Stationary phase, Laplace's method, and the Fourier transform for Gaussian integrals

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1 Critical points

Let U be a nonempty open subset of $\mathbb{R}^n$ and let $\phi : U \to \mathbb{R}$ be smooth. Then $\phi' : U \to \mathscr{L}(\mathbb{R}^n; \mathbb{R}) = (\mathbb{R}^n)^*$. For each $x \in U$, $\operatorname{grad} \phi(x)$ is the unique element of $\mathbb{R}^n$ satisfying[1]

$$\langle \operatorname{grad} \phi(x), y \rangle = \phi'(x)(y), \qquad y \in \mathbb{R}^n,$$

and $\operatorname{grad} \phi : U \to \mathbb{R}^n$ is itself smooth. $\operatorname{Hess} \phi : U \to \mathscr{L}(\mathbb{R}^n; \mathbb{R}^n)$ is the derivative of $\operatorname{grad} \phi$. One checks that

$$\phi''(x)(u)(v) = \langle \operatorname{Hess} \phi(x)(u), v \rangle, \qquad x \in U, \quad u, v \in \mathbb{R}^n,$$

and $(\operatorname{Hess} \phi(x))^* = \operatorname{Hess} \phi(x)$.

We call $p \in U$ a **critical point of** ϕ when $\operatorname{grad} \phi(p) = 0$, and we denote the set of critical points of ϕ by C_ϕ. For $p \in C_\phi$ and $\lambda \in \mathbb{R}$ let $v(p, \lambda)$ denote the dimension of the kernel of $\operatorname{Hess} \phi(p) - \lambda$, and we then define the **Morse index of** p to be

$$m_\phi(p) = \sum_{\lambda < 0} v(p, \lambda).$$

In other words, $m_\phi(p)$ is the number of negative eigenvalues of $\operatorname{Hess} \phi(p)$ counted according to geometric multiplicity. We say that $p \in C_\phi$ is nondegenerate when $\operatorname{Hess} \phi(p) \in \mathscr{L}(\mathbb{R}^n; \mathbb{R}^n)$ is invertible.

For $A \in \mathscr{L}(\mathbb{R}^n; \mathbb{R}^n)$ self-adjoint and for $\lambda \in \mathbb{R}$, let $v(\lambda)$ be the dimension of the kernel of $A - \lambda$. Let $\nu_+ = \sum_{\lambda > 0} v(\lambda)$, let $\nu_- = \sum_{\lambda < 0} v(\lambda)$, and let $\nu_0 = v(0)$. Because A is self-adjoint, $\nu_+ + \nu_- + \nu_0 = n$. We define the **signature of** A as $\operatorname{sgn}(A) = \nu_+ - \nu_-$. In other words, $\operatorname{sgn}(A)$ is the number of positive eigenvalues of A counted according to geometric multiplicity minus the number of negative eigenvalues of A counted according to geometric multiplicity.[2]

[1] http://individual.utoronto.ca/jordanbell/notes/gradienthilbert.pdf

[2] cf. Sylvester's law of inertia, http://individual.utoronto.ca/jordanbell/notes/principalaxis.pdf

We can connect the notions of Morse index and signature. For $p \in C_\phi$, write $A = \operatorname{Hess}\phi(p)$. For p to be a nondegenerate critical point means that A is invertible and because $\mathbb{R}^n$ is finite-dimensional this is equivalent to $\nu_0 = 0$. Then $\nu_+ = n - \nu_-$ which yields $\operatorname{sgn}(A) = n - 2\nu_- = n - 2m_\phi(p)$.

The **Morse lemma**[3] states that if 0 is a nondegenerate critical point of ϕ then there is an open subset V of U with $0 \in V$ and a C^∞-diffeomorphism $\Phi : V \to V$, $\Phi(0) = 0$, such that

$$\phi(x) = \phi(0) + \frac{1}{2} \langle \operatorname{Hess}\phi(0)(\Phi(x)), \Phi(x) \rangle, \qquad x \in V.$$

2 Stationary phase

Let U be a nonempty connected open subset of $\mathbb{R}^n$, and let $a, \phi : U \to \mathbb{R}$ be smooth functions such that a has compact support. Suppose that each $p \in C_\phi \cap \operatorname{supp} a$ is nondegenerate.[4] The **stationary phase approximation** states that

$$\begin{aligned}\int_U a(x) e^{it\phi(x)} dx = &\sum_{p \in C_\phi \cap \operatorname{supp} a} \left(\frac{2\pi}{t}\right)^{n/2} \frac{e^{\frac{i\pi \operatorname{sgn}(\operatorname{Hess}\phi(p))}{4}}}{|\det \operatorname{Hess}\phi(p)|^{1/2}} e^{it\phi(p)} a(p) \\ &+ O(t^{-\frac{n}{2}-1})\end{aligned}$$

as $t \to \infty$.[5]

Let $A \in \mathscr{L}(\mathbb{R}^n; \mathbb{R}^n)$ be self-adjoint and invertible and define

$$\phi(x) = \frac{1}{2} \langle Ax, x \rangle, \qquad x \in U.$$

We calculate $\operatorname{grad}\phi(x) = Ax$, so $C_\phi = \{0\}$. The Hessian of ϕ is $\operatorname{Hess}\phi(x) = A$, and because A is invertible, 0 is indeed a nondegenerate critical point of ϕ. Thus we have the following.

Theorem 1. *For a nonempty connected open subset of $\mathbb{R}^n$ and for smooth functions $a, \phi : U \to \mathbb{R}$ such that a has compact support and such that each $p \in C_\phi$ is nondegenerate,*

$$\int_U a(x) e^{\frac{1}{2}\langle Ax, x\rangle} dx = \left(\frac{2\pi}{t}\right)^{n/2} \frac{e^{\frac{i\pi \operatorname{sgn}(A)}{4}}}{|\det A|^{1/2}} e^{\frac{1}{2}it\langle Ap, p\rangle} a(p) + O(t^{-\frac{n}{2}-1})$$

as $t \to \infty$.

[3] Serge Lang, *Differential and Riemannian Manifolds*, p. 182, chapter VII, Theorem 5.1.

[4] In particular, ϕ is called a Morse function if it has no degenerate critical points, and in this case of course each $p \in C_\phi \cap \operatorname{supp} a$ is nondegenerate.

[5] Liviu Nicolaescu, *An Invitation to Morse Theory*, second ed., p. 183, Proposition 3.88.

3 The Fourier transform

For $A \in \mathscr{L}(\mathbb{R}^n;\mathbb{R}^n)$ self-adjoint, the spectral theorem tells us that are $\lambda_1,\dots,\lambda_n \in \mathbb{R}$ and an orthonormal basis $\{v_1,\dots,v_n\}$ for $\mathbb{R}^n$ such that $Av_j = \lambda_j v_j$.

We call $A \in \mathscr{L}(\mathbb{R}^n;\mathbb{R}^n)$ **positive** when it is self-adjoint and satisfies $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$. In this case, the eigenvalues of A are nonnegative, thus the signature of A is $\sigma(A) = n$. Suppose furthermore that A is invertible, and let $P = (v_1,\dots,v_n)$ and $\Lambda = \operatorname{diag}(\lambda_1,\dots,\lambda_n)$. Then

$$P^T A P = \Lambda, \qquad \Lambda^{1/2} = \operatorname{diag}(\lambda_1^{1/2},\dots,\lambda_n^{1/2}), \qquad A^{1/2} = P\Lambda^{1/2}P^T.$$

For $\xi \in \mathbb{R}^n$ and $t > 0$, using the change of variables formula with the fact that $|\det P| = 1$ and then using Fubini's theorem,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}t\langle Ax, x\rangle - i\langle P\xi, x\rangle\right) dx\\
=&\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}t\left\langle \Lambda^{1/2}P^T x, \Lambda^{1/2}P^T x\right\rangle - i\langle P\xi, x\rangle\right) dx\\
=&\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}t\left\langle \Lambda^{1/2}P^T Py, \Lambda^{1/2}P^T Py\right\rangle - i\langle P\xi, Py\rangle\right) |\det P| dy\\
=&\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}t\left\|\Lambda^{1/2}y\right\|^2 - i\langle \xi, y\rangle\right) dy\\
=&\prod_{j=1}^n \int_{\mathbb{R}} \exp\left(-\frac{1}{2}t\lambda_j y_j^2 - i\xi_j y_j\right) dy_j.
\end{aligned}$$

Using[6]

$$\int_{\mathbb{R}} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}+c\right), \qquad \operatorname{Re} a > 0, b, c \in \mathbb{C},$$

gives

$$\int_{\mathbb{R}} \exp\left(-\frac{1}{2}t\lambda_j y_j^2 - i\xi_j y_j\right) dy_j = \frac{1}{\lambda_j^{1/2}}\left(\frac{2\pi}{t}\right)^{1/2}\exp\left(-\frac{\xi_j^2}{2t\lambda_j}\right),$$

and using $\det A = \prod_{j=1}^n \lambda_j$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}t\langle Ax, x\rangle - i\langle P\xi, x\rangle\right) dx\\
=&\prod_{j=1}^n \frac{1}{\lambda_j^{1/2}}\left(\frac{2\pi}{t}\right)^{1/2}\exp\left(-\frac{\xi_j^2}{2t\lambda_j}\right)\\
=&(\det A)^{-1/2}\left(\frac{2\pi}{t}\right)^{n/2}\exp\left(-\frac{1}{2t}\sum_{j=1}^n \frac{\xi_j^2}{\lambda_j}\right),
\end{aligned}$$

[6] http://individual.utoronto.ca/jordanbell/notes/bochnertheorem.pdf

and because

$$\Lambda^{-1}\xi = \sum_{j=1}^{n} \frac{\xi_j}{\lambda_j} e_j, \qquad \left\langle \Lambda^{-1}\xi, \xi \right\rangle = \sum_{j=1}^{n} \frac{\xi_j^2}{\lambda_j}$$

this becomes

$$\begin{aligned} &\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} t \langle Ax, x\rangle - i \langle P\xi, x\rangle\right) dx \\ =&(\det A)^{-1/2} \left(\frac{2\pi}{t}\right)^{n/2} \exp\left(-\frac{1}{2t} \left\langle \Lambda^{-1}\xi, \xi \right\rangle\right) \\ =&(\det A)^{-1/2} \left(\frac{2\pi}{t}\right)^{n/2} \exp\left(-\frac{1}{2t} \left\langle A^{-1}P\xi, P\xi \right\rangle\right), \end{aligned}$$

and so, as P is invertible we get the following.

Theorem 2. *When $A \in \mathscr{L}(\mathbb{R}^n; \mathbb{R}^n)$ is positive and invertible, for $t > 0$ and $\xi \in \mathbb{R}^n$ we have*

$$\begin{aligned} &\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} t \langle Ax, x\rangle - i \langle \xi, x\rangle\right) dx \\ =&(\det A)^{-1/2} \left(2\pi t^{-1}\right)^{n/2} \exp\left(-\frac{1}{2t} \left\langle A^{-1}\xi, \xi \right\rangle\right). \end{aligned}$$

4 Gaussian integrals

Let $A \in \mathscr{L}(\mathbb{R}^n; \mathbb{R}^n)$ be positive and invertible and let $b \in \mathbb{R}^n$. As above,

$$\begin{aligned} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} \langle Ax, x\rangle + \langle Pb, x\rangle\right) dx &= \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} \left\|\Lambda^{1/2}\right\|^2 + \langle b, y\rangle\right) dy \\ &= \prod_{j=1}^{n} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}\lambda_j y_j^2 + b_j y_j\right) dy_j \\ &= \prod_{j=1}^{n} \frac{(2\pi)^{1/2}}{\lambda_j^{1/2}} \exp\left(\frac{b_j^2}{2\lambda_j}\right) \\ &= (\det A)^{-1/2} (2\pi)^{n/2} \exp\left(\frac{1}{2} \sum_{j=1}^{n} \frac{b_j^2}{\lambda_j}\right) \\ &= (\det A)^{-1/2} (2\pi)^{n/2} \exp\left(\frac{1}{2} \left\langle A^{-1}Pb, Pb \right\rangle\right), \end{aligned}$$

which gives the following.[7]

[7] cf. Gaussian measures on $\mathbb{R}^n$: `http://individual.utoronto.ca/jordanbell/notes/gaussian.pdf`

Theorem 3. *If $A \in \mathscr{L}(\mathbb{R}^n; \mathbb{R}^n)$ is positive and invertible, then for $b \in \mathbb{R}^n$,*

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\langle Ax, x\rangle + \langle b, x\rangle\right) dx = (\det A)^{-1/2}(2\pi)^{n/2} \exp\left(\frac{1}{2}\langle A^{-1}b, b\rangle\right).$$

5 Laplace's method

Let D be the open ball in $\mathbb{R}^n$ with center 0 and radius 1 and let $S : D \to \mathbb{R}$ be smooth, attain its minimum value only at 0, and satisfy $\det \operatorname{Hess} S(x) > 0$ for all $x \in D$. Let $g : D \to \mathbb{R}$ be smooth and for $t > 0$ let

$$J(t) = \int_D e^{-tS(x)} g(x) dx.$$

Laplace's method[8] tells us

$$J(t) = (2\pi t^{-1})^{n/2} (\det \operatorname{Hess} S(0))^{-1/2} e^{-tS(0)} g(0)(1 + O(t^{-1}))$$

as $t \to \infty$.

Let $A \in \mathscr{L}(\mathbb{R}^n; \mathbb{R}^n)$ be positive and invertible. Define $S : D \to \mathbb{R}$ by

$$S(x) = \frac{1}{2}\langle Ax, x\rangle.$$

Then as above $P^T AP = \Lambda$, with which $S(x) = \frac{1}{2}\langle P\Lambda P^T x, x\rangle = \frac{1}{2}\left\|\Lambda^{1/2} P^T x\right\|^2$. We get the following from according Laplace's method.

Theorem 4. *Let $A \in \mathscr{L}(\mathbb{R}^n; \mathbb{R}^n)$ be positive and invertible and let $g : D \to \mathbb{R}$ be smooth. Then*

$$J(t) = (2\pi t^{-1})^{n/2} (\det A)^{-1/2} g(0)(1 + O(t^{-1})),$$

as $t \to \infty$.

[8] Peter D. Miller, *Applied Asymptotic Analysis*, p. 92, Exercise 3.16 and R. Wong, *Asymptotic Approximations of Integrals*, p. 495, Theorem 3.