The uniform metric on product spaces

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1 Metric topology

If (X, d) is a metric space, $a \in X$, and $r > 0$, then the *open ball with center a and radius r* is

$$B_r^d(a) = \{x \in X : d(x, a) < r\}.$$

The set of all open balls is a basis for the *metric topology induced by d*.

If (X, d) is a metric space, define

$$\overline{d}(a, b) = d(a, b) \wedge 1, \qquad a, b \in X,$$

where $x \wedge y = \min\{x, y\}$. It is straightforward to check that $\overline{d}$ is a metric on X, and one proves that d and $\overline{d}$ induce the same metric topologies.[1] The *diameter* of a subset S of a metric space (X, d) is

$$\mathrm{diam}(S, d) = \sup_{a,b \in S} d(a, b).$$

The subset S is said to be *bounded* if its diameter is finite. The metric space (X, d) might be unbounded, but the diameter of the metric space $(X, \overline{d})$ is

$$\mathrm{diam}(X, \overline{d}) = \sup_{a,b \in X} \overline{d}(a, b) = \mathrm{diam}(X, d) \wedge 1,$$

and thus the metric space $(X, \overline{d})$ is bounded.

2 Product topology

If J is a set and X_j are topological spaces for each $j \in J$, let $X = \prod_{j \in J} X_j$ and let $\pi_j : X \to X_j$ be the projection maps. A basis for the *product topology* on X are those sets of the form $\bigcap_{j \in J_0} \pi_j^{-1}(U_j)$, where J_0 is a finite subset of J and U_j is an open subset of X_j, $j \in J_0$. Equivalently, the product topology is the initial topology for the projection maps $\pi_j : X \to X_j$, $j \in J$, i.e. the coarsest topology on X such that each projection map is continuous. Each of the projection maps is open.[2] The following theorem characterizes convergent

[1] James Munkres, *Topology*, second ed., p. 121, Theorem 20.1.
[2] John L. Kelley, *General Topology*, p. 90, Theorem 2.

nets in the product topology.[3]

Theorem 1. *Let J be a set and for each $j \in J$ let X_j be a topological space. If $X = \prod_{j\in J} X_j$ has the product topology and $(x_\alpha)_{\alpha\in I}$ is a net in X, then $x_\alpha \to x$ if and only if $\pi_j(x_\alpha) \to \pi_j(x)$ for each $j \in J$.*

Proof. Let $(x_\alpha)_{\alpha\in I}$ be a net that converges to $x \in X$. Because each projection map is continuous, if $j \in J$ then $\pi_j(x_\alpha) \to \pi_j(x)$. On the other hand, suppose that $(x_\alpha)_{\alpha\in I}$ is a net, that $x \in X$, and that $\pi_j(x_\alpha) \to \pi_j(x)$ for each $j \in J$. Let $\mathscr{O}_j$ be the set of open neighborhoods of $\pi_j(x) \in X_j$. For $j \in J$ and $U \in \mathscr{O}_j$, because $\pi_j(x_\alpha) \to \pi_j(x)$ we have that $\pi_j(x_\alpha)$ is eventually in U. It follows that if $j \in J$ and $U \in \mathscr{O}_j$ then x_α is eventually in $\pi_j^{-1}(U)$. Therefore, if J_0 is a finite subset of J and $U_j \in \mathscr{O}_j$ for each $j \in J_0$, then x_α is eventually in $\bigcap_{j\in J_0} \pi_j^{-1}(U_j)$. This means that the net $(x_\alpha)_{\alpha\in I}$ is eventually in every basic open neighborhood of x, which implies that $x_\alpha \to x$. □

The following theorem states that if J is a countable set and (X, d) is a metric space, then the product topology on X^J is metrizable.[4]

Theorem 2. *If J is a countable set and (X, d) is a metric space, then*

$$\rho(x, y) = \sup_{j\in J} \frac{\overline{d}(x_j, y_j)}{j} = \sup_{j\in J} \frac{d(x_j, y_j) \wedge 1}{j}$$

is a metric on X^J that induces the product topology.

A topological space is *first-countable* if every point has a countable local basis; a *local basis* at a point p is a set $\mathscr{B}$ of open sets each of which contains p such that each open set containing p contains an element of $\mathscr{B}$. It is a fact that a metrizable topological space is first-countable. In the following theorem we prove that the product topology on an uncountable product of Hausdorff spaces each of which has at least two points is not first-countable.[5] From this it follows that if (X, d) is a metric space with at least two points and J is an uncountable set, then the product topology on X^J is not metrizable.

Theorem 3. *If J is an uncountable set and for each $j \in J$ we have that X_j is a Hausdorff space with at least two points, then the product topology on $\prod_{j\in J} X_j$ is not first-countable.*

Proof. Write $X = \prod_{j\in J} X_j$, and suppose that $x \in X$ and that $U_n, n \in \mathbb{N}$, are open subsets of X containing x. Since U_n is an open subset of X containing x, there is a basic open set B_n satisfying $x \in B_n \subseteq U_n$: by saying that B_n is a basic open set we mean that there is a finite subset F_n of J and open subsets $U_{n,j}$ of X_j, $j \in F_n$, such that

$$B_n = \bigcap_{j\in F_n} \pi_j^{-1}(U_{n,j}).$$

[3] John L. Kelley, *General Topology*, p. 91, Theorem 4.
[4] James Munkres, *Topology*, second ed., p. 125, Theorem 20.5.
[5] cf. John L. Kelley, *General Topology*, p. 92, Theorem 6.

Let $F = \bigcup_{n\in\mathbb{N}} F_n$, and because J is uncountable there is some $k \in J \setminus F$; this is the only place in the proof at which we use that J is uncountable. As X_k has at least two points and $x(k) \in X_k$, there is some $a \in X_k$ with $x(k) \neq a$. Since X_k is a Hausdorff space, there are disjoint open subsets N_1, N_2 of X_k with $x(k) \in N_1$ and $a \in N_2$. Define

$$V_j = \begin{cases} N_1 & j = k \\ X_j & j \neq k \end{cases}$$

and let $V = \prod_{j\in J} V_j$. We have $x \in V$. But for each $n \in \mathbb{N}$, there is some $y_n \in B_n$ with $y_n(k) = a \in N_2$, hence $y_n(k) \notin N_1$ and so $y_n \notin V$. Thus none of the sets B_n is contained in V, and hence none of the sets U_n is contained in V. Therefore $\{U_n : n \in \mathbb{N}\}$ is not a local basis at x, and as this was an arbitrary countable set of open sets containing x, there is no countable local basis at x, showing that X is not first-countable. (In fact, we have proved there is no countable local basis at any point in X; not to be first-countable merely requires that there be at least one point at which there is no countable local basis.) □

3 Uniform metric

If J is a set and (X, d) is a metric space, we define the *uniform metric* on X^J by

$$d_J(x, y) = \sup_{j\in J} \overline{d}(x_j, y_j) = \sup_{j\in J} d(x_j, y_j) \wedge 1.$$

It is apparent that $d_J(x, y) = 0$ if and only if $x = y$ and that $d_J(x, y) = d_J(y, x)$. If $x, y, z \in X$ then,

$$\begin{aligned} d_J(x, z) &= \sup_{j\in J} \overline{d}(x_j, z_j) \\ &\leq \sup_{j\in J} \overline{d}(x_j, y_j) + \overline{d}(y_j, z_j) \\ &\leq \sup_{j\in J} \overline{d}(x_j, y_j) + \sup_{j\in J} \overline{d}(y_j, z_j) \\ &= d_J(x, y) + d_J(y, z), \end{aligned}$$

showing that d_J satisfies the triangle inequality and thus that it is indeed a metric on X^J. The *uniform topology* on X^J is the metric topology induced by the uniform metric.

If (X, d) is a metric space, then X is a topological space with the metric topology, and thus $X^J = \prod_{j\in J} X$ is a topological space with the product topology. The following theorem shows that the uniform topology on X^J is finer than the product topology on X^J.[6]

[6] James Munkres, *Topology*, second ed., p. 124, Theorem 20.4.

Theorem 4. *If J is a set and (X, d) is a metric space, then the uniform topology on X^J is finer than the product topology on X^J.*

Proof. If $x \in X^J$, let $U = \prod_{j \in J} U_j$ be a basic open set in the product topology with $x \in U$. Thus, there is a finite subset J_0 of J such that if $j \in J \setminus J_0$ then $U_j = X$. If $j \in J_0$, then because U_j is an open subset of (X, d) with the metric topology and $x_j \in U_j$, there is some $0 < \epsilon_j < 1$ such that $B^d_{\epsilon_j}(x_j) \subseteq U_j$. Let $\epsilon = \min_{j \in J_0} \epsilon_j$. If $d_J(x, y) < \epsilon$ then $d(x_j, y_j) < \epsilon$ for all $j \in J$ and hence $d(x_j, y_j) < \epsilon_j$ for all $j \in J_0$, which implies that $y_j \in B^d_{\epsilon_j}(x_j) \subseteq U_j$ for all $j \in J_0$. If $j \in J \setminus J_0$ then $U_j = X$ and of course $y_j \in U_j$. Therefore, if $y \in B^{d_J}_{\epsilon}(x)$ then $y \in U$, i.e. $B^{d_J}_{\epsilon}(x) \subseteq U$. It follows that the uniform topology on X^J is finer than the product topology on X^J. □

The following theorem shows that if we take the product of a complete metric space with itself, then the uniform metric on this product space is complete.[7]

Theorem 5. *If J is a set and (X, d) is a complete metric space, then X^J with the uniform metric is a complete metric space.*

Proof. It is straightforward to check that (X, d) being a complete metric space implies that $(X, \overline{d})$ is a complete metric space. Let f_n be a Cauchy sequence in (X^J, d_J): if $\epsilon > 0$ then there is some N such that $n, m \geq N$ implies that

$$d_J(f_n, f_m) < \epsilon.$$

Thus, if $\epsilon > 0$, then there is some N such that $n, m \geq N$ and $j \in J$ implies that $\overline{d}(f_n(j), f_m(j)) \leq d_J(f_n, f_m) < \epsilon$. Thus, if $j \in J$ then $f_n(j)$ is a Cauchy sequence in $(X, \overline{d})$, which therefore converges to some $f(j) \in X$, and thus $f \in X^J$. If $n, m \geq N$ and $j \in J$, then

$$\begin{aligned} \overline{d}(f_n(j), f(j)) &\leq \overline{d}(f_n(j), f_m(j)) + \overline{d}(f_m(j), f(j)) \\ &\leq d_J(f_n, f_m) + \overline{d}(f_m(j), f(j)) \\ &< \epsilon + \overline{d}(f_m(j), f(j)). \end{aligned}$$

As the left-hand side does not depend on m and $\overline{d}(f_m(j), f(j)) \to 0$, we get that if $n \geq N$ and $j \in J$ then

$$\overline{d}(f_n(j), f(j)) \leq \epsilon.$$

Therefore, if $n \geq N$ then

$$d_J(f_n, f) \leq \epsilon.$$

This means that f_n converges to f in the uniform metric, showing that (X^J, d_J) is a complete metric space. □

[7] James Munkres, *Topology*, second ed., p. 267, Theorem 43.5.

4 Bounded functions and continuous functions

If J is a set and (X,d) is a metric space, a function $f:J\to X$ is said to be *bounded* if its image is a bounded subset of X, i.e. $f(J)$ has a finite diameter. Let $B(J,X)$ be the set of bounded functions $J\to(X,d)$; $B(J,X)$ is a subset of X^J. Since the diameter of $(X,\overline{d})$ is ≤ 1, any function $J\to(X,\overline{d})$ is bounded, but there might be unbounded functions $J\to(X,d)$. We prove in the following theorem that $B(J,X)$ is a closed subset of X^J with the uniform topology.[8]

Theorem 6. *If J is a set and (X,d) is a metric space, then $B(J,X)$ is a closed subset of X^J with the uniform topology.*

Proof. If $f_n\in B(J,Y)$ and f_n converges to $f\in X^J$ in the uniform topology, then there is some N such that $d_J(f_N,f)<\frac{1}{2}$. Thus, for all $j\in J$ we have $\overline{d}(f_N(j),f(j))<\frac{1}{2}$, which implies that

$$d(f_N(j),f(j))=\overline{d}(f_N(j),f(j))<\frac{1}{2}.$$

If $i,j\in J$, then

$$\begin{aligned}d(f(i),f(j))&\leq d(f(i),f_N(i))+d(f_N(i),f_N(j))+d(f_N(j),f(j))\\&\leq\frac{1}{2}+\operatorname{diam}(f_N(J),d)+\frac{1}{2}.\end{aligned}$$

$f_N\in B(J,X)$ means that $\operatorname{diam}(f_N(J),d)<\infty$, and it follows that $\operatorname{diam}(f(J),d)\leq\operatorname{diam}(f_N(J),d)+1<\infty$, showing that $f\in B(J,X)$. Therefore if a sequence of elements in $B(J,X)$ converges to an element of X^J, that limit is contained in $B(J,X)$. This implies that $B(J,X)$ is a closed subset of X^J in the uniform topology, as in a metrizable space the closure of a set is the set of limits of sequences of points in the set. □

If J is a set and Y is a complete metric space, we have shown in Theorem 5 that Y^J is a complete metric space with the uniform metric. If X and Y are topological spaces, we denote by $C(X,Y)$ the set of continuous functions $X\to Y$. $C(X,Y)$ is a subset of Y^X, and we show in the following theorem that if Y is a metric space then $C(X,Y)$ is a closed subset of Y^X in the uniform topology.[9] Thus, if Y is a complete metric space then $C(X,Y)$ is a closed subset of the complete metric space Y^X, and is therefore itself a complete metric space with the uniform metric.

Theorem 7. *If X is a topological space and let (Y,d) is a metric space, then $C(X,Y)$ is a closed subset of Y^X with the uniform topology.*

Proof. Suppose that $f_n\in C(X,Y)$ and $f_n\to f\in Y^X$ in the uniform topology. Thus, if $\epsilon>0$ then there is some N such that $n\geq N$ implies that $d_J(f_n,f)<\epsilon$, and so if $n\geq N$ and $x\in X$ then

$$\overline{d}(f_n(x),f(x))\leq d_J(f_n,f)<\epsilon.$$

[8] James Munkres, *Topology*, second ed., p. 267, Theorem 43.6.
[9] James Munkres, *Topology*, second ed., p. 267, Theorem 43.6.

This means that the sequence f_n converges uniformly in X to f in the uniform metric, and as each f_n is continuous this implies that f is continuous.[10] We have shown that if $f_n \in C(X, Y)$ and $f_n \to f \in Y^X$ in the uniform topology then $f \in C(X, Y)$, and therefore $C(X, Y)$ is a closed subset of Y^X in the uniform topology. □

5 Topology of compact convergence

Let X be a topological space and (Y, d) be a metric space. If $f \in Y^X$, C is a compact subset of X, and $\epsilon > 0$, we denote by $B_C(f, \epsilon)$ the set of those $g \in Y^X$ such that

$$\sup\{d(f(x), g(x)) : x \in C\} < \epsilon.$$

A basis for the *topology of compact convergence* on Y^X are those sets of the form $B_C(f, \epsilon)$, $f \in Y^X$, C a compact subset of X, and $\epsilon > 0$. It can be proved that the uniform topology on Y^X is finer than the topology of compact convergence on Y^X, and that the topology of compact convergence on Y^X is finer than the product topology on Y^X.[11] Indeed, we have already shown in Theorem 4 that the uniform topology on Y^X is finer than the product topology on Y^X. The significance of the topology of compact convergence on Y^X is that a sequence of functions $f_n : X \to Y$ converges in the topology of compact convergence to a function $f : X \to Y$ if and only if for each compact subset C of X the sequence of functions $f_n|C : C \to Y$ converges uniformly in C to the function $f|C : C \to Y$.

[10] See James Munkres, *Topology*, second ed., p. 132, Theorem 21.6.
[11] James Munkres, *Topology*, second ed., p. 285, Theorem 46.7.