The Schrödinger kernel, spherical surface measure, Fourier restriction, and the Strichartz inequality

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1 The Schrödinger equation

For $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$, suppose that $\psi : [0, \infty) \times \mathbb{R}^d \to \mathbb{C}$ satisfies

$$i \partial_t \psi + \Delta \psi = 0, \quad \psi(0, x) = \psi_0(x), \quad x \in \mathbb{R}^d. \quad (1)$$

We write

$$\hat{\psi}(\xi) = \hat{\psi}(0, \xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} \psi(x) dx.$$

Integrating by parts,

$$\hat{\partial_j \psi}(\xi) = 2\pi i \xi_j \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} \psi(x) dx = 2\pi i \xi_j \hat{\psi}(\xi)$$

and

$$\hat{\partial_j^2 \psi}(\xi) = -4\pi^2 \xi_j^2 \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} \psi(x) dx = -4\pi^2 \xi_j^2 \hat{\psi}(\xi)$$

and so

$$\hat{\Delta \psi}(\xi) = -4\pi^2 |\xi|^2 \hat{\psi}(\xi).$$

Then taking the Fourier transform of (1),

$$i \hat{\partial_t \psi} - 4\pi^2 |\xi|^2 \hat{\psi} = 0, \quad \hat{\psi}(0, \xi) = \hat{\psi_0}(\xi), \quad \xi \in \mathbb{R}^d.$$

For each $\xi \in \mathbb{R}^d$, the solution of the above initial value problem is

$$\hat{\psi}(t, \xi) = e^{-4\pi^2 it|\xi|^2} \hat{\psi_0}(\xi). \quad (2)$$

For $(t, \xi) \in [0, \infty) \times \mathbb{R}^d$, using the Fourier inversion theorem with (2),

$$\psi(t, x) = \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} \hat{\psi}(t, \xi) d\xi = \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} e^{-4\pi^2 it|\xi|^2} \hat{\psi_0}(\xi) d\xi.$$
Let 
\[ P = \{(-2\pi|\xi|^2, \xi) : \xi \in \mathbb{R}^d\} \]
For \( F \in C_b(\mathbb{R}^{d+1}) \), define
\[ I(F) = \int_{\mathbb{R}^d} F(-2\pi|\xi|^2, \xi)\hat{\psi}_0(\xi)d\xi, \]
which satisfies
\[ |I(F)| \leq \|F\|_{L^\infty} \|\hat{\psi}_0\|_{L^1}. \]
Therefore \( I \) is bounded linear functional on \( C_0(\mathbb{R}^{d+1}) \), so by the Riesz representation theorem there is a complex Borel measure \( \mu \) on \( \mathbb{R}^{d+1} \) such that
\[ \int_{\mathbb{R}^d} F(-2\pi|\xi|^2, \xi)\hat{\psi}_0(\xi)d\xi = \int_{\mathbb{R}^{d+1}} Fd\mu \]
for all \( F \in C_b(\mathbb{R}^{d+1}) \). For \((t, x) \in [0, \infty) \times \mathbb{R}^d \), there is a sequence \( F_n \in C_0(\mathbb{R}^{d+1}) \) that tends to \( F(\tau, \xi) = e^{2\pi i \xi \cdot x + 2\pi it\tau} \) in \( C_0(\mathbb{R}^{d+1}) \), and then
\[ \psi(t, x) = \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} e^{-4\pi^2 t|\xi|^2} \hat{\psi}_0(\xi)d\xi = \int_{\mathbb{R}^d} F(-2\pi|\xi|^2, \xi)\hat{\psi}_0(\xi)d\xi = \int_{\mathbb{R}^{d+1}} Fd\mu = \int_{\mathbb{R}^{d+1}} e^{2\pi i \xi \cdot x + 2\pi it\tau}d\mu(\tau, \xi). \]

**Theorem 1.** If \( \psi \) satisfies \([\ref{wolff:24:4.2}]\) then for \( t \geq 0 \) and \( x \in \mathbb{R}^d \),
\[ \psi(t, x) = \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} e^{-4\pi^2 t|\xi|^2} \hat{\psi}_0(\xi)d\xi = \int_{\mathbb{R}^{d+1}} e^{2\pi i \xi \cdot x + 2\pi it\tau}d\mu(\tau, \xi). \]

For an invertible real symmetric \( d \times d \) matrix \( T \), with index \( \sigma = \nu_+ - \nu_- \), where \( \nu_+ \) is the number of positive eigenvalues of \( T \) counted with multiplicity and likewise for \( \nu_- \), let
\[ G_T(x) = e^{-\pi i \langle Tx, x \rangle}, \quad x \in \mathbb{R}^d. \]
Because \( G_T \) is bounded and continuous, it is a tempered distribution. The Fourier transform of the tempered distribution is the following tempered distribution\([\ref{strichartz:50:3}]\)

**Lemma 2.**
\[ \hat{G}_T = e^{-\pi i \xi^2} |\det T|^{-1/2} \hat{G}_{-T^{-1}}, \]
so
\[ G_T = e^{-\pi t^2} |\det T|^{-1/2} G_{-T^{-1}} \]
and
\[ \hat{G}_T = e^{-\pi t^2} |\det T|^{-1/2} G_{-T^{-1}}. \]

For \( T = 4\pi tI \) and \( t > 0 \), [2] reads
\[ \hat{\psi} = G_T \hat{\psi}_0, \]

\( T \) is an invertible real symmetric \( d \times d \) matrix, with \( \sigma(T) = d \) and \( \det T = (4\pi t)^d \), so
\[ \hat{G}_T = e^{-\pi t^2} (4\pi t)^{-d/2} G_{-T^{-1}}, \]

and applying the inverse Fourier transform we obtain
\[ \psi = \hat{G}_T \ast \psi_0 = e^{-\pi t^2} (4\pi t)^{-d/2} G_{-T^{-1}} \ast \psi_0. \]

That is,
\[
\begin{align*}
\psi(t, x) &= e^{-\pi t^2} (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-\pi \langle -T^{-1}(x-y), x-y \rangle} \psi_0(y) dy \\
&= e^{-\pi t^2} (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{i \pi x \cdot y - \pi t^2} \psi_0(y) dy.
\end{align*}
\]

**Theorem 3.** If \( \psi \) satisfies [1], then for \( t > 0 \) and \( x \in \mathbb{R}^d \),

\[ \psi(t, x) = e^{-\pi t^2} (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{i \pi x \cdot y - \pi t^2} \psi_0(y) dy. \]

Then
\[
|\psi(t, x)| \leq (4\pi t)^{-d/2} \int_{\mathbb{R}^d} |\psi_0(y)| dy.
\]

**Corollary 4.** If \( \psi \) satisfies [1], then for \( t > 0 \),

\[ \|\psi(t)\|_{L^2} \leq (4\pi t)^{-d/2} \|\psi_0\|_{L^2}. \]

For a Hilbert space \( H \), let \( \mathcal{U}(H) \) be the set of unitary operators \( H \rightarrow H \). If \( A \) is a densely defined self-adjoint operator in \( H \) and \( t \in \mathbb{R} \), we make sense of the expression \( e^{itA} \) using the spectral theorem, and \( (e^{itA})_{t \in \mathbb{R}} \) is a strongly continuous one-parameter unitary group. The infinitesimal generator of this one-parameter group is \( iA \). Let \( A = \Delta \) on \( L^2(\mathbb{R}^d) \). Then \( (e^{it\Delta})' = i\Delta e^{it\Delta} \), for \( \psi_0 \in \mathcal{D}(\Delta) \) and for \( \psi(t, x) = (e^{it\Delta} \psi_0)(x) \),

\[
(\partial_t \psi)(t, x) = (e^{it\Delta} \psi_0)'(x) = (i\Delta e^{it\Delta} \psi_0)(x) = (i\Delta \psi(t))(x) = i(\Delta \psi)(t, x).
\]

Thus, for \( \psi_0 \in \mathcal{S}(\mathbb{R}^d) \), the function \( \psi(t) = e^{it\Delta} \psi_0 \) satisfies the initial value problem [1].

\[ \text{http://individual.utoronto.ca/jordanbell/notes/trotter.pdf} \]
\[ \text{http://individual.utoronto.ca/jordanbell/notes/laplaceoperator.pdf} \]
2 Spherical surface measure

Let \( \sigma \) be surface measure on the unit sphere \( S^{d-1} \) in \( \mathbb{R}^d \). It satisfies \( \sigma(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \). Define \( A : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) with \( A(x, y) = x + y \). For a Borel set \( E \) in \( \mathbb{R}^d \), by Fubini’s theorem,

\[
(\sigma \ast \sigma)(E) = A_x(\sigma \times \sigma)(E) = \int_{\mathbb{R}^d \times \mathbb{R}^d} 1_E(x + y) d(\sigma \times \sigma)(x, y)
\]

\[
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} 1_E(x + y) d\sigma(x) \right) d\sigma(y)
\]

\[
= \int_{\mathbb{R}^d} \sigma(E - y) d\sigma(y).
\]

For \( T \in SO(d) \), because \( \sigma \) is \( SO(d) \)-invariant,

\[
T_x(\sigma \ast \sigma)(E) = \int_{\mathbb{R}^d} \sigma(T^{-1}(E) - y) d\sigma(y) = \int_{\mathbb{R}^d} \sigma(E - Ty) d\sigma(y) = \int_{\mathbb{R}^d} \sigma(E - y) d\sigma(y).
\]

This shows that \( \sigma \ast \sigma \) is \( SO(d) \)-invariant.

Assume that there is a function \( \kappa \) such that

\[
d(\sigma \ast \sigma)(x) = \kappa(|x|)dx.
\]

We then calculate

\[
\int_0^\infty F(r^2) \kappa(r) r^{d-1} dr = \frac{1}{\sigma(S^{d-1})} \int_{S^{d-1}} \int_0^\infty F(r^2) \kappa(r) r^{d-1} dr d\sigma
\]

\[
= \frac{1}{\sigma(S^{d-1})} \int_{\mathbb{R}^d} F(|x|^2) \kappa(|x|) dx
\]

\[
= \frac{1}{\sigma(S^{d-1})} \int_{\mathbb{R}^d} F(|x|^2) d(\sigma \ast \sigma)(x)
\]

\[
= \frac{1}{\sigma(S^{d-1})} \int_{S^{d-1}} \int_{S^{d-1}} F(|x + y|^2) d\sigma(x) d\sigma(y)
\]

\[
= \frac{1}{\sigma(S^{d-1})} \int_{S^{d-1}} \int_{S^{d-1}} F(|x + e_1|^2) d\sigma(x) d\sigma(y)
\]

\[
= \frac{1}{\sigma(S^{d-1})} \int_{S^{d-1}} \int_{S^{d-1}} F(2 + 2 \langle x, e_1 \rangle) d\sigma(x) d\sigma(y)
\]

\[
= \int_{S^{d-1}} F(2 + 2x_1) d\sigma(x).
\]

\[\text{http://individual.utoronto.ca/jordanbell/notes/harmonicpolynomials.pdf} \ §2.\]
Now, using spherical coordinates on \( S^{d-1} \)

\[
\int_{S^{d-1}} g(x) \, d\sigma(x)
\]

\[
= \int_{\phi_1 = 0}^{\pi} \cdots \int_{\phi_{d-2} = 0}^{\pi} \int_{\phi_{d-1} = 0}^{2\pi} g(x(\phi))(\sin \phi_1)^{d-2} \cdots (\sin \phi_{d-3})^2(\sin \phi_{d-2})d\phi_{d-1} \cdots d\phi_1,
\]

where

\[
x_1 = \cos \phi_1
\]

\[
x_2 = \sin \phi_1 \cos \phi_2
\]

\[
\vdots
\]

\[
x_{d-1} = \sin \phi_1 \sin \phi_2 \cdots \sin \phi_{d-2} \cos \phi_{d-1}
\]

\[
x_d = \sin \phi_1 \sin \phi_2 \cdots \sin \phi_{d-2} \sin \phi_{d-1}.
\]

Therefore

\[
\int_{S^{d-1}} F(2 + 2x_1) \, d\sigma(x)
\]

\[
= \int_{\phi_1 = 0}^{\pi} \cdots \int_{\phi_{d-2} = 0}^{\pi} \int_{\phi_{d-1} = 0}^{2\pi} F(2 + 2 \cos \phi_1)
\]

\[
\cdot (\sin \phi_1)^{d-2} \cdots (\sin \phi_{d-3})^2(\sin \phi_{d-2})d\phi_{d-1} \cdots d\phi_1
\]

\[
= c_d \int_{\phi_1 = 0}^{\pi} F(2 + 2 \cos \phi_1) \cdot (\sin \phi_1)^{d-2} \, d\phi_1,
\]

where

\[
c_d = 2\pi \cdot \prod_{k=1}^{d-3} \int_{\phi=0}^{\pi} (\sin \phi)^k \, d\phi
\]

\[
= 2\pi \cdot \prod_{k=1}^{d-3} \pi^{1/2} \Gamma \left( \frac{1}{2} + \frac{k}{2} \right) \Gamma \left( 1 + \frac{k}{2} \right)
\]

\[
= 2\pi \cdot \pi^{d-3} \cdot \frac{\Gamma(1)}{\Gamma(1 + \frac{d-3}{2})}
\]

\[
= 2\pi \frac{d-1}{\Gamma \left( \frac{d-1}{2} \right)}.
\]

Now we have, doing the change of variable \( t^2 = 2 + 2 \cos \phi \), for which \( (\sin \phi)^2 = \frac{4t^2 - 4}{4} \)

\[
\int_{0}^{\infty} F(r^2) \kappa(r)r^{d-1} \, dr = 2\pi \frac{d-1}{\Gamma \left( \frac{d-1}{2} \right)} \int_{0}^{\pi} F(2 + 2 \cos \phi) \cdot (\sin \phi)^{d-2} \, d\phi
\]

\[
= 2\pi \frac{d-1}{\Gamma \left( \frac{d-1}{2} \right)} \int_{0}^{2} F(t^2) \cdot t \left( \frac{4t^2 - 4}{4} \right)^{\frac{d-3}{2}} \, dt,
\]

\footnote{Loukas Grafakos, \textit{Classical Fourier Analysis}, second ed., p. 441, Appendix D.1.}
yielding the expression
\[ \kappa(r) = \frac{1}{\Gamma(d/2)} \frac{r^{d-1}}{R^{d-1}} \phi^{d-3} (r) \chi_{[0,2]} (r) (4r^2 - r^4)^{\frac{d-3}{2}}, \]
that is,
\[ \kappa(r) = 2^{d+4} \pi^{-\frac{d+4}{2}} \frac{1}{\Gamma(d/2)} \chi_{[0,2]} (r) r^{1-d+1+3} (4 - r^2)^{\frac{d-3}{2}}. \]
Therefore
\[ d(\sigma \ast \sigma)(x) = \kappa(|x|) dx \]
\[ = 2^{d+4} \pi^{-\frac{d+4}{2}} \frac{1}{\Gamma(d/2)} \chi_{[0,2]} (|x|) |x|^{-1} (4 - |x|^2)^{\frac{d-3}{2}} dx. \]

**Theorem 5.** \( \sigma \ast \sigma \) is absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}^d \):

\[ d(\sigma \ast \sigma)(x) = 2^{d+4} \pi^{-\frac{d+4}{2}} \frac{1}{\Gamma(d/2)} \chi_{[0,2]} (|x|) |x|^{-1} (4 - |x|^2)^{\frac{d-3}{2}} dx. \]

We compute
\[ \int_0^2 r^{-1} (4 - r^2)^{\frac{d-3}{2}} \cdot r^{d-1} dr = 2^{-3+d} \pi^{1/2} \frac{\Gamma(d/2)}{\Gamma(d/2)}, \]
and thus using the above density of \( \sigma \ast \sigma \) with respect to Lebesgue measure and polar coordinates,
\[ (\sigma \ast \sigma)(\mathbb{R}^d) \]
\[ = 2^{d+4} \pi^{-\frac{d+4}{2}} \frac{1}{\Gamma(d/2)} \cdot \sigma(S^{d-1}) \cdot 2^{-3+d} \pi^{1/2} \frac{\Gamma(d/2)}{\Gamma(d/2)} \]
\[ = 2 \pi^{d/2} \sigma(S^{d-1}) \frac{1}{\Gamma(d/2)} \]
\[ = \sigma(S^{d-1})^2. \]

## 3 Radial functions

A function \( f : \mathbb{R}^d \rightarrow \mathbb{C} \) is called **radial** if there is a function \( f_0 : [0, \infty) \rightarrow \mathbb{C} \) such that \( f(x) = f_0(|x|) \). The following is an expression for the Fourier transform of a radial function \[ \text{[http://individual.utoronto.ca/jordanbell/notes/sphericalmeasure.pdf](http://individual.utoronto.ca/jordanbell/notes/sphericalmeasure.pdf)} \] Elias M. Stein and Guido Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, p. 155, Theorem 3.3.
Lemma 6. For $f \in L^1$ and $\xi \in \mathbb{R}^d$, 

$$\hat{f}(\xi) = 2\pi|\xi|^{-\frac{d}{2}+1}\int_0^\infty r^{d/2} J_{\frac{d}{2}-1}(2\pi r|\xi|)f_0(r)dr.$$ 

We prove conditions under which the Fourier transform of a radial function in $L^p$ is continuous on $\mathbb{R}^d \setminus \{0\}$.

Theorem 7. If $f : \mathbb{R}^d \to \mathbb{C}$ is radial, with $f(x) = f_0(|x|)$, and $f \in L^p$ with $1 < p < \frac{2d}{d+1}$, then $\hat{f} \in C_0(\mathbb{R}^d \setminus \{0\})$ and

$$\|\hat{f}\|_\infty \leq C_{d,p}\|f\|_{L^p}.$$

Proof. It is a fact that

$$J_{\frac{d}{2}-1}(s) = (\pi s/2)^{-1/2} \cos \left(s - \frac{\pi d}{2} + \frac{\pi}{4}\right) + O(s^{-3/2}), \quad s \to \infty,$$

and

$$J_{\frac{d}{2}-1}(s) \sim \frac{1}{\Gamma(d/2)}(s/2)^{\frac{d}{2}-1}, \quad s \downarrow 0.$$

For $s > 0$ we get from the above asymptotic formulas

$$\left|\frac{J_{(d-2)/2}(s)}{s^{(d-2)/2}}\right| \leq C_d(1+s)^{-(d-1)/2}.$$

Using this, for $\xi \neq 0$ and $M > 0$,

$$\left|2\pi|\xi|^{-\frac{d}{2}+1}\int_0^M r^{d/2} J_{\frac{d}{2}-1}(2\pi r|\xi|)f_0(r)dr\right|$$

$$=2\pi\left|\int_0^M |r|^{-(d-2)/2}r^{-(d-2)/2}r^{d/2} J_{(d-2)/2}(2\pi r|\xi|)f_0(r)dr\right|$$

$$\leq 2\pi\int_0^M |r|^{-(d-2)/2}J_{(d-2)/2}(2\pi r|\xi|)||f_0(r)||r^{d-1}dr$$

$$\leq 2\pi \cdot C_d \cdot (2\pi)^{(d-2)/2} \cdot \int_0^M \left(1 + 2\pi r|\xi|\right)^{-(d-1)/2}|f_0(r)|r^{d-1}dr.$$ 

Using Hölder’s inequality, for $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_0^M \left(1 + 2\pi r|\xi|\right)^{-(d-1)/2}|f_0(r)|r^{(d-1)(1/p+1/q)}dr$$

$$\leq \left(\int_0^M |f_0(r)|^p r^{d-1}dr\right)^{1/p} \left(\int_0^M (1 + 2\pi r|\xi|)^{-q(d-1)/2}r^{d-1}dr\right)^{1/q}.$$
If $-q(d-1)/2 + d - 1 > -1$ then the second integral converges as $M \to \infty$, i.e. when $\frac{1}{q} > \frac{d-1}{2d}$, equivalently $\frac{1}{p} < \frac{d+1}{2d}$. For $\xi_k \to \xi \neq 0$, by the dominated convergence theorem,

$$2\pi |\xi_k|^{-\frac{d}{2}+1} \int_0^\infty r^{d/2} J_{\frac{d}{2}-1}(2\pi r|\xi_k|) f_0(r)dr$$

$$-2\pi |\xi|^{-\frac{d}{2}+1} \int_0^\infty r^{d/2} J_{\frac{d}{2}-1}(2\pi r|\xi|) f_0(r)dr.$$

Namely,

$$\hat{f}(\xi) = 2\pi |\xi|^{-\frac{d}{2}+1} \int_0^\infty r^{d/2} J_{\frac{d}{2}-1}(2\pi r|\xi|) f_0(r)dr$$

$$= 2\pi \int_0^{\infty} (r|\xi|)^{-(d-2)/2} J_{(d-2)/2}(2\pi r|\xi|) f_0(r) r^{d-1}dr$$

is continuous on $\mathbb{R}^d \setminus \{0\}$.

Let $\epsilon > 0$. First,

$$\left|2\pi \int_0^{\epsilon} (r|\xi|)^{-(d-2)/2} J_{(d-2)/2}(2\pi r|\xi|) f_0(r) r^{d-1}dr \right|$$

$$\leq E_d \int_0^{\epsilon} |f_0(r)| r^{d-1} dr$$

$$= E_d \int_0^{\epsilon} |f_0(r)| r^{(d-1)/p, (d-1)/q} dr$$

$$\leq E_d \left( \int_0^{\epsilon} |f_0(r)|^p r^{d-1} dr \right)^{1/p} \left( \int_0^{\epsilon} r^{d-1} dr \right)^{1/q}$$

$$= E_d \frac{\epsilon^{d/q}}{d^{1/q}} \left( \int_0^{\epsilon} |f_0(r)|^p r^{d-1} dr \right)^{1/p} .$$

Second,

$$\left|2\pi \int_\epsilon^{\infty} (r|\xi|)^{-(d-2)/2} J_{(d-2)/2}(2\pi r|\xi|) f_0(r) r^{d-1}dr \right|$$

$$\leq F_d \int_\epsilon^{\infty} (r|\xi|)^{-(d-2)/2} (2\pi r|\xi|)^{-1/2} |f_0(r)| r^{d-1} dr$$

$$= F_d |\xi|^{-(d-1)/2} \int_\epsilon^{\infty} r^{-(d-1)/2} |f_0(r)| r^{d-1} dr$$

$$= F_d |\xi|^{-(d-1)/2} \int_\epsilon^{\infty} r^{-(d-1)(1/p+1/q)} r^{-(d-1)/2} |f_0(r)| dr$$

$$\leq F_d |\xi|^{-(d-1)/2} \left( \int_\epsilon^{\infty} |f_0(r)|^p r^{d-1} dr \right)^{1/p} \left( \int_\epsilon^{\infty} r^{d-1} r^{-(d-1)/2} dr \right)^{1/q}$$

$$= F_d |\xi|^{-(d-1)/2} \left( \frac{\epsilon^{d-q(d-1)/2}}{q(d-1)/2 - d} \right)^{1/q} \left( \int_\epsilon^{\infty} |f_0(r)|^p r^{d-1} dr \right)^{1/p} .$$
4 The Tomas-Stein theorem

For $f \in L^p$, $p > 1$, the Fourier transform of $f$ is a tempered distribution. The Hausdorff-Young inequality states that for $1 \leq p \leq 2$,

$$\|\hat{f}\|_{L^q} \leq \|f\|_{L^p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. In fact, the Babenko-Beckner inequality\(^9\) states that for $1 < p \leq 2$ and with $A_p = \frac{\frac{2}{d} - \frac{1}{2}}{q^2/2}$,

$$\|\hat{f}\|_{L^q} \leq A_p \|f\|_{L^p}.$$

If $f$ and $g$ are equal as elements of $L^p$, $1 \leq p \leq 2$, the Hausdorff-Young inequality tells us that their Fourier transforms are equal as elements of $L^q$, and so are equal almost everywhere with respect to Lebesgue measure. But $\sigma$ is not absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$, so merely knowing that two functions are equal almost everywhere with respect to Lebesgue measure does not suffice for them to be equal almost everywhere with respect to $\sigma$. The **Tomas-Stein theorem** shows that this is the case when $1 \leq p \leq \frac{2d+2}{d+3}$\(^10\).

**Theorem 8** (Tomas-Stein theorem). For $1 \leq p \leq \frac{2d+2}{d+3}$, there is some $A_{d,p}$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\left(\int_{S^{d-1}} |\hat{f}|^2 d\sigma\right)^{1/2} \leq A_{d,p} \|f\|_{L^p(\mathbb{R}^d)}.$$

We prove a subset of this theorem, for $1 \leq p < \frac{4d}{3d+1}$\(^11\).

**Theorem 9** (Tomas theorem). For $1 \leq p < \frac{4d}{3d+1}$ and for all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\left(\int_{S^{d-1}} |\hat{f}|^2 d\sigma\right)^{1/2} \leq \|\hat{\sigma}\|_{L^p(2^{p-2})}^{1/2} \cdot \|f\|_{L^p(\mathbb{R}^d)}.$$

**Proof.** The Fourier transform of $\sigma$ is\(^12\)

$$\hat{\sigma}(\xi) = 2\pi |\xi|^{-\frac{d}{2}+1} J_{\frac{d}{2}-1}(2\pi|\xi|).$$

Thus,

$$\hat{\sigma}(\xi) = O(|\xi|^{-(d-1)/2}), \quad |\xi| \to \infty.$$
Because the Fourier transform of a complex Borel measure is continuous, using polar coordinates we check that
\[ p \sigma P L q r d q \text{ if and only if } q \mu P L q \]
and the Fourier transform of \( u \) is the tempered distribution defined by
\[ \hat{u}(\phi) = u(\hat{\phi}). \]

It is a fact that if \( u \) is a tempered distribution then
\[ \hat{\phi} \ast \hat{u} = \hat{\phi u} \]
and
\[ (u \ast \phi) \ast \psi = u \ast (\phi \ast \psi). \]

Writing \( N f = \tilde{f} \), for which we check \( N(\tilde{f} \ast f)(x) = (\tilde{f} \ast f)(x) \), we get
\[
\int_{\mathbb{R}^d} |\tilde{f}|^2 d\sigma = (\tilde{f} \ast \tilde{\sigma})(f) = (\tilde{f} \ast (f \ast \tilde{\sigma}))(0) = ((\tilde{f} \ast f) \ast \tilde{\sigma})(0) = \tilde{\sigma}(N(\tilde{f} \ast f)) = \tilde{\sigma}(\tilde{f} \ast f) = \int_{\mathbb{R}^d} \tilde{f} \ast f \cdot \tilde{\sigma} dx.
\]

Therefore by Hölder’s inequality with \( \frac{1}{r} + \frac{1}{r'} = 1 \) and then Young’s inequality with \( \frac{1}{r} + \frac{1}{r'} = \frac{1}{q} + 1 \), we get
\[
\int_{\mathbb{R}^d} |\tilde{f}|^2 d\sigma \leq \|\tilde{\sigma}\|_{L_q} \cdot \|\tilde{f} \ast f\|_{L_{q'}} \leq \|\tilde{\sigma}\|_{L_q} \cdot \|\tilde{f}\|_{L_r} \cdot \|f\|_{L_r'} = \|\tilde{\sigma}\|_{L_q} \cdot \|f\|_{L_r}^2.
\]

For \( r = p \) we have \( \frac{2}{p} = 2 - \frac{1}{q} \), i.e. \( q = \frac{2p}{2p-2} \). Now from above, \( \hat{\mu} \in L^q \) if and only if \( q > \frac{2d}{d-1} \), i.e. \( \frac{1}{q} < \frac{d-1}{2d} \), which is equivalent to \( \frac{2}{p} > \frac{3d+1}{2d} \), which is the case by hypothesis.

5 The Hardy-Littlewood-Sobolev inequality

For $0 < \alpha < d$ and $f \in \mathcal{S}(\mathbb{R}^d)$, we define the Riesz potential $I_\alpha$ by

$$(I_\alpha f)(x) = \frac{1}{c_\alpha} \int_{\mathbb{R}^d} |x-y|^{-d+\alpha} f(y) dy,$$

where

$$c_\alpha = \pi^{d/2} 2^\alpha \frac{\Gamma(\alpha/2)}{\Gamma((d-\alpha)/2)}.$$ 

The Fourier transform of the tempered distribution $K_\alpha(x) = \frac{1}{c_\alpha} |x|^{-d+\alpha}$ is the tempered distribution

$$\hat{K}_\alpha(\xi) = |2\pi\xi|^{-\alpha}.$$

As $I_\alpha f = K_\alpha * f$, we have

$$\widehat{I_\alpha f}(\xi) = \hat{K}_\alpha(\xi) \cdot \hat{f}(\xi) = |2\pi\xi|^{-\alpha} \cdot \hat{f}(\xi).$$

If $0 < \alpha, \beta, \alpha + \beta < d$ then from the above formula we get

$$I_\alpha I_\beta = I_{\alpha+\beta}.$$

Using $\Delta(I_\alpha f) = \Delta(K_\alpha * f) = K_\alpha * (\Delta f)$\footnote{Walter Rudin, *Functional Analysis*, second ed., p. 195, Theorem 7.19.} we calculate for $2 < \alpha < d$, \footnote{cf. \url{http://individual.utoronto.ca/jordanbell/notes/dirac.pdf}}

$$\Delta(\widehat{I_\alpha f})(\xi) = \widehat{I_\alpha(\Delta f)}(\xi) = |2\pi\xi|^{-\alpha} \cdot \widehat{\Delta f}(\xi) = |2\pi\xi|^{-\alpha} \cdot (-4\pi^2|\xi|^2) \cdot \hat{f}(\xi) = -|2\pi\xi|^{-\alpha+2} \cdot \hat{f}(\xi) = -\widehat{I_{\alpha-2} f}(\xi).$$

This implies that $\Delta I_\alpha f = -I_{\alpha-2} f$.

**Theorem 10.** For $0 < \alpha < d$ and for $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\widehat{I_\alpha f}(\xi) = |2\pi\xi|^{-\alpha} \cdot \hat{f}(\xi).$$

$I_\alpha$ satisfies, for $0 < \alpha, \beta, \alpha + \beta < d$,

$$I_\alpha I_\beta = I_{\alpha+\beta}$$

and for $2 < \alpha < d$,

$$\Delta I_\alpha f = -I_{\alpha-2} f.$$

\footnote{Loukas Grafakos, *Classical Fourier Analysis*, second ed., p. 128, Theorem 2.4.6.}
The following is the **Hardy-Littlewood-Sobolev inequality**.

**Theorem 11** (Hardy-Littlewood-Sobolev inequality). Suppose that \(0 < \alpha < d\) and \(1 \leq p < q < \infty\) with \(\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}\). There is \(C(d, \alpha, p) < \infty\) such that

\[
\|I_{\alpha} f\|_{L^q} \leq C(d, \alpha, p) \|f\|_{L^p}
\]

when \(p > 1\), and

\[
m(x) = \int \mathbb{1}_{|I_{\alpha} f(x)| > \lambda} \|f\|_{L^1} \lambda, \quad \lambda > 0,
\]

when \(p = 1\).

### 6 The Strichartz inequality

We now prove a case of the **Strichartz inequality**, when \(p = q\).

**Theorem 12** (Strichartz inequality). Suppose that \(\{U(t)\}_{t \in \mathbb{R}}\) is a family of linear operators on \(L^1(\mathbb{R}^d) + L^2(\mathbb{R}^d)\) and that there are \(C, \gamma > 0\) such that

\[
\|U(t)U(s)^*\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C
\]

and

\[
\|U(t)U(s)^*\|_{L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)} \leq C|t - s|^{-\gamma}.
\]

When

\[
\frac{2}{p} + \frac{2\sigma}{q} = \sigma, \quad 2 \leq p \leq \infty, \quad 1 \leq q \leq \infty, \quad (p, q) \neq (2, \infty),
\]

it holds that

\[
\left( \int_{\mathbb{R}} \|U(t) f\|_{L^p(\mathbb{R}^d)}^p \, dt \right)^{1/p} \leq C \|f\|_{L^2(\mathbb{R}^d)}.
\]

**Proof.** For \(p = q\), we wish to prove that

\[
\|(U(t)f)(t, x)\|_{L^{p_r}(\mathbb{R} \times \mathbb{R}^d)} \leq C \|f\|_{L^2(\mathbb{R}^d)}
\]

with \(p = \frac{2(1 + \sigma)}{\sigma}\).

Let \(s, t \in \mathbb{R}\) and define \(T = U(t)U(s)^*\). For \(p_0 = 2, q_0 = 2, p_1 = 1, q_1 = \infty\), for \(0 < r < 1\) let

\[
\frac{1}{p_r} = \frac{1 - r}{p_0} + \frac{r}{p_1}, \quad \frac{1}{q_r} = \frac{1 - r}{q_0} + \frac{r}{q_1}.
\]

\[\text{Loukas Grafakos, Modern Fourier Analysis, second ed., p. 3, Theorem 6.1.3; Elias M. Stein, Singular Integrals and Differentiability Properties of Functions, p. 119, Theorem 1; Elliott H. Lieb and Michael Loss, Analysis, second ed., p. 106, Theorem 4.3.}
\[\text{Maciej Zworski, Semiclassical Analysis, p. 230, Theorem 10.7.}\]
that is, $p_r = \frac{2}{1+\sigma}$ and $q_r = \frac{2}{1-\sigma}$. To have $q_r = p$ is equivalent to

$$r = 1 - \frac{2}{p} = \frac{1}{1+\sigma}.$$ 

Because $\|T\|_{L^p \rightarrow L^q} \leq C$ and $\|T\|_{L^1 \rightarrow L^1} \leq C|t-s|^{-\sigma}$, by the Riesz-Thorin interpolation theorem,

$$\|T\|_{L^p \rightarrow L^q} \leq C^{1-r} C^r |t-s|^{-\sigma r} = C|t-s|^{-\sigma \left(\frac{1}{2} - \frac{1}{2}r\right)}.$$ 

For functions $F(t, x)$ and $G(t, x)$, using Hölder’s inequality with $\frac{1}{p} + \frac{1}{p'} = 1$,

$$|\langle U(t)^* G(t, \cdot), U(s)^* F(s, \cdot) \rangle| \leq \|G(t, \cdot)\|_{L^{p'}} \cdot \|U(t)U(s)^* F(s, \cdot)\|_{L^p} \leq \|G(t, \cdot)\|_{L^{p'}} \cdot |t-s|^{-\sigma \left(1-2\frac{1}{p}\right)} \cdot \|F(s, \cdot)\|_{L^p} = \|F(s, \cdot)\|_{L^{p'}} \cdot |G(t, \cdot)\|_{L^p} \cdot |t-s|^{-\sigma \left(1-2\frac{1}{p}\right)}.$$ 

For $-1 + \alpha = -\sigma \left(-1 + 2\frac{1}{p}\right)$, applying Hölder’s inequality,

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \langle U(t)^* G(t, \cdot), U(s)^* F(s, \cdot) \rangle \, dt \, ds \right| \leq C \int_{\mathbb{R}} \int_{\mathbb{R}} |t-s|^{-\sigma \left(-1+\frac{2}{p}\right)} \cdot \|F(s, \cdot)\|_{L^{p'}} \cdot \|G(t, \cdot)\|_{L^{p'}} \, dt \, ds \leq C \cdot c_\alpha \int_{\mathbb{R}} \left( \frac{1}{c_\alpha} \int_{\mathbb{R}} |t-s|^{-1+\alpha} \|G(t, \cdot)\|_{L^{p'}} \, dt \right) \|F(s, \cdot)\|_{L^{p'}} \, ds \leq C \cdot c_\alpha \left( \int_{\mathbb{R}} I_\alpha(\|G(t, \cdot)\|_{L^{p'}})(s) \, ds \right)^{1/p} \left( \int_{\mathbb{R}} \|F(s, \cdot)\|_{L^{p'}}^{p'} \, ds \right)^{1/p'},$$

By the Hardy-Littlewood-Sobolev inequality, when $0 < \alpha < 1$ and $\frac{1}{p} = \frac{1}{\beta} - \alpha$, for which $\beta = p'$,

$$\left( \int_{\mathbb{R}} I_\alpha(\|G(t, \cdot)\|_{L^{p'}})(s) \, ds \right)^{1/p} \leq C(\alpha, p) \left( \int_{\mathbb{R}} \|G(s, \cdot)\|_{L^{p'}}^{p'} \, ds \right)^{1/p'}.$$ 

Therefore we have

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \langle U(t)^* G(t, \cdot), U(s)^* F(s, \cdot) \rangle \, dt \, ds \right| \leq C \cdot c_\alpha \|F\|_{L^{p'}(\mathbb{R} \times \mathbb{R}^d)} \cdot \|G\|_{L^{p'}(\mathbb{R} \times \mathbb{R}^d)}.$$ 

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