Sobolev spaces in one dimension and absolutely continuous functions

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1 Locally integrable functions and distributions

Let λ be Lebesgue measure on $\mathbb{R}$. We denote by $\mathscr{L}^1_{\mathrm{loc}}(\lambda)$ the collection of Borel measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that for each compact subset K of $\mathbb{R}$,

$$N_K(f) = \int_K |f| d\lambda = \int_{\mathbb{R}} 1_K |f| d\lambda < \infty.$$

We denote by $L^1_{\mathrm{loc}}(\lambda)$ the collection of equivalence classes of elements of $\mathscr{L}^1_{\mathrm{loc}}(\lambda)$ where $f \sim g$ when $f = g$ almost everywhere.

Write $B(x, r) = \{y \in \mathbb{R} : |y - x| < r\} = (x - r, x + r)$. For $f \in \mathscr{L}^1_{\mathrm{loc}}(\lambda)$ and $x \in \mathbb{R}$, we say that x is a **Lebesgue point of** f if

$$\lim_{r \to 0} \frac{1}{\lambda(B(x, r))} \int_{B(x,r)} |f(y) - f(x)| d\lambda(y) = 0.$$

It is immediate that if f is continuous at x then x is a Lebesgue point of f. The **Lebesgue differentiation theorem**[1] states that for $f \in \mathscr{L}^1_{\mathrm{loc}}(\lambda)$, almost every $x \in \mathbb{R}$ is a Lebesgue point of f. A sequence of Borel sets E_n is said to **shrink nicely to** x if there is some $\alpha > 0$ and a sequence $r_n \to 0$ such that $E_n \subset B(x, r_n)$ and $\lambda(E_n) \geq \alpha \cdot \lambda(B(x, r_n))$. The sequence $B(x, n^{-1}) = (x - n^{-1}, x + n^{-1})$ shrinks nicely to x, the sequence $[x, x + n^{-1}]$ shrinks nicely to x, and the sequence $[x - n^{-1}, x]$ shrinks nicely to x. It is proved that if $f \in \mathscr{L}^1_{\mathrm{loc}}(\lambda)$ and for each $x \in \mathbb{R}$, $E_n(x)$ is a sequence that shrinks nicely to x, then

$$f(x) = \lim_{n \to \infty} \frac{1}{\lambda(E_n)} \int_{E_n(x)} f d\lambda$$

at each Lebesgue point of f.[2]

[1] Walter Rudin, *Real and Complex Analysis*, third ed., p. 138, Theorem 7.7.
[2] Walter Rudin, *Real and Complex Analysis*, third ed., p. 140, Theorem 7.10.

For a nonempty open set Ω in $\mathbb{R}$, we denote by $C_c^k(\Omega)$ the collection of C^k functions $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\operatorname{supp} \phi = \overline{\{x \in \mathbb{R} : \phi(x) \neq 0\}}$$

is compact and is contained in Ω. We write $\mathscr{D}(\Omega) = C_c^\infty(\Omega)$, whose elements are called called **test functions**. The following statement is called the **fundamental lemma of the calculus of variations** or the **Du Bois-Reymond Lemma**.[3]

Theorem 1. *If $f \in \mathscr{L}^1_{\mathrm{loc}}(\lambda)$ and $\int_{\mathbb{R}} f\phi d\lambda = 0$ for all $\phi \in \mathscr{D}(\mathbb{R})$, then $f = 0$ almost everywhere.*

Proof. There is some $\eta \in \mathscr{D}(-1,1)$ with $\int_{\mathbb{R}} \eta d\lambda = 1$. We can explicitly write this out:

$$\eta(x) = \begin{cases} c^{-1} \exp\left(\frac{1}{x^2-1}\right) & |x| < 1 \\ 0 & |x| \geq 1, \end{cases}$$

where

$$c = \int_{-1}^{1} \exp\left(\frac{1}{y^2-1}\right) d\lambda(y) = 0.443994\ldots.$$

For x a Lebesgue point of f and for $0 < r < 1$,

$$\begin{aligned} f(x) &= f(x) \cdot \int_{\mathbb{R}} \eta(y) d\lambda(y) \\ &= f(x) \cdot \frac{1}{r} \int_{\mathbb{R}} \eta\left(\frac{y}{r}\right) d\lambda(y) \\ &= f(x) \cdot \frac{1}{r} \int_{\mathbb{R}} \eta\left(\frac{x-y}{r}\right) d\lambda(y) \\ &= \frac{1}{r} \int_{\mathbb{R}} (f(x) - f(y))\eta\left(\frac{x-y}{r}\right) d\lambda(y) + \frac{1}{r} \int_{\mathbb{R}} f(y)\eta\left(\frac{x-y}{r}\right) d\lambda(y) \\ &= \frac{1}{r} \int_{\mathbb{R}} (f(x) - f(y))\eta\left(\frac{x-y}{r}\right) d\lambda(y) \\ &= \frac{1}{r} \int_{(x-r,x+r)} (f(x) - f(y))\eta\left(\frac{x-y}{r}\right) d\lambda(y). \end{aligned}$$

Then

$$|f(x)| \leq \|\eta\|_\infty \cdot \frac{1}{r} \int_{(x-r,x+r)} |f(y) - f(x)| d\lambda(y) \to 0, \qquad r \to 0,$$

meaning that $f(x) = 0$. This is true for almost all $x \in \mathbb{R}$, showing that $f = 0$ almost everywhere. ∎

[3] Lars Hörmander, *The Analysis of Linear Partial Differential Operators I*, second ed., p. 15, Theorem 1.2.5.

For $f \in \mathscr{L}^1_{\text{loc}}(\lambda)$, define $\Lambda_f : \mathscr{D}(\mathbb{R}) \to \mathbb{R}$ by

$$\Lambda_f(\phi) = \int_{\mathbb{R}} f\phi d\lambda.$$

$\mathscr{D}(\mathbb{R})$ is a locally convex space, and one proves that Λ_f is continuous and thus belongs to the dual space $\mathscr{D}'(\mathbb{R})$, whose elements are called **distributions**.[4] We say that a distribution Λ is **induced** by $f \in \mathscr{L}^1_{\text{loc}}(\lambda)$ if $\Lambda = \Lambda_f$. For $\Lambda \in \mathscr{D}'(\mathbb{R})$, we define $D\Lambda : \mathscr{D}(\mathbb{R}) \to \mathbb{R}$ by

$$(D\Lambda)(\phi) = -\Lambda(\phi').$$

It is proved that $D\Lambda \in \mathscr{D}'(\mathbb{R})$.[5]

Let $f, g \in \mathscr{L}^1_{\text{loc}}(\lambda)$. If $D\Lambda_f = \Lambda_g$, we call g a **distributional derivative of** f. In other words, for $f \in \mathscr{L}^1_{\text{loc}}(\lambda)$ to have a distributional derivative means that there is some $g \in \mathscr{L}^1_{\text{loc}}(\lambda)$ such that for all $\phi \in \mathscr{D}(\mathbb{R})$,

$$-\int_{\mathbb{R}} f\phi' d\lambda = \int_{\mathbb{R}} g\phi d\lambda.$$

If $g_1, g_2 \in \mathscr{L}^1_{\text{loc}}(\lambda)$ are distributional derivatives of f then $\int_{\mathbb{R}} (g_1 - g_2)\phi d\lambda = 0$ for all $\phi \in \mathscr{D}(\mathbb{R})$, which by Theorem 1 implies that $g_1 = g_2$ almost everywhere. It follows that if f has a distributional derivative then the distributional derivative is unique in $L^1_{\text{loc}}(\lambda)$, and is denoted $Df \in L^1_{\text{loc}}(\lambda)$:

$$-\int_{\mathbb{R}} f\phi' d\lambda = \int_{\mathbb{R}} (Df) \cdot \phi d\lambda, \qquad \phi \in \mathscr{D}(\mathbb{R}).$$

2 The Sobolev space $H^1(\mathbb{R})$

We denote by $\mathscr{L}^2(\lambda)$ the collection of Borel measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} |f|^2 d\lambda < \infty$, and we denote by $L^2(\lambda)$ the collection of equivalence classes of elements of $\mathscr{L}^2(\lambda)$ where $f \sim g$ when $f = g$ almost everywhere, and write

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{R}} fg d\lambda.$$

It is a fact that $L^2(\lambda)$ is a Hilbert space.

We define the **Sobolev space** $H^1(\mathbb{R})$ to be the set of $f \in L^2(\lambda)$ that have a distributional derivative that satisfies $Df \in L^2(\lambda)$. We remark that the elements of $H^1(\mathbb{R})$ are equivalence classes of elements of $\mathscr{L}^2(\lambda)$. We define

$$\langle f, g \rangle_{H^1} = \langle f, g \rangle_{L^2} + \langle Df, Dg \rangle_{L^2}.$$

[4] Walter Rudin, *Functional Analysis*, second ed., p. 157, §6.11.
[5] Walter Rudin, *Functional Analysis*, second ed., p. 158, §6.12.

Let $f, g \in H^1(\mathbb{R})$ and let $\phi \in \mathscr{D}(\mathbb{R})$. Because f, g have distributional derivatives Df, Dg,

$$\begin{aligned} -\int_{\mathbb{R}} (f+g)\phi' d\lambda &= -\int_{\mathbb{R}} f\phi' d\lambda - \int_{\mathbb{R}} g\phi' d\lambda \\ &= \int_{\mathbb{R}} Df \cdot \phi d\lambda + \int_{\mathbb{R}} Dg \cdot \phi d\lambda \\ &= \int_{\mathbb{R}} (Df + Dg)\phi d\lambda. \end{aligned}$$

This means that $f+g$ has a distributional derivative, $D(f+g) = Df + Dg$. Thus $H^1(\mathbb{R})$ is a linear space. If $\langle f, f\rangle_{H^1} = 0$ then $\int_{\mathbb{R}} |f|^2 d\lambda = 0$, which implies that $f = 0$ as an element of $L^2(\lambda)$. Therefore $\langle \cdot, \cdot \rangle_{H^1}$ is an inner product on $H^1(\mathbb{R})$.

If f_n is a Cauchy sequence in $H^1(\mathbb{R})$, then f_n is a Cauchy sequence in $L^2(\lambda)$ and Df_n is a Cauchy sequence in $L^2(\lambda)$, and hence these sequences have limits $f, g \in L^2(\lambda)$. For $\phi \in \mathscr{D}(\mathbb{R})$,

$$\begin{aligned} -\int_{\mathbb{R}} f\phi' d\lambda &= -\lim_{n\to\infty} \int_{\mathbb{R}} f_n \phi' d\lambda \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} (Df_n) \cdot \phi d\lambda \\ &= \int_{\mathbb{R}} g\phi d\lambda. \end{aligned}$$

This means that f has distributional derivative, $Df = g$. Because $f, Df \in L^2(\lambda)$ it is the case that $f \in H^1(\mathbb{R})$. Furthermore,

$$\|f_n - f\|_{H^1}^2 = \|f_n - f\|_{L^2}^2 + \|Df_n - Df\|_{L^2}^2 = \|f_n - f\|_{L^2}^2 + \|Df_n - g\|_{L^2}^2 \to 0,$$

meaning that $f_n \to f$ in $H^1(\mathbb{R})$, which shows that $H^1(\mathbb{R})$ is a Hilbert space.

3 Absolutely continuous functions

We prove a lemma that gives conditions under which a function, for which integration by parts needs not make sense, is equal to a particular constant almost everywhere.[6]

Lemma 2. *If $f \in \mathscr{L}^1_{\mathrm{loc}}(\lambda)$ and*

$$\int_{\mathbb{R}} f\phi' d\lambda = 0, \qquad \phi \in \mathscr{D}(\mathbb{R}),$$

then there is some $c \in \mathbb{R}$ such that $f = c$ almost everywhere.

[6] Haim Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, p. 204, Lemma 8.1.

Proof. Fix $\eta \in \mathscr{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \eta d\lambda = 1$. Let $w \in \mathscr{D}(\mathbb{R})$ and define

$$h = w - \eta \cdot \int_{\mathbb{R}} w d\lambda,$$

which belongs to $\mathscr{D}(\mathbb{R})$ and satisfies $\int_{\mathbb{R}} h d\lambda = 0$. Define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \int_{-\infty}^{x} h d\lambda.$$

Using $\phi'(x) = h(x)$ for all x and $\phi(x) \to \int_{\mathbb{R}} h d\lambda = 0$ as $x \to \infty$, check that $\phi \in \mathscr{D}(\mathbb{R})$. Then by hypothesis, $\int_{\mathbb{R}} f\phi' d\lambda = 0$, i.e.

$$\begin{aligned} 0 &= \int_{\mathbb{R}} f h d\lambda \\ &= \int_{\mathbb{R}} \left(f w - f\eta \cdot \int_{\mathbb{R}} w d\lambda \right) d\lambda \\ &= \int_{\mathbb{R}} \left(f - \int_{\mathbb{R}} f\eta d\lambda \right) \cdot w d\lambda. \end{aligned}$$

Because this is true for all $w \in \mathscr{D}(\mathbb{R})$, by Theorem 1 we get that $f = \int_{\mathbb{R}} f\eta d\lambda$ almost everywhere. □

Lemma 3. *Let $g \in \mathscr{L}^1_{\text{loc}}(\lambda)$, let $a \in \mathbb{R}$, and define $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \int_{a}^{x} g(y) d\lambda(y).$$

Then

$$\int_{\mathbb{R}} f\phi' d\lambda = -\int_{\mathbb{R}} g\phi d\lambda$$

for all $\phi \in \mathscr{D}(\mathbb{R})$.

Proof. Using Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}} f(x)\phi'(x) d\lambda(x) &= -\int_{-\infty}^{a} \left(\int_{x}^{a} g(y) d\lambda(y) \right) \phi'(x) d\lambda(x) \\ &+ \int_{a}^{\infty} \left(\int_{a}^{x} g(y) d\lambda(y) \right) \phi'(x) d\lambda(x) \\ &= -\int_{-\infty}^{a} \left(\int_{-\infty}^{y} \phi'(x) d\lambda(x) \right) g(y) d\lambda(y) \\ &+ \int_{a}^{\infty} \left(\int_{y}^{\infty} \phi'(x) d\lambda(x) \right) g(y) d\lambda(y) \\ &= -\int_{-\infty}^{a} \phi(y) g(y) d\lambda(y) - \int_{a}^{\infty} \phi(y) g(y) d\lambda(y) \\ &= -\int_{\mathbb{R}} g(y)\phi(y) d\lambda(y). \end{aligned}$$

□

For real numbers a, b with $a < b$, we say that a function $f : [a, b] \to \mathbb{R}$ is **absolutely continuous** if for all $\epsilon > 0$ there is some $\delta > 0$ such that whenever $(a_1, b_1), \ldots, (a_n, b_n)$ are disjoint intervals each contained in $[a, b]$ with $\sum(b_k - a_k) < \delta$ it holds that $\sum |f(b_k) - f(a_k)| < \epsilon$. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is **locally absolutely continuous** if for each nonempty compact interval $[a, b]$, the restriction of f to $[a, b]$ is absolutely continuous. We denote the collection of locally absolutely continuous by $AC_{\text{loc}}(\mathbb{R})$.

Let $f \in H^1(\mathbb{R})$, let $a \in \mathbb{R}$, and define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \int_a^x Df d\lambda.$$

By Lemma 3 and by the definition of a distributional derivative,

$$\int_{\mathbb{R}} h\phi' d\lambda = -\int_{\mathbb{R}} (Df) \cdot \phi d\lambda = \int_{\mathbb{R}} f\phi' d\lambda, \qquad \phi \in \mathscr{D}(\mathbb{R}).$$

Hence $\int_{\mathbb{R}} (f - h)\phi' d\lambda = 0$ for all $\phi \in \mathscr{D}(\mathbb{R})$, which by Lemma 2 implies that there is some $c \in \mathbb{R}$ such that $f - h = c$ almost everywhere. Let $\widetilde{f} = c + h$. On the one hand, the fact that $Df \in L^1_{\text{loc}}(\lambda)$ implies that $h \in AC_{\text{loc}}(\mathbb{R})$ and so $\widetilde{f} \in AC_{\text{loc}}(\mathbb{R})$. On the other hand, $\widetilde{f} = f$ almost everywhere. Furthermore, because $\widetilde{f}$ is locally absolutely continuous, integration by parts yields

$$\int_{\mathbb{R}} \widetilde{f}\phi' d\lambda = -\int_{\mathbb{R}} \widetilde{f}'\phi d\lambda,$$

and by definition of a distributional derivative,

$$\int_{\mathbb{R}} \widetilde{f}\phi' d\lambda = -\int_{\mathbb{R}} (D\widetilde{f})\phi d\lambda.$$

Therefore by Theorem 1, $\widetilde{f}' = D\widetilde{f}$ almost everywhere. But the fact that $\widetilde{f} = f$ almost everywhere implies that $D\widetilde{f} = Df$ almost everywhere, so $\widetilde{f}' = Df$ almost everywhere. In particular, $\widetilde{f}' \in L^2(\lambda)$.

Theorem 4. *For $f \in H^1(\mathbb{R})$, there is a function $\widetilde{f} \in AC_{\text{loc}}(\mathbb{R})$ such that $\widetilde{f} = f$ almost everywhere and $\widetilde{f}' = Df$ almost everywhere. The function $\widetilde{f}$ is $\frac{1}{2}$-Hölder continuous.*

Proof. For $x, y \in \mathbb{R}$,[7]

$$\widetilde{f}(x) - \widetilde{f}(y) = \int_y^x \widetilde{f}' d\lambda,$$

and using the Cauchy-Schwarz inequality,

$$\begin{aligned} |\widetilde{f}(x) - \widetilde{f}(y)| &\le \int_y^x |\widetilde{f}'| d\lambda \\ &\le |x - y|^{1/2} \left(\int_y^x |\widetilde{f}'|^2 d\lambda \right)^{1/2} \\ &\le \|Df\|_{L^2} |x - y|^{1/2}. \end{aligned}$$

[7] cf. Giovanni Leoni, *A First Course in Sobolev Spaces*, p. 222, Theorem 7.13.

$\square$