The Segal-Bargmann transform and the Segal-Bargmann space

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July 30, 2015

1 The Fourier transform

Let $dm_n(x) = (2\pi)^{-n/2}dx$. For Borel measurable functions $f, g : \mathbb{R}^n \to \mathbb{C}$, when $y \mapsto f(x-y)g(y)$ is integrable we define

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dm_n(y).$$

For $f \in L^1$,

$$\hat{f}(\xi) = (\mathscr{F} f)(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i\langle \xi, x\rangle} dm_n(x), \qquad \xi \in \mathbb{R}^n.$$

For $f, g \in L^1$, for almost all $x \in \mathbb{R}^n$, $y \mapsto f(x-y)g(y)$ is integrable,[1] and using Fubini's theorem one checks that

$$\widehat{f * g} = \widehat{f}\widehat{g}.$$

Let $\mathscr{S}$ be the Schwartz functions $\mathbb{R}^n \to \mathbb{C}$. For a multi-index α and $\phi \in \mathscr{S}$ define $X^\alpha \phi : \mathbb{R}^n \to \mathbb{C}$ by

$$(X^\alpha \phi)(x) = x^\alpha \phi(x).$$

Define $\Delta\phi : \mathbb{R}^n \to \mathbb{C}$ by

$$(\Delta\phi)(x) = \sum_{j=1}^n (\partial_j^2 \phi)(x).$$

One proves that

$$\mathscr{F} D^\alpha = i^{|\alpha|} X^\alpha \mathscr{F}, \qquad D^\alpha \mathscr{F} = (-i)^{|\alpha|} \mathscr{F} X^\alpha$$

[1] Walter Rudin, *Real and Complex Analysis*, third ed., p. 170, Theorem 8.14.

and

$$\mathscr{F}(\Delta\phi)(\xi) = -|\xi|^2(\mathscr{F}\phi)(\xi).$$

Parseval's formula states that for $f, g \in L^2$,

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{R}^n} f\overline{g}dm_n = \int_{\mathbb{R}^n} (\mathscr{F}f)\overline{(\mathscr{F}g)}dm_n = \langle \mathscr{F}f, \mathscr{F}g\rangle_{L^2},$$

thus

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^n} |f|^2 dm_n = \int_{\mathbb{R}^n} |\mathscr{F}f|^2 dm_n = \|\mathscr{F}f\|_{L^2}^2.$$

For $z \in \mathbb{C}^n$, using Cauchy's integral theorem we obtain

$$\int_{\mathbb{R}^n} F(x+iy)e^{-i\langle\xi,x\rangle}dx = e^{-\langle\xi,y\rangle}\int_{\mathbb{R}^n} F(x)e^{-i\langle\xi,x\rangle}dx. \tag{1}$$

2 The heat kernel

For $t \geq 0$ and $f \in L^2$, define $H_t f : \mathbb{R}^n \to \mathbb{C}$ by

$$(H_t f)(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{-t|\xi|^2}e^{i\langle\xi,x\rangle}dm_n(\xi).$$

For $t \in \mathbb{R}_{>0}$ let

$$h_t(x) = (4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}, \qquad x \in \mathbb{R}^n,$$

and we calculate

$$\partial_t h_t = (4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}\left(-\frac{n}{2t} + \frac{|x|^2}{4t^2}\right) = \Delta h_t,$$

which yields

$$\partial_t(f * h_t) = f * (\partial_t h_t) = f * (\Delta h_t) = \Delta(f * h_t).$$

The Fourier transform of h_t is[2]

$$\begin{aligned}\widehat{h}_t(\xi) &= \int_{\mathbb{R}^n} (4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}e^{-i\langle\xi,x\rangle}dm_n(x)\\ &= (4\pi t)^{-n/2}\cdot(2\pi)^{-n/2}(4\pi t)^{n/2}\exp(-t|\xi|^2)\\ &= (2\pi)^{-n/2}\exp(-t|\xi|^2).\end{aligned}$$

Using $\widehat{h_t * f} = \widehat{p}_t \cdot \widehat{f}$ and the Fourier inversion theorem,

$$\begin{aligned}(h_t * f)(x) &= \int_{\mathbb{R}^n} \widehat{h_t * f}(\xi)e^{i\langle\xi,x\rangle}dm_n(\xi)\\ &= \int_{\mathbb{R}^n} \widehat{p}_t(\xi)\widehat{f}(\xi)e^{i\langle\xi,x\rangle}dm_n(\xi)\\ &= \int_{\mathbb{R}^n} (2\pi)^{-n/2}\exp(-t|\xi|^2)\widehat{f}(\xi)e^{i\langle\xi,x\rangle}dm_n(\xi)\\ &= (H_t f)(x).\end{aligned}$$

[2] `http://individual.utoronto.ca/jordanbell/notes/stationaryphase.pdf`, Theorem 2.

For $t > 0$ and for $z \in \mathbb{C}^n$,

$$(H_t f)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{-t|\xi|^2} e^{i\langle \xi, z\rangle} dm_n(\xi)$$

and

$$h_t(z) = (4\pi t)^{-n/2} \exp\left(-\frac{z_1^2 + \cdots + z_n^2}{4t}\right).$$

It is apparent that $h_t : \mathbb{C}^n \to \mathbb{C}$ is holomorphic. By the dominated convergence theorem,

$$\frac{dH_t f}{dz_j}(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{-t|\xi|^2} i\xi_j e^{i\langle \xi, z\rangle} dm_n(\xi),$$

and $H_t f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic.

3 The Segal-Bargmann transform and the Segal-Bargmann space

Let λ_n be Lebesgue measure on $\mathbb{R}^n$, for $t > 0$ let

$$\omega_t(y) = t^{-n/2} e^{-\frac{|y|^2}{2t}},$$

and let μ_t be the Borel measure on $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$ whose density with respect to $\lambda_n \times \lambda_n$ is $x + iy \mapsto \omega_t(y)$. We define $\mathcal{H}_t(\mathbb{C}^n)$ to be the set of those holomorphic functions $F : \mathbb{C}^n \to \mathbb{C}$ satisfying

$$\|F\|_{\mathcal{H}_t}^2 = \int_{\mathbb{C}^n} |F|^2 d\mu_t < \infty,$$

and for $G, H \in \mathcal{H}_t$ we define

$$\langle F, G\rangle_{\mathcal{H}_t} = \int_{\mathbb{C}^n} F\overline{G} d\mu_t = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} F(x+iy)\overline{G(x+iy)}\omega_t(y) dy \right) dx.$$

We call $\mathcal{H}_t$ the **Segal-Bargmann space**. It can be proved that it is a Hilbert space.

For $y \in \mathbb{R}^n$ write $g(x) = (H_t f)(x + iy)$, and applying Parseval's formula and (1) yields

$$\begin{aligned}
\int_{\mathbb{R}^n} |(H_t f)(x+iy)|^2 dm_n(x) &= \int_{\mathbb{R}^n} |g(x)|^2 dm_n(x) \\
&= \int_{\mathbb{R}^n} |\widehat{g}(\xi)|^2 dm_n(\xi) \\
&= \int_{\mathbb{R}^n} |e^{-\langle \xi, y\rangle} \widehat{H_t f}(\xi)|^2 dm_n(\xi).
\end{aligned}$$

Using this with $\widehat{H_t f} = \widehat{h}_t \widehat{f}$ and then using Fubini's theorem and an identity for Gaussian integrals[3] we get

$$\begin{aligned}
\|H_t f\|_{\mathcal{H}_t}^2 &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |(H_t f)(x+iy)|^2 \omega_t dy \right) dx \\
&= (2\pi)^n \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |(H_t f)(x+iy)|^2 dm_n(x) \right) \omega_t(y) dm_n(y) \\
&= (2\pi)^n \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-2\langle \xi, y\rangle} |\widehat{H_t f}(\xi)|^2 dm_n(\xi) \right) \omega_t(y) dm_n(y) \\
&= (2\pi)^n \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-2\langle \xi, y\rangle} (2\pi)^{-n} \exp(-2t|\xi|^2) |\widehat{f}(\xi)|^2 dm_n(\xi) \right) \omega_t(y) dm_n(y) \\
&= \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \exp(-2t|\xi|^2) \left(t^{-n/2} \int_{\mathbb{R}^n} e^{-2\langle \xi, y\rangle} e^{-\frac{|y|^2}{2t}} dm_n(y) \right) dm_n(x) \\
&= \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \exp(-2t|\xi|^2) \cdot \exp(2t|\xi|^2) dm_n(x) \\
&= \|\mathscr{F} f\|_{L^2}^2 \\
&= \|f\|_{L^2}^2 .
\end{aligned}$$

Therefore $H_t : L^2(\mathbb{R}^n) \to \mathcal{H}_t(\mathbb{C}^n)$ is a linear isometry. We call H_t the **Segal-Bargmann transform**. It can be proved that H_t is a Hilbert space isomorphism.[4]

For $F \in \mathcal{H}_t$ and $z \in \mathbb{C}^n$, write

$$\mathrm{ev}_z(F) = F(z)$$

and

$$(T_w F)(z) = F(z - w).$$

For $f \in L^2(\mathbb{R}^n)$ and $t > 0$ let $F = H_t f \in \mathcal{H}_t(\mathbb{C}^n)$, and for $w \in \mathbb{C}^n$, using

$$\overline{h_t(w-x)} = \overline{h_t(x-w)} = h_t(x - \overline{w}) = (T_{\overline{w}} h_t)(x),$$

we get

$$\begin{aligned}
\mathrm{ev}_w(F) &= (f * h_t)(w) \\
&= \int_{\mathbb{R}^n} f(x) \overline{(T_{\overline{w}} h_t)(x)} dm_n(x) \\
&= \langle f, T_{\overline{w}} h_t \rangle_{L^2} \\
&= \langle H_t f, H_t T_{\overline{w}} h_t \rangle_{L^2} \\
&= \langle F, H_t T_{\overline{w}} h_t \rangle_{L^2} .
\end{aligned}$$

Then $(w, z) \mapsto (H_t T_{\overline{w}} h_t)(z)$ is a **reproducing kernel** for the Hilbert space $\mathcal{H}_t$.

[3] `http://individual.utoronto.ca/jordanbell/notes/stationaryphase.pdf`, Theorem 3.
[4] cf. `https://www.math.lsu.edu/~olafsson/pdf_files/ht.pdf`