Test functions, distributions, and Sobolev’s lemma

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1 Introduction

If $X$ is a topological vector space, we denote by $X^*$ the set of continuous linear functionals on $X$. With the weak-* topology, $X^*$ is a locally convex space, whether or not $X$ is a locally convex space. (But in this note, we only talk about locally convex spaces.)

The purpose of this note is to collect the material given in Walter Rudin, *Functional Analysis*, second ed., chapters 6 and 7, involved in stating and proving Sobolev’s lemma.

2 Test functions

Suppose that $\Omega$ is an open subset of $\mathbb{R}^n$. We denote by $\mathcal{D}(\Omega)$ the set of all $\phi \in C^\infty(\Omega)$ such that $\text{supp} \phi$ is a compact subset of $\Omega$. Elements of $\mathcal{D}(\Omega)$ are called *test functions*. For $N = 0, 1, \ldots$ and $\phi \in \mathcal{D}(\Omega)$, write

$$\|\phi\|_N = \sup\{|(D^\alpha \phi)(x)| : x \in \Omega, |\alpha| \leq N\},$$

where

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$  

For each compact subset $K$ of $\Omega$, we define

$$\mathcal{D}_K = \{\phi \in \mathcal{D}(\Omega) : \text{supp} \phi \subseteq K\},$$

and define $\tau_K$ to be the locally convex topology on $\mathcal{D}_K$ determined by the family of seminorms $\{\|\cdot\|_N : N \geq 0\}$. One proves that $\mathcal{D}_K$ with the topology $\tau_K$ is a Fréchet space. As sets,

$$\mathcal{D}(\Omega) = \bigcup_K \mathcal{D}_K.$$
Define $\beta$ to be the collection of all convex balanced subsets $W$ of $D(\Omega)$ such that for every compact subset $K$ of $\Omega$ we have $W \cap D_K \in \tau_K$; to say that $W$ is balanced means that if $c$ is a complex number with $|c| \leq 1$ then $cW \subseteq W$. One proves that $\{ \phi + W : \phi \in D(\Omega), W \in \beta \}$ is a basis for a topology $\tau$ on $D(\Omega)$, that $\beta$ is a local basis at 0 for this topology, and that with the topology $\tau$, $D(\Omega)$ is a locally convex space.\footnote{Walter Rudin, Functional Analysis, second ed., p. 152, Theorem 6.4; cf. Helmut H. Schaefer, Topological Vector Spaces, p. 57.}

For each compact subset $K$ of $\Omega$, one proves that the topology $\tau_K$ is equal to the subspace topology on $D_K$ inherited from $D(\Omega)$.

We write $D'(\Omega) = (D(\Omega))^*$, and elements of $D'(\Omega)$ are called distributions. With the weak-* topology, $D'(\Omega)$ is a locally convex space.\footnote{Walter Rudin, Functional Analysis, second ed., p. 153, Theorem 6.5.}

It is a fact that a linear functional $\Lambda$ on $D(\Omega)$ is continuous if and only if for every compact subset $K$ of $\Omega$ there is a nonnegative integer $N$ and a constant $C$ such that $|\Lambda \phi| \leq C \|\phi\|_N$ for all $\phi \in D_K$.\footnote{Walter Rudin, Functional Analysis, second ed., p. 156, Theorem 6.8.}

For $\Lambda \in D'(\Omega)$ and $\alpha$ a multi-index, we define

$$D^\alpha (f \Lambda)(\phi) = (-1)^{|\alpha|} \Lambda(D^\alpha f), \quad \phi \in D(\Omega).$$

Let $K$ be a compact subset of $\Omega$. As $\Lambda$ is continuous, there is a nonnegative integer $N$ and a constant $C$ such that $|\Lambda \phi| \leq C \|\phi\|_N$ for all $\phi \in D_K$. Then

$$|D^\alpha \Lambda(\phi)| = |\Lambda(D^\alpha f)| \leq C \|D^\alpha \phi\|_N \leq C \|\phi\|_{N+|\alpha|},$$

which shows that $D^\alpha \Lambda \in D'(\Omega)$.

The Leibniz formula is the statement that for all $f, g \in C^\infty(\mathbb{R}^n)$,

$$D^\alpha (fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta} f)(D^\beta g),$$

where $\binom{\alpha}{\beta}$ are multinomial coefficients.

For $\Lambda \in D'(\Omega)$ and $f \in C^\infty(\Omega)$, we define

$$(f \Lambda)(\phi) = \Lambda(f \phi), \quad \phi \in D(\Omega);$$

this makes sense because $f \phi \in D(\Omega)$ when $\phi \in D(\Omega)$. It is apparent that $f \Lambda$ is linear, and in the following lemma we prove that $f \Lambda$ is continuous.\footnote{Walter Rudin, Functional Analysis, second ed., p. 159, §6.15.}

**Lemma 1.** If $\Lambda \in D'(\Omega)$ and $f \in C^\infty(\Omega)$, then $f \Lambda \in D'(\Omega)$.

**Proof.** Suppose that $K$ is a compact subset of $\Omega$. Because $\Lambda$ is continuous, there is some nonnegative integer $N$ and some constant $C$ such that

$$|\Lambda \phi| \leq C \|\phi\|_N, \quad \phi \in D_K.$$ 

For $|\alpha| \leq N$, by the Leibniz formula, for all $\phi \in D_K$,

$$D^\alpha (f \phi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta} f)(D^\beta \phi).$$
Because \( f \in C^\infty(\Omega) \), there is some \( C_\alpha \) such that \( |(D^{\alpha-\beta}f)(x)| \leq C_\alpha \) for \( \beta \leq \alpha \) and for \( x \in K \). Using \( \phi(x) = 0 \) for \( x \notin K \), the above statement of the Leibniz formula, and the inequality just obtained, it follows that there is some \( C'_\alpha \) such that \( |(D^{\alpha}(f\phi))(x)| \leq C'_\alpha \|\phi\|_N \) for all \( x \in \Omega \). This gives
\[
\|f\phi\|_N = \sup_{|\alpha| \leq N} \sup_{x \in \Omega} |(D^{\alpha}(f\phi))(x)| \leq \sup_{|\alpha| \leq N} C'_\alpha \|\phi\|_N = C' \|\phi\|_N ;
\]
the last equality is how we define \( C' \), which is a maximum of finitely many \( C'_\alpha \) and so finite. Then,
\[
|(f\Lambda)(\phi)| = |\Lambda(f\phi)| \leq C \|f\phi\|_N \leq C'C' \|\phi\|_N , \quad \phi \in \mathcal{D}_K.
\]
This bound shows that \( f\Lambda \) is continuous. \( \square \)

The above lemma shows that \( f\Lambda \in \mathcal{D}'(\Omega) \) when \( f \in C^\infty(\Omega) \) and \( \Lambda \in \mathcal{D}'(\Omega) \). Therefore \( D^{\alpha}(f\Lambda) \in \mathcal{D}(\Omega) \), and the following lemma, proved in Rudin, states that the Leibniz formula can be used with \( f\Lambda \).

**Lemma 2.** If \( f \in C^\infty(\Omega) \) and \( \Lambda \in \mathcal{D}'(\Omega) \), then
\[
D^{\alpha}(f\Lambda) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta}f)(D^{\beta}\Lambda).
\]

If \( f : \Omega \to \mathbb{C} \) is locally integrable, define
\[
\Lambda \phi = \int_{\Omega} \phi(x)f(x)dx , \quad \phi \in \mathcal{D}(\Omega).
\]
For \( \phi \in \mathcal{D}_K \),
\[
|\Lambda \phi| \leq \|\phi\|_0 \int_K |f|dx ,
\]
from which it follows that \( \Lambda \) is continuous. If \( \mu \) is a complex Borel measure on \( \mathbb{R}^n \) or a positive Borel measure on \( \mathbb{R}^n \) that assigns finite measure to compact sets, define
\[
\Lambda \phi = \int_{\Omega} \phi d\mu , \quad \phi \in \mathcal{D}(\Omega).
\]
For \( \phi \in \mathcal{D}_K \),
\[
|\Lambda \phi| \leq \|\phi\|_0 |\mu|(K) ,
\]
from which it follows that \( \Lambda \) is continuous. Thus, we can encode certain functions and measures as distributions. I will dare to say that we can encode most functions and measures that we care about as distributions.

If \( \Lambda_1, \Lambda_2 \in \mathcal{D}'(\Omega) \) and \( \omega \) is an open subset of \( \Omega \), we say that \( \Lambda_1 = \Lambda_2 \text{ in } \omega \) if \( \Lambda_1 \phi = \Lambda_2 \phi \) for all \( \phi \in \mathcal{D}(\omega) \).

Let \( \Lambda \in \mathcal{D}'(\Omega) \) and let \( \omega \) be an open subset of \( \Omega \). We say that \( \Lambda \text{ vanishes on } \omega \) if \( \Lambda \phi = 0 \) for all \( \phi \in \mathcal{D}(\omega) \). Taking \( W \) to be the union of all open subsets \( \omega \) of \( \Omega \) on which \( \Lambda \) vanishes, we define the **support of \( \Lambda \)** to be the set \( \Omega \setminus W \).

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3 The Fourier transform

Let $C_0(\mathbb{R}^n)$ be the set of those continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ such that for every $\epsilon > 0$, there is some compact set $K$ such that $|f(x)| < \epsilon$ for $x \notin K$. With the supremum norm $\| \cdot \|_\infty$, $C_0(\mathbb{R}^n)$ is a Banach space.

Let $m_n$ be normalized Lebesgue measure on $\mathbb{R}^n$:

$$dm_n(x) = (2\pi)^{-n/2}dx.$$

Using $m_n$, we define

$$\| f \|_{L^p} = \left( \int_{\mathbb{R}^n} |f|^p dm_n \right)^{1/p}, \quad 1 \leq p < \infty$$

and

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dm_n(y).$$

For $t \in \mathbb{R}^n$, define $e_t : \mathbb{R}^n \to \mathbb{C}$ by

$$e_t(x) = \exp(it \cdot x), \quad x \in \mathbb{R}^n.$$

The Fourier transform of $f \in L^1(\mathbb{R}^n)$ is the function $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$(\mathcal{F}f)(t) = \hat{f}(t) = \int_{\mathbb{R}^n} fe^{-it}dm_n, \quad t \in \mathbb{R}^n.$$ 

Using the dominated convergence theorem, one shows that $\hat{f}$ is continuous.

For $f \in C^\infty(\mathbb{R}^n)$ and $N$ a nonnegative integer, write

$$p_N(f) = \sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N|\langle D^\alpha f \rangle(x)|,$$

and let $\mathcal{S}_n$ be the set of those $f \in C^\infty(\mathbb{R}^n)$ such that for every nonnegative integer $N$, $p_N(f) < \infty$. $\mathcal{S}_n$ is a vector space, and with the locally convex topology determined by the family of seminorms $\{p_N : N \geq 0\}$ it is a Fréchet space. Further, one proves that $\mathcal{F} : \mathcal{S}_n \to \mathcal{S}_n$ is a continuous linear map.

The Riemann-Lebesgue lemma is the statement that if $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$.

The inversion theorem is the statement that if $g \in \mathcal{S}_n$ then

$$g(x) = \int_{\mathbb{R}^n} \hat{g} e_x dm_n, \quad x \in \mathbb{R}^n,$$

and that if $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$, and we define $f_0 \in C_0(\mathbb{R}^n)$ by

$$f_0(x) = \int_{\mathbb{R}^n} \hat{f} e_x dm_n, \quad x \in \mathbb{R}^n,$$

\[6\text{Walter Rudin, Functional Analysis, second ed., p. 184, Theorem 7.4.}\]
\[7\text{Walter Rudin, Functional Analysis, second ed., p. 184, Theorem 7.4.}\]
\[8\text{Walter Rudin, Functional Analysis, second ed., p. 185, Theorem 7.5.}\]
\[9\text{Walter Rudin, Functional Analysis, second ed., p. 186, Theorem 7.7.}\]
then \( f(x) = f_0(x) \) for almost all \( x \in \mathbb{R}^n \). For \( g \in \mathcal{S}_n \), as \( \hat{g} \in \mathcal{S}_n \), the function \( f(t) = \hat{g}(-t) \) belongs to \( \mathcal{S}_n \). The inversion theorem tells us that for all \( x \in \mathbb{R}^n \),

\[
g(x) = \int_{\mathbb{R}^n} \hat{g}(t)e_x(t)dm_n(t) = \int_{\mathbb{R}^n} \hat{g}(-t)e_{-x}(t)dm_n(t) = \int_{\mathbb{R}^n} f(t)e_{-x}(t)dm_n(t),
\]

and hence that \( g = \hat{f} \). This shows that \( \mathcal{F} : \mathcal{S}_n \to \mathcal{S}_n \) is onto. Using the inversion theorem, one checks that

\[
\int_{\mathbb{R}^n} f\hat{g}dm_n = \int_{\mathbb{R}^n} \hat{f}gdm_n, \quad f, g \in \mathcal{S}_n,
\]

and so \( \|f\|_{L^2} = \|\mathcal{F}f\|_{L^2} \) for \( f \in \mathcal{S}_n \). It is a fact that \( \mathcal{S}_n \) is a dense subset of the Hilbert space \( L^2(\mathbb{R}^n) \), and it follows that there is a unique bounded linear operator \( L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \), that is equal to \( \mathcal{F} \) on \( \mathcal{S}_n \), and that is unitary. We denote this \( \mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \).

It is a fact that \( \mathcal{D}(\mathbb{R}^n) \) is a dense subset of \( \mathcal{S}_n \) and that the identity map \( i : \mathcal{D}(\mathbb{R}^n) \to \mathcal{S}_n \) is continuous. If \( L_1, L_2 \in (\mathcal{S}_n)^* \) are distinct, then there is some \( f \in \mathcal{S}_n \) such that \( L_1f \neq L_2f \), and as \( \mathcal{D}(\mathbb{R}^n) \) is dense in \( \mathcal{S}_n \), there is a sequence \( f_j \in \mathcal{D}(\mathbb{R}^n) \) with \( f_j \to f \) in \( \mathcal{S}_n \). As

\[
(L_1 \circ i)(f_j) - (L_2 \circ i)(f_j) = L_1f_j - L_2f_j \to L_1f - L_2f \neq 0,
\]

there is some \( f_j \) with \( (L_1 \circ i)(f_j) \neq (L_2 \circ i)(f_j) \), and hence \( L_1 \circ i \neq L_2 \circ i \). This shows that \( L \to L \circ i \) is a one-to-one linear map \( (\mathcal{S}_n)^* \to \mathcal{D}(\mathbb{R}^n) \). Elements of \( \mathcal{D}(\mathbb{R}^n) \) of the form \( L \circ i \) for \( L \in (\mathcal{S}_n)^* \) are called tempered distributions, and we denote the set of tempered distributions by \( \mathcal{S}_n' \). It is a fact that every distribution with compact support is tempered.

### 4 Sobolev’s lemma

Suppose that \( \Omega \) is an open subset of \( \mathbb{R}^n \). We say that a measurable function \( f : \Omega \to \mathbb{C} \) is locally \( L^2 \) if \( \int_K |f|^2dm_n < \infty \) for every compact subset \( K \) of \( \Omega \). We say that \( \Lambda \in \mathcal{D}'(\Omega) \) is locally \( L^2 \) if there is a function \( g \) that is locally \( L^2 \) in \( \Omega \) such that \( \Lambda \phi = \int_{\Omega} \hat{\phi}gdm_n \) for every \( \phi \in \mathcal{D}(\Omega) \).

The following proof of Sobolev’s lemma follows Rudin.

**Theorem 3** (Sobolev’s lemma). Suppose that \( n, p, r \) are integers, \( n > 0 \), \( p \geq 0 \), and

\[
r > p + \frac{n}{2},
\]

Suppose that \( \Omega \) is an open subset of \( \mathbb{R}^n \), that \( f : \Omega \to \mathbb{C} \) is locally \( L^2 \), and that the distribution derivatives \( D_k^j f \) are locally \( L^2 \) for \( 1 \leq j \leq n, 1 \leq k \leq r \). Then there is some \( f_0 \in C^0(\Omega) \) such that \( f_0(x) = f(x) \) for almost all \( x \in \Omega \).

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Proof. To say that the distribution derivative $D^k_j f$ is locally $L^2$ means that there is some $g_{j,k} : \Omega \to \mathbb{C}$ that is locally $L^2$ such that

$$D^k_j \Lambda f = \Lambda g_{j,k}.$$ 

Suppose that $\omega$ is an open subset of $\Omega$ whose closure $K$ is a compact subset of $\Omega$. There is some $\psi \in \mathcal{D}(\Omega)$ with $\psi(x) = 1$ for $x \in K$, and we define $F : \mathbb{R}^n \to \mathbb{C}$ by

$$F(x) = \begin{cases} \psi(x) f(x) & x \in \Omega, \\ 0 & x \notin \Omega; \end{cases}$$

in particular, for $x \in K$ we have $F(x) = f(x)$, and for $x \notin \text{supp} \psi$ we have $F(x) = 0$. Because $\text{supp} \psi \subset \Omega$ is compact and $f$ is locally $L^2$,

$$\|F\|_{L^2} = \left( \int_{\text{supp } \psi} |\psi f|^2 dm_n \right)^{1/2} \leq \|\psi\|_0 \left( \int_{\text{supp } \psi} |f|^2 dm_n \right)^{1/2} < \infty,$$

and using the Cauchy-Schwarz inequality, $\|F\|_1 \leq \|F\|_{L^2} m_n(\text{supp } \psi)^{1/2} < \infty$, so $F \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

Then,

$$\int_{\mathbb{R}^n} |\hat{F}|^2 dm_n < \infty. \quad (1)$$

Because $\Lambda F = \psi \Lambda f$ in $\Omega$, the Leibniz formula tells us that in $\Omega$,

$$D^r_j \Lambda_F = D^r_j (\psi \Lambda f) = \sum_{s=0}^r \binom{r}{s} (D^r_j \psi)(D^s_j \Lambda f) = \sum_{s=0}^r \binom{r}{s} (D^r_j \psi)(\Lambda g_{s,\cdot}),$$

hence, defining $H_j : \mathbb{R}^n \to \mathbb{C}$ by

$$H_j(x) = \begin{cases} \sum_{s=0}^r \binom{r}{s} (D^r_j \psi)(x) g_{s,\cdot}(x) & x \in \Omega, \\ 0 & x \notin \Omega, \end{cases}$$

we have $D^r_j \Lambda_F = \Lambda H_j$ in $\Omega$. It is apparent that $H_j \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

Let $\phi \in \mathcal{D}(\mathbb{R}^n)$. There are $\phi_1, \phi_2 \in \mathcal{D}(\mathbb{R}^n)$ with $\phi = \phi_1 + \phi_2$ and $\text{supp} \phi_1 \subset \Omega$, $\text{supp} \phi_2 \subset \mathbb{R}^n \setminus \text{supp } \psi$. We have just established that $(D^r_j \Lambda_F) \phi_1 = \Lambda H_j \phi_1$. For $\phi_2$, it is apparent that

$$(D^r_j \Lambda_F) \phi_2 = \Lambda F (D^r_j \phi_2) = \int_{\mathbb{R}^n} (D^r_j \phi_2)(x) F(x) dm_n(x) = 0$$

and

$$\Lambda_{H_j} \phi_2 = \int_{\mathbb{R}^n} \phi_2(x) H_j(x) dm_n(x) = 0.$$

$^{13}\phi_1$ and $\phi_2$ are constructed using a partition of unity. See Walter Rudin, Functional Analysis, second ed., p. 162, Theorem 6.20.
Hence $(D_j^r \Lambda_F)(\phi) = \Lambda_{H_j} \phi$. It is apparent that $\Lambda_{H_j}$ has compact support, so $D_j^r \Lambda_F = \Lambda_{H_j}$ are tempered distributions. Let $\xi \in \mathcal{S}'_n$, and take $\phi \in \mathcal{S}_n$ with $\xi = \hat{\phi}$. Then,

\[
(D_j^r \Lambda_F)\phi = \Lambda_F D_j^r \phi
= \int_{\mathbb{R}^n} (D_j^r \phi)(x)F(x)dm_n(x)
= \int_{\mathbb{R}^n} \mathcal{F}(D_j^r \phi)(y)\hat{F}(y)dm_n(y)
= \int_{\mathbb{R}^n} (iy_j)^r \xi(y)\hat{F}(y)dm_n(y),
\]

and

\[
\Lambda_{H_j} \phi = \int_{\mathbb{R}^n} \phi(x)H_j(x)dm_n(x) = \int_{\mathbb{R}^n} \xi(y)\hat{H}_j(y)dm_n(y).
\]

It follows that $(iy_j)^r \hat{F}(y) = \hat{H}_j(y)$ for all $y \in \mathbb{R}^n$. But $\hat{H}_j \in L^2(\mathbb{R}^n)$, so

\[
\int_{\mathbb{R}^n} y_i^{2r}|\hat{F}(y)|^2dm_n(y) < \infty, \quad 1 \leq i \leq n. \quad (2)
\]

Using (1), (2), and the inequality

\[
(1 + |y|)^{2r} < (2n + 2)^r(1 + y_1^{2r} + \cdots + y_n^{2r}), \quad y \in \mathbb{R}^n,
\]

we get

\[
J = \int_{\mathbb{R}^n} (1 + |y|)^{2r}|\hat{F}(y)|^2dm_n(y) < \infty.
\]

Let $\sigma_{n-1}$ be surface measure on $S^{n-1}$, with $\sigma_{n-1}(S^{n-1}) = 2\pi^{n/2}/\Gamma(n/2)$. Using the Cauchy-Schwarz inequality and the change of variable $y = tu$, $u \in S^{n-1}$, $t \geq 0$,

\[
\left(\int_{\mathbb{R}^n} (1 + |y|)^p |\hat{F}(y)|dm_n(y) \right)^2
= \left(\int_{\mathbb{R}^n} (1 + |y|)^r |\hat{F}(y)|(1 + |y|)^{p-r}dm_n(y) \right)^2
\leq J \int_{\mathbb{R}^n} (1 + |y|)^{2p-2r}dm_n(y)
= J(2\pi)^{-n/2} \int_0^\infty \int_{S^{n-1}} (1 + t)^{2p-2r}t^{n-1}d\sigma_{n-1}(u)dt
= \frac{2J}{\Gamma(n/2)} \int_0^\infty (1 + t)^{2p-2r}t^{n-1}dt.
\]

This integral is finite if and only if $2p - 2r + n - 1 < -1$, and we have assumed that $r > p + \frac{n}{2}$. Therefore,

\[
\int_{\mathbb{R}^n} (1 + |y|)^p |\hat{F}(y)|dm_n(y) < \infty,
\]

from which we get that $y^\alpha \hat{F}(y)$ is in $L^1(\mathbb{R}^n)$ for $|\alpha| \leq p$. 

7
Define
\[ F_\omega(x) = \int_{\mathbb{R}^n} \hat{F}e_x dm_n, \quad x \in \mathbb{R}^n. \]
(Note that \( F \) depends on \( \omega \).) \( F, \hat{F} \in L^1(\mathbb{R}^n) \) so by the inversion theorem we have \( F(x) = F_\omega(x) \) for almost all \( x \in \mathbb{R}^n \). \( F_\omega \in C_0(\mathbb{R}^n) \). If \( p \geq 1 \), then we shall show that \( F_\omega \in C^p(\Omega) \). Take \( \varepsilon_k \) to be the standard basis for \( \mathbb{R}^n \). For \( 1 \leq k_1 \leq n \) and \( \varepsilon \neq 0 \),
\[
\frac{F_\omega(x + \varepsilon e_{k_1}) - F_\omega(x)}{\varepsilon} = \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \hat{F}(y) (\exp(\varepsilon e_{k_1} \cdot y) - 1) \exp(ix \cdot y) dm_n(y)
\]
\[
= \int_{\mathbb{R}^n} i y_{k_1} \hat{F}(y) \frac{e^{ix y_{k_1}} - 1}{i \varepsilon y_k} e_x(y) dm_n(y).
\]
But \( |iy_{k_1} \hat{F}(y) \frac{e^{ix y_{k_1}} - 1}{i \varepsilon y_k} e_x(y)| \leq |y_{k_1} \hat{F}(y)| \) and \( y_{k_1} \hat{F}(y) \) belongs to \( L^1(\mathbb{R}^n) \) (supposing \( p \geq 1 \)) so we can apply the dominated convergence theorem, which gives us
\[
(D_{k_1} F_\omega)(x) = \lim_{\varepsilon \to 0} \frac{F_\omega(x + \varepsilon e_{k_1}) - F_\omega(x)}{\varepsilon} = \int_{\mathbb{R}^n} i y_{k_1} \hat{F}(y) e_x(y) dm_n(y).
\]
From the above expression, it is apparent that \( D_{k_1} F_\omega \) is continuous. This is true for all \( 1 \leq k_1 \leq n \), so \( F_\omega \in C^1(\mathbb{R}^n) \). If \( p \geq 2 \), then \( y_{k_1} y_{k_2} \hat{F}(y) \) is in \( L^1(\mathbb{R}^n) \) for any \( 1 \leq k_2 \leq n \), and repeating the above argument we get \( F_\omega \in C^2(\mathbb{R}^n) \). In this way, \( F_\omega \in C^p(\mathbb{R}^n) \).

For all \( x \in \omega \), \( f(x) = F(x) \), so \( f(x) = F_\omega(x) \) for almost all \( x \in \omega \). If \( \omega' \) is an open subset of \( \Omega \) whose closure is a compact subset of \( \Omega \) and \( \omega \cap \omega' \neq \emptyset \), then \( F_\omega, F_{\omega'} \in C^p(\mathbb{R}^n) \) satisfy \( f(x) = F_\omega(x) \) for almost all \( x \in \omega \) and \( f(x) = F_{\omega'}(x) \) for almost all \( x \in \omega' \), so \( F_{\Omega}(x) = F_{\omega'}(x) \) for almost all \( x \in \omega \cap \omega' \). Since \( F_\omega, F_{\omega'} \) are continuous, this implies that \( F_\omega(x) = F_{\omega'}(x) \) for all \( x \in \omega \cap \omega' \). Thus, it makes sense to define \( f_0(x) = F_\omega(x) \) for \( x \in \omega \). Because every point in \( \Omega \) has an open neighborhood of the kind \( \omega \) and the restriction of \( f_0 \) to each \( \omega \) belongs to \( C^p(\omega) \), it follows that \( f_0 \in C^p(\Omega) \). \( \square \)