Germs of smooth functions

Jordan Bell
jordan.bell@gmail.com
Department of Mathematics, University of Toronto
April 4, 2016

1 Sheaves

Let $M = \mathbb{R}^m$. For an open set $U$ in $M$, write $\mathcal{F}(U) = C^\infty(U)$, which is a commutative ring with unity $1_M(x) = 1$. For open sets $V \subset U$ in $M$, define $r_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ by $r_{U,V} f = f|_V$, which is a homomorphism of rings. $\mathcal{F}$ is a presheaf, a contravariant functor from the category of open sets in $M$ to the category of commutative unital rings. For $\mathcal{F}$ to be a sheaf means the following:

1. If $U_i$, $i \in I$, is an open cover of an open set $U$ and if $f, g \in \mathcal{F}(U)$ satisfy $r_{U,U_i} f = r_{U,U_i} g$ for all $i \in I$, then $f = g$.
2. If $U_i, i \in I$, is an open cover of an open set $U$ and for each $i \in I$ there is some $f_i \in \mathcal{F}(U_i)$ such that for all $i, j \in I$, $r_{U_i, U_i \cap U_j} f_i = r_{U_j, U_i \cap U_j} f_j$, then there is some $f \in \mathcal{F}(U)$ such that $r_{U,U_i} f = f_i$ for each $i \in I$.

For the first condition, let $p \in U$. As $U_i$ is an open cover of $U$, there is some $i$ for which $p \in U_i$. As $f|_{U_i} = g|_{U_i}$, $f(p) = g(p)$. Therefore $f = g$. For the second condition, let $p \in U$. If $p \in U_i$ and $p \in U_j$, then $f_i(p) = f_j(p)$. This shows that it makes sense to define $f : U \to \mathbb{R}$ by $f(p) = f_i(p)$, for any $i$ such that $p \in U_i$. Then $f|_{U_i} = f_i$, which implies that $f \in \mathcal{F}(U)$: for each $p \in U$, there is some open neighborhood $U_i$ of $p$ on which $f$ is smooth. Therefore $\mathcal{F}$ is a sheaf.

2 Stalks and germs

For $p \in M$, let $\mathcal{U}_p$ be the set of open neighborhoods of $p$. For $U, V \in \mathcal{U}_p$, say $U \leq V$ when $V \subset U$. For $U \leq V \leq W$ and $f \in \mathcal{F}(U)$,

$$(r_{V,W} \circ r_{U,V})(f) = r_{V,W} f|_V = f_W = r_{U,W} f.$$ 

For $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$, say $f \sim_p g$ if there is some $W \in \mathcal{U}_p$, $W \geq U$, $W \geq V$, such that $r_{U,W} f = r_{V,W} g$. Let

$$\mathcal{R}_p = \bigsqcup_{U \in \mathcal{U}_p} \mathcal{F}(U),$$

1
and let $F_p$ be the direct limit of the direct system $F(U)$, $r_{U,V}$ of commutative unital rings:

$$F_p = R_p/\sim_p.$$  

We call $F_p$ the **stalk** of $F$ at $p$. An element of $F_p$ is called a **germ** of $F$ at $p$.

In other words, for $f \in R_p$, let $[f]_p$ be the set of those $g \in R_p$ such that $f \sim_p g$, equivalently, $f|_{U_f \cap U_g} = g|_{U_f \cap U_g}$. A germ of $F$ at $p$ is such an equivalence class $[f]_p$, and

$$F_p = \{[f]_p : f \in R_p\}.$$

3 Maximal ideals

For $p \in M$, and $f, g \in R_p$ with $f \sim_p g$, $f(p) = g(p)$. Thus it makes sense to define $ev_p : F_p \to \mathbb{R}$ by $ev_p[f]_p = f(p)$. Now, for $[f]_p, [g]_p \in F_p$,

$$ev_p([f]_p + [g]_p) = ev_p([f + g]_p) = (f + g)(p) = f(p) + g(p) = ev_p[f]_p + ev_p[g]_p,$$

$$ev_p([f]_p[g]_p) = ev_p((fg)_p) = (fg)(p) = f(p)g(p) = ev_p[f]_p \cdot ev_p[g]_p.$$  

$ev_p[1_M]_p = 1$. This means that $ev_p : F_p \to \mathbb{R}$ is a homomorphism of unital rings. It is straightforward that $ev_p$ is surjective. Write $m_p = ker ev_p$. By the first isomorphism theorem, there is an isomorphism of unital rings $F_p/m_p \to \mathbb{R}$.

Therefore $m_p$ is a maximal ideal in $F_p$. Now, if $[f]_p \in F_p \setminus m_p$ then $ev_p[f]_p \neq 0$, hence $f(p) \neq 0$. Then there is some $U \in \mathcal{U}_p$ such that $f(x) \neq 0$ for $x \in U$, and $(1/f)(x) = \frac{1}{f(x)}$ belongs to $F(U)$. Then $[1/f]_p \in F_p$ and $[f]_p : [1/f]_p = [f \cdot 1/f]_p = [1_M]_p$, which shows that if $[f]_p \in F_p \setminus m_p$ then $[f]_p$ has an inverse $[1/f]_p$ in $F_p$. This means $m_p$ is the set of noninvertible elements of $F_p$, which means that $F_p$ is a **local ring**.

For $1 \leq i \leq m$ define the coordinate function $x^i : M \to \mathbb{R}$ by $x^i(p) = p_i$, which belongs to $F(M)$. Because $ev_0 x^i = 0$, $[x^i]_0 \in m_0$. We prove Hadamard’s lemma, that the ring $m_0$ is generated by the germs of the coordinate functions at $0$.

**Lemma 1 (Hadamard’s lemma).** The ideal $m_0$ is generated by the set $\{[x^i]_0 : 1 \leq i \leq m\}$.

**Proof.** Let $[f]_0 \in m_0$ with $f \in F(B_r)$ for some $r > 0$. For $y \in B_r$, using the fundamental theorem of calculus and using the chain rule,

$$f(y) - f(0) = \int_0^1 \frac{d}{ds} f(sy) ds = \int_0^1 \sum_{i=1}^m x^i(y) (\partial_i f)(sy) ds = \sum_{i=1}^m x^i(y) u_i(y),$$

and $u_i \in F(B_r)$. This means that $[f]_0 = \sum_{i=1}^m [x^i]_0 [u_i]_0$, which shows that $[f]_0$ belongs to the ideal generated by the set $\{[x^i]_0 : 1 \leq i \leq m\}$.  

---

For a multi-index \( \alpha \in \mathbb{Z}_{\geq 0}^m \), write
\[
|\alpha| = \sum_{i=1}^{m} \alpha_i, \quad \alpha! = \alpha_1! \cdots \alpha_m!
\]
and
\[
\partial^{\alpha} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m}, \quad x^{\alpha} = (x_1)^{\alpha_1} \cdots (x_m)^{\alpha_m},
\]
and say \( \alpha \leq \beta \) if \( \alpha_i \leq \beta_i \) for each \( i \). We shall use the fact that
\[
\partial^{\alpha} x^{\beta} = \begin{cases} 
\beta! (\beta - \alpha)!\ x^{\beta - \alpha} & \alpha \leq \beta \\
0 & \text{otherwise}.
\end{cases}
\]

**Lemma 2.** For \( f \in \mathcal{R}_0 \), if \( (\partial^{\alpha} f)(0) = 0 \) for all \( |\alpha| < k \), then \( [f]_0 \in \mathfrak{m}_0^k \).

**Proof.** For \( k = 1 \), if \( (\partial^{\alpha} f)(0) = 0 \) for \( \alpha = (0, \ldots, 0) \) then \( \text{ev}_0 f = f(0) = 0 \), hence \( [f]_0 \in \mathfrak{m}_0 \). Suppose the claim is true for some \( k \geq 1 \), and suppose that \( f \in \mathcal{R}_0 \) and that \( (\partial^{\alpha} f)(0) = 0 \) for all \( |\alpha| < k+1 \). A fortiori, \( (\partial^{\alpha} f)(0) = 0 \) for all \( |\alpha| < k \) and then by the induction hypothesis we get \( [f]_0 \in \mathfrak{m}_0^k \). Now, Lemma 1 tells us that the ideal \( \mathfrak{m}_0 \) is generated by the set \( \{x_i^\alpha : 1 \leq i \leq m\} \), and then the product ideal \( \mathfrak{m}_0^k \) is generated by the set
\[
\{x_i^{\alpha} : 1 \leq i \leq m\} = \{x^{\alpha} : |\alpha| = k\},
\]
for \( x^{\alpha} = (x_1)^{\alpha_1} \cdots (x_m)^{\alpha_m} \). As \( [\partial^{\alpha} f]_0 = \sum_{|\alpha| = k} [u_\alpha]_0 [x^{\alpha}]_0 \).

For \( |\alpha| = k \), on some set in \( \mathcal{U}_0 \), using the Leibniz rule,
\[
\partial^{\alpha} f = \sum_{|\beta| = k} \partial^{\alpha} (u_\beta x^\beta) = \sum_{|\beta| = k} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} (\partial^{\alpha - \gamma} u_\beta) (\partial^{\gamma} x^\beta).
\]
And for \( \gamma \neq \beta \), \( (\partial^{\gamma} x^\beta)(0) = 0 \), so
\[
\partial^{\alpha} f \in u_\alpha \partial^{\alpha} x^\alpha + h, \quad [h]_0 \in \mathfrak{m}_0.
\]
But \( (\partial^{\alpha} f)(0) = 0 \) so \( u_\alpha(0) = 0 \), which means that \( u_\alpha \in \mathfrak{m}_0 \). And
\[
[x^{\alpha}]_0 = [x_1]_0^{\alpha_1} \cdots [x_m]_0^{\alpha_m} \in \mathfrak{m}_0[\alpha] = \mathfrak{m}_0^k,
\]
so \( [u_\alpha]_0 [x^{\alpha}]_0 \in \mathfrak{m}_0^{k+1} \), showing that \( [f]_0 \in \mathfrak{m}_0^{k+1} \). This completes the proof by induction. \( \square \)
4 Hessians

For an open set $U$ in $\mathbb{R}^m$ and $\phi \in F(U)$, $\phi' : U \to \mathcal{L}(\mathbb{R}^m, \mathbb{R})$, and $\nabla \phi : U \to \mathbb{R}^m$ satisfies

$$\langle \nabla \phi(x), v \rangle = \phi'(x)(v), \quad x \in U, \quad v \in \mathbb{R}^m.$$ 

$x \in U$ is a critical point of $\phi$ if $\phi'(x) = 0$, equivalently $\nabla \phi(x) = 0$. Define $\text{Hess} \phi : U \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ by

$$\text{Hess} \phi = (\nabla \phi)'.$$

This satisfies

$$\phi''(x)(u)(v) = \langle v, \text{Hess} \phi(x)(u) \rangle, \quad x \in U, \quad u, v \in \mathbb{R}^m.$$ 

A critical point $x$ of $\phi$ is called nondegenerate if $\text{Hess} \phi(x)$ is invertible in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$.

For $\phi \in R_p$, let $J_\phi$ be the ideal in the ring $F_p$ generated by the set

$$\{[\partial_i \phi]_p : 1 \leq i \leq m\}.$$ 

We call $J_\phi$ the Jacobian ideal of $\phi$ at $p$. If $p$ is a critical point of $\phi$, then $(\partial_i \phi)(p) = 0$ for each $i$, hence $[\partial_i \phi]_p \in m_p$ for each $i$.

If $0$ is a nondegenerate critical point of $\phi$, we prove that $m_0 \subset J_\phi$.\footnote{Liviu Nicolaescu, An Invitation to Morse Theory, second ed., p. 15, Lemma 1.15.}

**Theorem 3.** Let $U$ be an open set in $\mathbb{R}^m$ containing $0$ and let $\phi \in F(U)$. If $0$ is a nondegenerate critical point of $\phi$, then $J_\phi = m_0$.\footnote{Serge Lang, Real and Functional Analysis, third ed., p. 361, chapter XIV, Theorem 1.2.}

**Proof.** Let $f = \nabla \phi$, which is a smooth function $U \to \mathbb{R}^m$. Because $0$ is a nondegenerate critical point of $\phi$, $f'(0)$ is invertible in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ and hence by the inverse function theorem\footnote{http://individual.utoronto.ca/jordanbell/notes/gradienthilbert.pdf} $f$ is a local $C^\infty$ isomorphism at $x$: there is some open set $V, x \in V$ and $V \subset U$, such that $W = f(V)$ is open in $\mathbb{R}^m$, and there is a smooth function $g : W \to V$ such that $g \circ f = \text{id}_V$ and $f \circ g = \text{id}_W$. \qed