satisfies \( f' = g \) almost everywhere, which shows that \( f \in H \). Then \( f_n \to f \) in \( H \), which proves that \( H \) is a Hilbert space. For \( t \in [0,T] \), by the Cauchy-Schwarz inequality,
\[
    |f(t)|^2 = \left| \int_0^t f'(s)ds \right|^2 \leq \int_0^T f'(s)^2ds \leq T \int_0^T f'(s)^2ds = T \|f\|_H^2,
\]
i.e. \( |L_t f| \leq T^{1/2} \|f\|_H \), which shows that \( L_t \in H^* \). Therefore \( H \) is a reproducing kernel Hilbert space.

For \( a \in [0,T] \) define \( h_a : [0,T] \to \mathbb{R} \) by \( h_a(s) = 1_{[0,a]}(s) \), which belongs to \( L^2 \), and define \( g_a : [0,T] \to \mathbb{R} \) by
\[
    g_a(t) = \int_0^t h_a(s)ds = \min(t,a),
\]
which belongs to \( H \). For \( f \in H \),
\[
    \langle f, g_a \rangle_H = \int_0^T f'(s)g'_a(s)ds = \int_0^T f'(s)1_{[0,a]}(s)ds = \int_0^a f'(s)ds = f(a).
\]
This means that \( K_a = g_a \). For \( a, b \in [0,T] \),
\[
    \langle K_a, K_b \rangle_H = \int_0^T g'_a(s)g'_b(s)ds = \int_0^T 1_{[0,a]}(s)1_{[0,b]}(s)ds = \int_0^T 1_{[0,\min(a,b)]}(s)ds.
\]
That is, the reproducing kernel of \( H \) is \( K : [0,T] \times [0,T] \to \mathbb{R} \),
\[
    K(a,b) = \langle K_a, K_b \rangle_H = \min(a,b).
\]
3 Sobolev spaces on $\mathbb{R}$

Let $\lambda$ be Lebesgue measure on $\mathbb{R}$. Let $L^2(\lambda)$ be the collection of Borel measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $|f|^2$ is integrable, and let $L^2_2(\lambda)$ be the Hilbert space of equivalence classes of elements of $L^2(\lambda)$ where $f \sim g$ when $f = g$ almost everywhere, with

$$\langle f, g \rangle_{L^2} = \int_\mathbb{R} fg d\lambda.$$

Let $H^1(\mathbb{R})$ be the set of locally absolutely continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f, f' \in L^2_2(\lambda)$. This is a Hilbert space with the inner product

$$\langle f, g \rangle_{H^1} = \langle f, g \rangle_{L^2} + \langle f', g' \rangle_{L^2}.$$

Define $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$K(x, y) = \frac{1}{2} \exp(-|x - y|), \quad x, y \in \mathbb{R}.$$

Let $x \in \mathbb{R}$. For $y < x$, $K'_x(y) = K_x(y)$ and for $y > x$, $K'_x(y) = -K_x(y)$, which shows that $K_x \in H^1(\mathbb{R})$. For $f \in H^1(\mathbb{R})$, doing integration by parts,

$$\langle f, K_x \rangle_{H^1} = \int_{-\infty}^\infty fK_x d\lambda + \int_{-\infty}^x f'(y)K_x(y)d\lambda(y) - \int_x^\infty f'(y)K_x(y)d\lambda(y)$$

$$= \int_{-\infty}^\infty fK_x d\lambda + f(x)K_x(x) - \int_{-\infty}^x f(y)K'_x(y)d\lambda(y)$$

$$+ f(x)K_x(x) + \int_x^\infty f(y)K'_x(y)d\lambda(y)$$

$$= \int_{-\infty}^\infty fK_x d\lambda + \frac{1}{2} f(x) - \int_{-\infty}^x f(y)K_x(y)d\lambda(y)$$

$$+ \frac{1}{2} f(x) - \int_x^\infty f(y)K_x(y)d\lambda(y)$$

$$= f(x) = T_x f.$$

This shows that $H^1(\mathbb{R})$ is a reproducing kernel Hilbert space. We calculate, for $x < y$,

$$\langle T_x, T_y \rangle_{H^1} = \int_{-\infty}^x K_x K_y d\lambda + \int_x^y K_x K_y d\lambda + \int_y^\infty K_x K_y d\lambda$$

$$+ \int_{-\infty}^x K_x K_y d\lambda - \int_x^y K_x K_y d\lambda + \int_y^\infty K_x K_y d\lambda$$

$$= 4 \cdot \frac{1}{8} \exp(x - y)$$

$$= K(x, y).$$

\[\text{http://individual.utoronto.ca/jordanbell/notes/sobolevid.pdf}\]
This shows that $K(x, y) = \frac{1}{2} \exp(-|x - y|)$ is the reproducing kernel of $H^1(\mathbb{R})$.\footnote{cf. Alain Berlinet and Christine Thomas-Agnan, \textit{Reproducing Kernel Hilbert Spaces in Probability and Statistics}, pp. 8–9, Example 5.}