A series of secants

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November 3, 2014

Let $\mathfrak{H} = \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}$. Define $C : \mathfrak{H} \to \mathbb{C}$ by

$$C(\tau) = 2 \sum_{n=-\infty}^{\infty} \frac{1}{e^{\pi i n\tau} + q^{-\pi i n\tau}} = \sum_{n=-\infty}^{\infty} \sec \pi n\tau, \qquad \tau \in \mathfrak{H}.$$

We take as granted that C is holomorphic on $\mathfrak{H}$.

First we calculate the Fourier transform of $x \mapsto \operatorname{sech} \pi x$.[1]

Lemma 1. *For $\xi \in \mathbb{R}$,*

$$\int_{-\infty}^{\infty} e^{-2\pi i \xi x} \operatorname{sech} \pi x dx = \operatorname{sech} \pi \xi.$$

Proof. Let $\xi \in \mathbb{R}$ and define

$$f(z) = \frac{e^{-2\pi i z\xi}}{\cosh \pi z}.$$

The poles of f are those z at which $\cosh \pi z = 0$, thus $z = ni + \frac{i}{2}$, $n \in \mathbb{Z}$. Taking γ_R to be the contour going from $-R$ to R, from R to $R + 2i$, from $R + 2i$ to $-R+2i$, and from $-R+2i$ to $-R$, the poles of f inside γ_R are $\frac{i}{2}$ and $\frac{3i}{2}$. Because $(\cosh \pi z)' = \pi \sinh \pi z$, we work out

$$\operatorname{Res}_{z=i/2} f(z) = \frac{e^{-2\pi i \cdot \frac{i}{2} \xi}}{\pi \sinh \pi \frac{i}{2}} = \frac{e^{\pi \xi}}{\pi i \sin \frac{\pi}{2}} = \frac{e^{\pi \xi}}{\pi i}$$

and

$$\operatorname{Res}_{z=3i/2} f(z) = \frac{e^{-2\pi i \cdot \frac{3i}{2} \xi}}{\pi \sinh \pi \frac{3i}{2}} = \frac{e^{3\pi \xi}}{\pi i \sin \frac{3\pi}{2}} = \frac{e^{3\pi \xi}}{-\pi i}.$$

We bound the integrals on the vertical sides as follows. For $z = -R + iy$,

$$|\cosh \pi z| = \frac{|e^{\pi z} + e^{-\pi z}|}{2} \geq \frac{||e^{\pi z}| - |e^{-\pi z}||}{2} = \frac{|e^{-R\pi} - e^{R\pi}|}{2} = \frac{e^{R\pi} - e^{-R\pi}}{2},$$

[1] Elias M. Stein and Rami Shakarchi, *Complex Analysis*, p. 81, Example 3.

and, for $0 \le y \le 2$,

$$|e^{-2\pi iz\xi}| = e^{2\pi y\xi} = e^{4\pi\xi}.$$

For $z = R + iy$,

$$|\cosh \pi z| = \frac{|e^{\pi z} + e^{-\pi z}|}{2} \ge \frac{||e^{\pi z}| - |e^{-\pi z}||}{2} = \frac{|e^{R\pi} - e^{-R\pi}|}{2} = \frac{e^{R\pi} - e^{-R\pi}}{2},$$

and, for $0 \le y \le 2$,

$$|e^{-2\pi iz\xi}| = e^{2\pi y\xi} = e^{4\pi\xi}.$$

Therefore

$$\left|\int_{-R}^{-R+2i} f(z)dz\right| \le \int_{-R}^{-R+2i} |f(z)|dz \le 2 \cdot e^{4\pi\xi} \cdot \frac{2}{e^{R\pi} - e^{-R\pi}} = \frac{e^{4\pi\xi}}{e^{R\pi} - e^{-R\pi}}$$

and likewise

$$\left|\int_{R}^{R+2i} f(z)dz\right| \le \frac{e^{4\pi\xi}}{e^{R\pi} - e^{-R\pi}}.$$

As $R \to \infty$, each of these tends to 0. Therefore,

$$\int_{-\infty}^{\infty} f(z)dz + \int_{\infty+2i}^{-\infty+2i} f(z)dz = 2\pi i\left(\frac{e^{\pi\xi}}{\pi i} + \frac{e^{3\pi\xi}}{-\pi i}\right) = -2e^{2\pi\xi}(e^{\pi\xi} - e^{-\pi\xi}),$$

i.e.,

$$\int_{-\infty}^{\infty} f(z)dz = \int_{-\infty+2i}^{\infty+2i} f(z)dz - 2e^{2\pi\xi}(e^{\pi\xi} - e^{-\pi\xi}).$$

For the top horizontal side,

$$\begin{aligned}
\int_{-R+2i}^{R+2i} f(z)dz &= \int_{-R}^{R} \frac{e^{-2\pi i(x+2i)\xi}}{\cosh(\pi x + 2\pi i)}dx \\
&= \int_{-R}^{R} \frac{e^{-2\pi ix\xi}e^{4\pi\xi}}{\cosh(\pi x)\cosh(2\pi i) + \sinh(\pi x)\sinh(2\pi i)}dx \\
&= e^{4\pi\xi}\int_{-R}^{R} \frac{e^{-2\pi ix\xi}}{\cosh \pi x}dx \\
&= e^{4\pi\xi}\int_{-R}^{R} f(x)dx.
\end{aligned}$$

Writing

$$I = \int_{-\infty}^{\infty} f(z)dz,$$

this gives us

$$I = e^{4\pi\xi}I - 2e^{2\pi\xi}(e^{\pi\xi} - e^{-\pi\xi}),$$

and so

$$I = -2e^{2\pi\xi}\frac{e^{\pi\xi} - e^{-\pi\xi}}{1 - e^{4\pi\xi}} = 2\frac{e^{\pi\xi} - e^{-\pi\xi}}{e^{2\pi\xi} - e^{-2\pi\xi}} = 2\frac{e^{\pi\xi} - e^{-\pi\xi}}{(e^{\pi\xi} - e^{-\pi\xi})(e^{\pi\xi} + e^{-\pi\xi})} = \operatorname{sech} \pi\xi,$$

which is what we wanted to show. □

Corollary 2. *For $t > 0$ and $a \in \mathbb{R}$,*

$$\int_{-\infty}^{\infty} e^{-2\pi i\xi x} e^{-2\pi iax} \operatorname{sech} \frac{\pi x}{t} dx = t\operatorname{sech}(\pi(\xi + a)t), \qquad \xi \in \mathbb{R}.$$

Proof.

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-2\pi ix\xi} e^{-2\pi iax} \operatorname{sech} \frac{\pi x}{t} dx &= \int_{-\infty}^{\infty} e^{-2\pi i(\xi+a)x} \operatorname{sech} \frac{\pi x}{t} dx \\ &= t \int_{-\infty}^{\infty} e^{-2\pi i(\xi+a)tx} \operatorname{sech} \pi x dx \\ &= t\operatorname{sech}(\pi(\xi + a)t).\end{aligned}$$

□

Theorem 3. *For all $\tau \in \mathfrak{H}$,*

$$C(\tau) = \frac{i}{\tau} C\left(-\frac{1}{\tau}\right).$$

Proof. For $f \in L^1(\mathbb{R})$, we define $\widehat{f} : \mathbb{R} \to \mathbb{C}$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi x} f(x) dx, \qquad \xi \in \mathbb{R}.$$

Following Stein and Shakarchi, for $a > 0$, define $\mathfrak{F}_a$ to be the set of those functions f defined on some neighborhood of $\mathbb{R}$ in $\mathbb{C}$ such that f is holomorphic on the set $\{z \in \mathbb{C} : |\operatorname{Im} z| < a\}$ and for which there is some $A > 0$ such that

$$|f(x + iy)| \le \frac{A}{1 + x^2}, \qquad x \in \mathbb{R}, \quad |y| < a,$$

and we set $\mathfrak{F} = \bigcup_{a>0} \mathfrak{F}_a$. The **Poisson summation formula**[2] states that for $f \in \mathfrak{F}$,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n).$$

For $z = x + iy$ with $|y| < \frac{1}{2}$,

$$\begin{aligned}|\operatorname{sech} \frac{\pi z}{t}| &= \frac{2}{|e^{\pi(x+iy)} - e^{-\pi(x+iy)}|} \\ &\le \frac{2}{||e^{\pi(x+iy)}| - |e^{-\pi(x+iy)}||} \\ &= \frac{2}{|e^{\pi x} - e^{-\pi x}|} \\ &= \operatorname{sech} \pi|x|.\end{aligned}$$

[2] Elias M. Stein and Rami Shakarchi, *Complex Analysis*, p. 118, Theorem 2.4.

Let $t > 0$. Because the zeros of $\cosh \pi z$ are $ni + \frac{i}{2}$, $n \in \mathbb{Z}$, the function $f(z) = \operatorname{sech} \frac{\pi z}{t}$ belongs to $\mathfrak{F}_{\frac{t}{2}}$. Corollary 2 with $a = 0$ gives us

$$\widehat{f}(\xi) = t \operatorname{sech} \pi \xi t,$$

so applying the Poisson summation formula we get

$$\sum_{n\in\mathbb{Z}} \operatorname{sech} \frac{\pi n}{t} = t \sum_{n\in\mathbb{Z}} \operatorname{sech} \pi n t,$$

or,

$$\sum_{n\in\mathbb{Z}} \sec \frac{\pi i n}{t} = t \sum_{n\in\mathbb{Z}} \sec \pi i n t,$$

i.e.,

$$C\left(\frac{i}{t}\right) = tC(it).$$

For $\tau = it$ this reads

$$C(\tau) = \frac{i}{\tau} C\left(-\frac{1}{\tau}\right).$$

But $\tau \mapsto C(\tau)$ and $\tau \mapsto \frac{i}{\tau} C\left(-\frac{1}{\tau}\right)$ are holomorphic on $\mathfrak{H}$, so by analytic continuation this identity is true for all $\tau \in \mathfrak{H}$. □

Theorem 4.

$$C\left(1 - \frac{1}{\tau}\right) \sim \frac{4\tau}{i} e^{\frac{\pi i \tau}{2}}, \qquad \operatorname{Im} \tau \to +\infty.$$

Proof. Let $t > 0$ and define $f(z) = e^{-\pi i z} \operatorname{sech} \frac{\pi z}{t}$, which we check belongs to $\mathfrak{F}_{\frac{t}{2}}$. Corollary 2 with $a = \frac{1}{2}$ tells us that for $t > 0$,

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi x} e^{-\pi i x} \operatorname{sech} \frac{\pi x}{t} dx = t \operatorname{sech}\left(\pi\left(\xi + \frac{1}{2}\right) t\right), \qquad \xi \in \mathbb{R}.$$

Thus the Poisson summation formula gives, as $(-1)^n = e^{-i\pi n}$,

$$\sum_{n\in\mathbb{Z}} (-1)^n \operatorname{sech} \frac{\pi n}{t} = t \sum_{n\in\mathbb{Z}} \operatorname{sech}\left(\pi\left(n + \frac{1}{2}\right) t\right),$$

or

$$\sum_{n\in\mathbb{Z}} (-1)^n \sec \frac{\pi i n}{t} = t \sum_{n\in\mathbb{Z}} \sec\left(\pi i\left(n + \frac{1}{2}\right) t\right).$$

For $\tau = it$ this reads

$$\sum_{n\in\mathbb{Z}} (-1)^n \sec \frac{\pi n}{\tau} = \frac{\tau}{i} \sum_{n\in\mathbb{Z}} \sec\left(\pi\left(n + \frac{1}{2}\right) \tau\right).$$

Now,

$$\sec\left(\pi n\left(1-\frac{1}{\tau}\right)\right) = \frac{1}{\cos\pi n\cos\frac{-\pi n}{\tau} - \sin\pi n\sin\frac{-\pi n}{\tau}} = (-1)^n\sec\frac{\pi n}{\tau},$$

so the above states that for $\tau = it$, $t > 0$,

$$C\left(1-\frac{1}{\tau}\right) = \frac{\tau}{i}\sum_{n\in\mathbb{Z}}\sec\left(\pi\left(n+\frac{1}{2}\right)\tau\right). \tag{1}$$

We assert that both sides of (1) are holomorphic on $\mathfrak{H}$, and thus by analytic continuation that (1) is true for all $\tau\in\mathfrak{H}$.

Write $\tau = \sigma + it$. For $\nu > 0$,

$$\sec\pi\nu\tau = \frac{2}{e^{i\pi\nu\tau}+e^{-i\pi\nu\tau}} = \frac{2}{e^{-i\pi\nu\tau}(e^{2\pi i\nu\tau}+1)} = 2e^{i\pi\nu\tau}(1+O(|e^{2\pi i\nu\tau}|)),$$

or,

$$\sec\pi\nu\tau = 2e^{i\pi\nu\tau} + O(|e^{3\pi i\nu\tau}|).$$

Now,

$$|e^{\frac{3\pi i\tau}{2}}| = e^{\frac{-3\pi t}{2}},$$

so,

$$\sec\pi\nu\tau = 2e^{i\pi\nu\tau} + O(e^{\frac{-3\pi t}{2}}).$$

For $\nu < 0$,

$$\sec\pi\nu\tau = \sec(-\pi\nu\tau) = 2e^{-i\pi\nu\tau} + O(e^{\frac{-3\pi t}{2}}).$$

For $\nu = \frac{1}{2}$,

$$\sec\pi\nu\tau = 2e^{\frac{i\pi\tau}{2}} + O(e^{\frac{-3\pi t}{2}}),$$

and for $\nu = -\frac{1}{2}$,

$$\sec\pi\nu\tau = 2e^{\frac{i\pi\tau}{2}} + O(e^{\frac{-3\pi t}{2}}).$$

It follows that

$$\sum_{n\in\mathbb{Z}}\sec\left(\pi\left(n+\frac{1}{2}\right)\tau\right) = 2e^{\frac{i\pi\tau}{2}} + 2e^{\frac{i\pi\tau}{2}} + O(e^{\frac{-3\pi t}{2}}) = 4e^{\frac{i\pi\tau}{2}} + O(e^{\frac{-3\pi t}{2}}).$$

Using this with (1) yields

$$C\left(1-\frac{1}{\tau}\right) = \frac{4\tau}{i}e^{\frac{i\pi\tau}{2}} + O(|\tau|e^{\frac{-3\pi t}{2}}), \qquad \tau = \sigma + it,$$

proving the claim. □

Define $\theta : \mathfrak{H}\to\mathbb{C}$ by

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau}, \qquad \tau\in\mathfrak{H}.$$

By proving that $\frac{C}{\theta^2}$ is a modular form of weight 0, it follows that it is constant, and one thus finds that $C = \theta^2$.[3] One reason that θ is significant is that, for $q = e^{i\pi\tau}$,

$$\begin{aligned}\theta(\tau)^2 &= \left(\sum_{n_1\in\mathbb{Z}} q^{n_1^2}\right)\left(\sum_{n_2\in\mathbb{Z}} q^{n_2^2}\right)\\ &= \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} q^{n_1^2+n_2^2}\\ &= \sum_{n=0}^{\infty} r_2(n)q^n,\end{aligned}$$

where $r_2(n)$ denotes the number of ways that n can be expressed as a sum of two squares. We can write $C(\tau)$ as

$$\begin{aligned}C(\tau) &= 2\sum_{n=-\infty}^{\infty}\frac{1}{q^n+q^{-n}}\\ &= 2\sum_{n=-\infty}^{\infty}\frac{q^n}{1+q^{2n}}\\ &= 1+4\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\\ &= 1+4\sum_{n=1}^{\infty}q^n\frac{1-q^{2n}}{1-q^{4n}}\\ &= 1+4\sum_{n=1}^{\infty}\left(\frac{q^n}{1-q^{4n}}-\frac{q^{3n}}{1-q^{4n}}\right).\end{aligned}$$

Therefore the identity $\theta(\tau)^2 = C(\tau)$ can be written as

$$\sum_{n=0}^{\infty} r_2(n)q^n = 1+4\sum_{n=1}^{\infty}\left(\frac{q^n}{1-q^{4n}}-\frac{q^{3n}}{1-q^{4n}}\right).$$

We write

$$\sum_{n=1}^{\infty}\frac{q^n}{1-q^{4n}} = \sum_{n=1}^{\infty}q^n\sum_{m=0}^{\infty}(q^{4n})^m = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q^{n(4m+1)} = \sum_{k=1}^{\infty}a(k)q^k,$$

where $a(k)$ denotes the number of divisors of k of the form $4m+1$, and

$$\sum_{n=1}^{\infty}\frac{q^{3n}}{1-q^{4n}} = \sum_{n=1}^{\infty}q^{3n}\sum_{m=0}^{\infty}(q^{4n})^m = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q^{n(4m+3)} = \sum_{k=1}^{\infty}b(k)q^k,$$

where $b(k)$ denotes the number of divisors of k of the form $4m+3$. Thus for $n \geq 1$,

$$r_2(n) = 4(a(n)-b(n)).$$

[3] Elias M. Stein and Rami Shakarchi, *Complex Analysis*, p. 304.