Stationary phase, Laplace’s method, and the Fourier transform for Gaussian integrals

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1 Critical points

Let \( U \) be a nonempty open subset of \( \mathbb{R}^n \) and let \( \phi : U \to \mathbb{R} \) be smooth. Then \( \phi' : U \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}) = (\mathbb{R}^n)^* \). For each \( x \in U \), grad \( \phi(x) \) is the unique element of \( \mathbb{R}^n \) satisfying

\[
\langle \text{grad} \phi(x), y \rangle = \phi'(x)(y), \quad y \in \mathbb{R}^n,
\]

and grad \( \phi : U \to \mathbb{R}^n \) is itself smooth. Hess \( \phi : U \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \) is the derivative of grad \( \phi \). One checks that

\[
\phi''(x)(u)(v) = \langle \text{Hess} \phi(x)(u), v \rangle, \quad x \in U, \ u, v \in \mathbb{R}^n,
\]

and \( (\text{Hess} \phi(x))^* = \text{Hess} \phi(x) \).

We call \( p \in U \) a critical point of \( \phi \) when \( \text{grad} \phi(p) = 0 \), and we denote the set of critical points of \( \phi \) by \( C_\phi \). For \( p \in C_\phi \) and \( \lambda \in \mathbb{R} \) let \( v(p, \lambda) \) denote the dimension of the kernel of \( \text{Hess} \phi(p) - \lambda \), and we then define the Morse index of \( p \) to be

\[
m_\phi(p) = \sum_{\lambda < 0} v(p, \lambda).
\]

In other words, \( m_\phi(p) \) is the number of negative eigenvalues of \( \text{Hess} \phi(p) \) counted according to geometric multiplicity. We say that \( p \in C_\phi \) is nondegenerate when \( \text{Hess} \phi(p) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \) is invertible.

For \( A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \) self-adjoint and for \( \lambda \in \mathbb{R} \), let \( v(\lambda) \) be the dimension of the kernel of \( A - \lambda \). Let \( \nu_+ = \sum_{\lambda > 0} v(\lambda), \) let \( \nu_- = \sum_{\lambda < 0} v(\lambda), \) and let \( \nu_0 = v(0) \). Because \( A \) is self-adjoint, \( \nu_+ + \nu_- + \nu_0 = n \). We define the signature of \( A \) as \( \text{sgn}(A) = \nu_+ - \nu_- \). In other words, \( \text{sgn}(A) \) is the number of positive eigenvalues of \( A \) counted according to geometric multiplicity minus the number of negative eigenvalues of \( A \) counted according to geometric multiplicity.

\[\text{http://individual.utoronto.ca/jordanbell/notes/gradienthalbert.pdf}\]
We can connect the notions of Morse index and signature. For \( p \in C_\phi \), write \( A = \text{Hess} \phi(p) \). For \( p \) to be a nondegenerate critical point means that \( A \) is invertible and because \( \mathbb{R}^n \) is finite-dimensional this is equivalent to \( \nu_0 = 0 \).

Then \( \nu_0 = n - \nu_- \) which yields \( \text{sgn}(A) = n - 2\nu_- = n - 2m_\phi(p) \).

The Morse lemma\(^3\) states that if 0 is a nondegenerate critical point of \( \phi \) then there is an open subset \( V \) of \( U \) with \( 0 \in V \) and a \( C^\infty \)-diffeomorphism \( \Phi : V \to V, \Phi(0) = 0 \), such that

\[
\phi(x) = \phi(0) + \frac{1}{2} \langle \text{Hess} \phi(0)(\Phi(x)), \Phi(x) \rangle, \quad x \in V.
\]

## 2 Stationary phase

Let \( U \) be a nonempty connected open subset of \( \mathbb{R}^n \), and let \( a, \phi : U \to \mathbb{R} \) be smooth functions such that \( a \) has compact support. Suppose that each \( p \in C_\phi \cap \text{supp} \ a \) is nondegenerate\(^4\). The stationary phase approximation states that

\[
\int_U a(x)e^{it\phi(x)}dx = \sum_{p \in C_\phi \cap \text{supp} \ a} \left( \frac{2\pi}{t} \right)^{n/2} e^{\frac{\pi \text{sgn}(\text{Hess} \phi(p))}{4} / |\det \text{Hess} \phi(p)|^{1/2}} e^{it\phi(p)}a(p) + O(t^{-\frac{n}{2}-1})
\]

as \( t \to \infty \)^5

Let \( A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \) be self-adjoint and invertible and define

\[
\phi(x) = \frac{1}{2} \langle Ax, x \rangle, \quad x \in U.
\]

We calculate \( \text{grad} \phi(x) = Ax \), so \( C_\phi = \{0\} \). The Hessian of \( \phi \) is \( \text{Hess} \phi(x) = A \), and because \( A \) is invertible, 0 is indeed a nondegenerate critical point of \( \phi \). Thus we have the following.

**Theorem 1.** For a nonempty connected open subset of \( \mathbb{R}^n \) and for smooth functions \( a, \phi : U \to \mathbb{R} \) such that \( a \) has compact support and such that each \( p \in C_\phi \) is nondegenerate,

\[
\int_U a(x)e^{it\phi(x)}dx = \left( \frac{2\pi}{t} \right)^{n/2} \frac{e^{\frac{\pi \text{sgn}(A)}{4}}}{|\det A|^{1/2}} e^{\frac{1}{2}it\langle Ap,p \rangle} a(p) + O(t^{-\frac{n}{2}-1})
\]

as \( t \to \infty \).

\(^3\)Serge Lang, *Differential and Riemannian Manifolds*, p. 182, chapter VII, Theorem 5.1.

\(^4\)In particular, \( \phi \) is called a Morse function if it has no degenerate critical points, and in this case of course each \( p \in C_\phi \cap \text{supp} \ a \) is nondegenerate.

3 The Fourier transform

For \( A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \) self-adjoint, the spectral theorem tells us that are \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) and an orthonormal basis \( \{v_1, \ldots, v_n\} \) for \( \mathbb{R}^n \) such that \( Av_j = \lambda_j v_j \).

We call \( A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \) \textbf{positive} when it is self-adjoint and satisfies \( \langle Ax, x \rangle \geq 0 \) for all \( x \in \mathbb{R}^n \). In this case, the eigenvalues of \( A \) are nonnegative, thus the signature of \( A \) is \( \sigma(A) = n \). Suppose furthermore that \( A \) is invertible, and let \( P = (v_1, \ldots, v_n) \) and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \). Then

\[ P^T A P = \Lambda, \quad \Lambda^{1/2} = \text{diag}(\lambda_1^{1/2}, \ldots, \lambda_n^{1/2}), \quad A^{1/2} = P \Lambda^{1/2} P^T. \]

For \( \xi \in \mathbb{R}^n \) and \( t > 0 \), using the change of variables formula with the fact that \( |\det P| = 1 \) and then using Fubini’s theorem,

\[
\int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} t \langle Ax, x \rangle - i \langle P\xi, x \rangle \right) dx = \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} t \langle \Lambda^{1/2} P^T x, \Lambda^{1/2} P^T x \rangle - i \langle P\xi, x \rangle \right) dx
\]

\[
= \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} t \left\langle \Lambda^{1/2} P^T y, \Lambda^{1/2} P^T y \right\rangle - i \langle P\xi, Py \rangle \right) |\det P| dy
\]

\[
= \prod_{j=1}^n \int_{\mathbb{R}} \exp \left( -\frac{1}{2} t \lambda_j y_j^2 - i \xi_j y_j \right) dy_j.
\]

Using\[6\]

\[
\int_{\mathbb{R}} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}} \exp \left( \frac{b^2}{4a} + c \right), \quad \text{Re } a > 0, b, c \in \mathbb{C},
\]
gives

\[
\int_{\mathbb{R}} \exp \left( -\frac{1}{2} t \lambda_j y_j^2 - i \xi_j y_j \right) dy_j = \frac{1}{\lambda_j^{1/2}} \left( \frac{2\pi}{t} \right)^{1/2} \exp \left( -\frac{\xi_j^2}{2t\lambda_j} \right),
\]

and using \( \det A = \prod_{j=1}^n \lambda_j \) we have

\[
\int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} t \langle Ax, x \rangle - i \langle P\xi, x \rangle \right) dx
\]

\[
= \prod_{j=1}^n \frac{1}{\lambda_j^{1/2}} \left( \frac{2\pi}{t} \right)^{1/2} \exp \left( -\frac{\xi_j^2}{2t\lambda_j} \right)
\]

\[
= (\det A)^{-1/2} \left( \frac{2\pi}{t} \right)^{n/2} \exp \left( -\frac{1}{2t} \sum_{j=1}^n \frac{\xi_j^2}{\lambda_j} \right),
\]

\[6\text{http://individual.utoronto.ca/jordanbell/notes/bochnertheorem.pdf} \]
and because
\[ \Lambda^{-1} \xi = \sum_{j=1}^{n} \frac{\xi_j}{\lambda_j} e_j, \quad \langle \Lambda^{-1} \xi, \xi \rangle = \sum_{j=1}^{n} \frac{\xi_j^2}{\lambda_j} \]
this becomes
\[
\int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} t \langle Ax, x \rangle - i \langle P\xi, x \rangle \right) dx
= (\det A)^{-1/2} \left( \frac{2\pi}{t} \right)^{n/2} \exp \left( -\frac{1}{2t} \langle \Lambda^{-1} \xi, \xi \rangle \right)
= (\det A)^{-1/2} \left( \frac{2\pi}{t} \right)^{n/2} \exp \left( -\frac{1}{2t} \langle A^{-1} P\xi, P\xi \rangle \right),
\]
and so, as \( P \) is invertible we get the following.

**Theorem 2.** When \( A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \) is positive and invertible, for \( t > 0 \) and \( \xi \in \mathbb{R}^n \) we have
\[
\int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} t \langle Ax, x \rangle - i \langle P\xi, x \rangle \right) dx
= (\det A)^{-1/2} \left( 2\pi t^{-1} \right)^{n/2} \exp \left( -\frac{1}{2t} \langle A^{-1} P\xi, P\xi \rangle \right).
\]

4 Gaussian integrals

Let \( A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \) be positive and invertible and let \( b \in \mathbb{R}^n \). As above,
\[
\int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} \langle Ax, x \rangle + \langle Pb, x \rangle \right) dx = \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} \| A^{1/2} \| ^2 + \langle b, y \rangle \right) dy
= \prod_{j=1}^{n} \int_{\mathbb{R}} \exp \left( -\frac{1}{2} \lambda_j y_j^2 + b_j y_j \right) dy_j
= \prod_{j=1}^{n} \left( \frac{2\pi}{\lambda_j^{1/2}} \right) \exp \left( \frac{b_j^2}{2\lambda_j} \right)
= (\det A)^{-1/2}(2\pi)^{n/2} \exp \left( \frac{1}{2} \sum_{j=1}^{n} \frac{b_j^2}{\lambda_j} \right)
= (\det A)^{-1/2}(2\pi)^{n/2} \exp \left( \frac{1}{2} \langle A^{-1} Pb, Pb \rangle \right),
\]
which gives the following\[7\]

\[7\text{cf. Gaussian measures on } \mathbb{R}^n: \text{http://individual.utoronto.ca/jordanbell/notes/gaussian.pdf}\]
Theorem 3. If $A \in \mathcal{L} (\mathbb{R}^n; \mathbb{R}^n)$ is positive and invertible, then for $b \in \mathbb{R}^n$, \[
int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} \langle Ax, x \rangle + \langle b, x \rangle \right) \, dx = (\det A)^{-1/2} (2\pi)^{n/2} \exp \left( \frac{1}{2} \left( A^{-1}b, b \right) \right). \]

5 Laplace’s method

Let $D$ be the open ball in $\mathbb{R}^n$ with center 0 and radius 1 and let $S: D \to \mathbb{R}$ be smooth, attain its minimum value only at 0, and satisfy $\det \text{Hess} S(x) > 0$ for all $x \in D$. Let $g: D \to \mathbb{R}$ be smooth and for $t > 0$ let \[
J(t) = \int_D e^{-tS(x)} g(x) \, dx. \]
Laplace’s method tells us \[
J(t) = (2\pi t^{-1})^{n/2} (\det \text{Hess} S(0))^{-1/2} e^{-tS(0)} g(0) (1 + O(t^{-1})) \]
as $t \to \infty$.

Let $A \in \mathcal{L} (\mathbb{R}^n; \mathbb{R}^n)$ be positive and invertible. Define $S: D \to \mathbb{R}$ by \[
S(x) = \frac{1}{2} \langle Ax, x \rangle. \]
Then as above $P^TAP = \Lambda$, with which $S(x) = \frac{1}{2} \langle PAP^T x, x \rangle = \frac{1}{2} \| \Lambda^{1/2} P^T x \|^2$. We get the following from according Laplace’s method.

Theorem 4. Let $A \in \mathcal{L} (\mathbb{R}^n; \mathbb{R}^n)$ be positive and invertible and let $g: D \to \mathbb{R}$ be smooth. Then \[
J(t) = (2\pi t^{-1})^{n/2} (\det A)^{-1/2} g(0) (1 + O(t^{-1})), \]
as $t \to \infty$.

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