SCALING FOR THE NONLINEAR SCHRÖDINGER EQUATION

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The nonlinear Schrödinger equation is

$$i\partial_t u + \Delta u = \mu |u|^{p-1} u. \tag{1}$$

Let $u_\lambda(t,x) = \lambda^\alpha u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$, for $\lambda > 0$.

Then

$$\partial_t u_\lambda(t,x) = \lambda^{\alpha-2}(\partial_t u)\Big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\Big)$$

and

$$\Delta u_\lambda(t,x) = \lambda^{\alpha-2}(\Delta u)\Big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\Big).$$

For u_λ to satisfy (1) means that

$$i\lambda^{\alpha-2}(\partial_t u)\Big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\Big) + \lambda^{\alpha-2}(\Delta u)\Big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\Big) = \lambda^{p\alpha}\mu\Big|u\Big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\Big)\Big|^{p-1} u\Big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\Big),$$

i.e.,

$$i\lambda^{\alpha-2}\partial_t u + \lambda^{\alpha-2}\Delta u = \lambda^{p\alpha}\mu|u|^{p-1}u.$$

This will hold if $\lambda^{\alpha-2} = \lambda^{p\alpha}$, so if $\alpha - 2 = p\alpha$, which happens when $\alpha = \frac{-2}{p-1}$. Therefore if u is a solution of (1) then $u_\lambda(t,x) = \lambda^\alpha u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ is also a solution of (1), and we say that we obtained u_λ by *scaling* the solution u, and we also say that (1) is scaling invariant.

Now let's consider the $\dot{H}^s(\mathbb{R}^d)$ norm of u_λ.

$$\begin{aligned}
\|u_\lambda\|_{\dot{H}^s(\mathbb{R}^d)} &= \Big(\int (\nabla^s u_\lambda(x))^2 dx\Big)^{1/2} \\
&= \Big(\int \Big(\lambda^\alpha \frac{1}{\lambda^s}(\nabla^s u)\Big(\frac{x}{\lambda}\Big)\Big)^2 dx\Big)^{1/2} \\
&= \lambda^{\alpha-s}\Big(\int \Big((\nabla^s u)\Big(\frac{x}{\lambda}\Big)\Big)^2 dx\Big)^{1/2} \\
&= \lambda^{\alpha-s}\Big(\int (\nabla^s u)^2 \lambda^d dy\Big)^{1/2} \\
&= \lambda^{\alpha-s+\frac{d}{2}}\|u\|_{\dot{H}^s(\mathbb{R}^d)}
\end{aligned}$$

If $\alpha - s + \frac{d}{2} = 0$ then the scaled solution u_λ has the same $\dot{H}^s(\mathbb{R}^d)$ norm as u has.

We say that (1) is L^2 scaling invariant ($s = 0$) if $\frac{-2}{p-1} + \frac{d}{2} = 0$, i.e., $p = 1 + \frac{4}{d}$.

We say that (1) is $\dot{H}^1$ scaling invariant ($s = 1$) if $\frac{-2}{p-1} - 1 + \frac{d}{2} = 0$, i.e., $p = 1 + \frac{4}{d-2}$.

For (1) to be $\dot{H}^s$ *critical* is another way of saying that it is $\dot{H}^s$ scaling invariant.

Date: June 4, 2012.

Let's consider the relation between the norm of u_λ and the norm of u that we obtained. It is

$$\|u_\lambda\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{\alpha-s+\frac{d}{2}}\|u\|_{\dot{H}^s(\mathbb{R}^d)}. \tag{2}$$

If u blows up at time t^* then u_λ blows up at time t such that $\frac{t}{\lambda^2} = t^*$, so $t = \lambda^2 t^*$, which will be larger than t^* if $\lambda > 1$. Also, if $\alpha - s + \frac{d}{2} < 0$ then the norm of u_λ is smaller than the norm of u.

In the case $\alpha - s + \frac{d}{2} < 0$ we say that (1) is $\dot{H}^s$ *subcritical.*

In the case $\alpha - s + \frac{d}{2} = 0$ we say that (1) is $\dot{H}^s$ *critical.*

In the case $\alpha - s + \frac{d}{2} > 0$ we say that (1) is $\dot{H}^s$ *supercritical.*

In the $\dot{H}^s$ subcritical case, we can make the solution strictly nicer by scaling it with large λ, whereas in the $\dot{H}^s$ supercritical case if we scale it with large λ the time of existence will grow but the norm will also grow, and if we scale it with small λ the norm will decrease but the time of existence will also decrease. This is our reason for considering the $\dot{H}^s$ subcritical case to be nicer than the $\dot{H}^s$ supercritical case.

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