Christoffel Symbols

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1 In General Topologies

We have a metric tensor g_{nm} defined by,

$$ds^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b \tag{1}$$

which tells us how the distance is measured between two points in a manifold M. Note g_{ab} is a function of only x^a and x^b. Say we wish to investigate what an observer will experience as she moves on a world line $\mathfrak{W}$ in M, then we will take intuition from classical mechanics and extremize the action of the Lagrangian L along $\mathfrak{W}$. First we say $\mathfrak{W} : \lambda \to \mathbb{R}^n = x^\mu(\lambda)$ so that the world line is parametrized. Here λ plays a role similar to time in classical mechanics except our time in incorporated into the 4-position x^c. Now we define the Lagrangian as,

$$L = \left(\frac{\mathrm{d}s}{\mathrm{d}\lambda}\right)^2$$

We need to extremize the following functional for some *affine* parameter λ,

$$\int_{\mathfrak{W}} L\,\mathrm{d}\lambda = 0$$

From the familiar Euler-Lagrange equations we get the result,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) = \frac{\partial L}{\partial x^c} \tag{2}$$

Note by some handy theorem that for almost any continuous function $F(L)$, equation 2 still holds. Now we work out an explict form of equation 2.

$$\begin{aligned} L &= \left(\frac{\mathrm{d}s}{\mathrm{d}\lambda}\right)^2 \\ &= g_{ab}\dot{x}^a\dot{x}^b \end{aligned}$$

Where we note that the metric tensor is independent of λ. Thus,

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}^c} &= g_{ab}\frac{\partial \dot{x}^a}{\partial \dot{x}^c}\dot{x}^b + g_{ab}\frac{\partial \dot{x}^b}{\partial \dot{x}^c}\dot{x}^a \\ &= 2g_{ab}\frac{\partial \dot{x}^a}{\partial \dot{x}^c}\dot{x}^b \\ &= 2g_{ab}\delta^a_c\dot{x}^b \\ &= 2g_{cb}\dot{x}^b \end{aligned}$$

Where we used the symmetry of g_{nm}. Note that indices are arbitrary and that we can swap them at will for each other if it simplifies the procedure. Now we apply the λ derivative.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) &= 2g_{cb}\frac{\mathrm{d}\dot{x}^b}{\mathrm{d}\lambda} + 2\frac{\partial g_{cb}}{\partial x^d}\frac{\mathrm{d}x^d}{\mathrm{d}\lambda}\dot{x}^b \\ &= 2g_{cb}\ddot{x}^b + 2g_{cb|d}\dot{x}^d\dot{x}^b \end{aligned}$$

Now for the right hand side,

$$\frac{\partial L}{\partial x^c} = g_{ab|c}\dot{x}^a\dot{x}^b$$

Then from equation 2 we have,

$$\begin{aligned}
&2g_{cb}\ddot{x}^b + 2g_{cb|d}\dot{x}^d\dot{x}^b - g_{ab|c}\dot{x}^a\dot{x}^b = 0\\
\Longrightarrow &2g^{ec}g_{cb}\ddot{x}^b + g^{ec}(2g_{cb|a} - g_{ab|c})\dot{x}^a\dot{x}^b\\
&= 2\delta^e_b\ddot{x}^b + g^{ec}(g_{cb|a} + g_{ca|b} - g_{ab|c})\dot{x}^a\dot{x}^b\\
&= \ddot{x}^e + \frac{1}{2}g^{ec}(g_{bc|a} + g_{ca|b} - g_{ab|c})\dot{x}^a\dot{x}^b = 0\\
&= \ddot{x}^e + \Gamma^e_{ab}\dot{x}^a\dot{x}^b
\end{aligned}$$

Where $\Gamma^e_{ab} = \frac{1}{2}g^{ec}(g_{bc|a} + g_{ca|b} - g_{ab|c})$ are called the Christoffel Symbols *of the second kind*[1].

2 Γ^e_{ab} is not a tensor

A tensor is simply a geometrical object which represents some locally isomorphic operation on a manifold M. This implies that the null space is preserved in all representations of the same tensor. Let's try to take Γ^e_{ab} into a different representation via,

$$\chi^{e'}_{a'b'} = A^{e'}_e A^a_{a'} A^b_{b'} \chi^e_{ab} \tag{3}$$

where,

$$A^t_s = \frac{\partial x^t}{\partial x^s} \tag{4}$$

is the jacobian of the new representation, x^t, with respect to the old representation, x^s. If Γ^e_{ab} is indeed a tensorial object then it must invariantly transform via equation 3. And so we begin by expanding the metric tensor and then applying the partial derivative, like $g_{i'j'|k'} = \frac{\partial g_{i'j'}}{\partial x^{k'}} = \frac{\partial A^i_{i'}A^j_{j'}g_{ij}}{\partial x^{k'}}$.

$$\Gamma^{e'}_{a'b'} = \frac{1}{2}g^{e'c'}(\frac{\partial A^b_{b'}A^c_{c'}g_{bc}}{\partial x^{a'}} + \frac{\partial A^a_{a'}A^c_{c'}g_{ca}}{\partial x^{b'}} - \frac{\partial A^a_{a'}A^b_{b'}g_{ab}}{\partial x^{c'}}) \tag{5}$$

We need to deal with these partial derivatives,

$$\begin{aligned}
\frac{\partial A^i_{i'}A^j_{j'}g_{ij}}{\partial x^{k'}} &= \frac{\partial \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}g_{ij}}{\partial x^{k'}}\\
&= \frac{\partial^2 x^i}{\partial x^{i'}\partial x^{k'}}\frac{\partial x^j}{\partial x^{j'}}g_{ij} + \frac{\partial x^i}{\partial x^{i'}}\frac{\partial^2 x^j}{\partial x^{j'}\partial x^{k'}}g_{ij}\\
&+ \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial g_{ij}}{\partial x^k}\\
&= 2\frac{\partial^2 x^i}{\partial x^{i'}\partial x^{k'}}g_{ij'} + A^i_{i'}A^j_{j'}A^k_{k'}g_{ij|k}
\end{aligned}$$

Where we used symmetry of i and j to combine terms. Therefore, equation 5 becomes,

$$\begin{aligned}
\Gamma^{e'}_{a'b'} = \frac{1}{2}g^{e'c'}(&2\frac{\partial^2 x^b}{\partial x^{b'}\partial x^{a'}}g_{bc'} + A^b_{b'}A^c_{c'}A^a_{a'}g_{bc|a}\\
&+2\frac{\partial^2 x^c}{\partial x^{c'}\partial x^{b'}}g_{ca'} + A^c_{c'}A^a_{a'}A^b_{b'}g_{ca|b}\\
&-2\frac{\partial^2 x^a}{\partial x^{a'}\partial x^{c'}}g_{ab'} - A^a_{a'}A^b_{b'}A^c_{c'}g_{ab|c})
\end{aligned}$$

Carrying the factors in and seeing the Kronecker deltas that appear we get,

[1] Christoffel Symbols of the *first* kind are almost never seen or used.

$$
\begin{aligned}
\Gamma^{e'}_{a'b'} &= \frac{1}{2}(2\frac{\partial^2 x^b}{\partial x^{b'}\partial x^{a'}}\delta^{e'}_b + A^b_{b'}A^c_{c'}A^a_{a'}g_{bc|a}g^{e'c'} \\
&+ 2\frac{\partial^2 x^c}{\partial x^{c'}\partial x^{b'}}A^a_{a'}\delta^{e'}_a + A^c_{c'}A^a_{a'}A^b_{b'}g_{ca|b}g^{e'c'} \\
&- 2\frac{\partial^2 x^a}{\partial x^{a'}\partial x^{c'}}A^b_{b'}A^{c'}_b\delta^{e'}_a - A^a_{a'}A^b_{b'}A^c_{c'}g_{ab|c}g^{e'c'}) \\
&= \frac{1}{2}(2\frac{\partial^2 x^{e'}}{\partial x^{b'}\partial x^{a'}} + A^b_{b'}A^c_{c'}A^a_{a'}g_{bc|a}g^{e'c'} \\
&+ 2\frac{\partial^2 x^c}{\partial x^{c'}\partial x^{b'}}A^{e'}_{a'} + A^c_{c'}A^a_{a'}A^b_{b'}g_{ca|b}g^{e'c'} \\
&- 2\frac{\partial^2 x^{e'}}{\partial x^{a'}\partial x^{b'}} - A^a_{a'}A^b_{b'}A^c_{c'}g_{ab|c}g^{e'c'}) \\
&= \frac{\partial^2 x^{c'}}{\partial x^{a'}\partial x^{b'}}A^c_{c'}A^{e'}_{c'} + \frac{1}{2}(A^b_{b'}A^a_{a'}g_{bc|a}A^{e'}_e g^{ec} \\
&+ A^a_{a'}A^b_{b'}g_{ca|b}A^{e'}_e g^{ec} \\
&- A^a_{a'}A^b_{b'}g_{ab|c}A^{e'}_e g^{ec})
\end{aligned}
$$

Or,

$$\Gamma^{e'}_{a'b'} = \frac{\partial^2 x^{c'}}{\partial x^{a'}\partial x^{b'}}A^c_{c'}A^{e'}_{c'} + A^a_{a'}A^b_{b'}A^{e'}_e\Gamma^{e'}_{a'b'}) \tag{6}$$

3 Christoffel Symbols of Flat Space-Time in Spherical Coordinates

Say we have a Minkowski space-time with euclidean coordinates $x^\mu = (t, x, y, z)$, which has metric,

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{7}$$

$$\Longrightarrow \mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{8}$$

Let's now look in spherical coordinates $x^\sigma = (t', \rho, \theta, \phi)$. We have the relations,

$$t = t' \tag{9}$$

$$x = \rho\cos\theta\sin\phi \tag{10}$$

$$y = \rho\sin\theta\sin\phi \tag{11}$$

$$z = \rho\cos\phi \tag{12}$$

and,

$$t' = t \tag{13}$$

$$\rho = \sqrt{x^2 + y^2 + z^2} \tag{14}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{15}$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) \tag{16}$$

The jacobian $A^n_m = \frac{\partial x^\mu}{\partial x^\sigma}$ of which is,

$$A^n_m = \begin{pmatrix} \Delta_{t'}x^\mu \\ \Delta_\rho x^\mu \\ \Delta_\theta x^\mu \\ \Delta_\phi x^\mu \end{pmatrix} \tag{17}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ 0 & -\rho\sin\theta\sin\phi & \rho\cos\theta\sin\phi & 0 \\ 0 & \rho\cos\theta\cos\phi & \rho\sin\theta\cos\phi & -\rho\sin\phi \end{pmatrix} \tag{18}$$

Clearly then, in the spherical coordinate system the unit vectors would be just the rows of our above jacobian. That is

$$\hat{\boldsymbol{t}}' = \frac{\mathrm{d}x^\mu}{\mathrm{d}t'} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{19}$$

$$\hat{\boldsymbol{\rho}} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\rho} = \begin{pmatrix} 0 \\ \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{pmatrix} \tag{20}$$

$$\hat{\boldsymbol{\theta}} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\theta} = \begin{pmatrix} 0 \\ -\rho\sin\theta\sin\phi \\ \rho\cos\theta\sin\phi \\ 0 \end{pmatrix} \tag{21}$$

$$\hat{\boldsymbol{\phi}} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\phi} = \begin{pmatrix} 0 \\ \rho\cos\theta\cos\phi \\ \rho\sin\theta\cos\phi \\ -\rho\sin\phi \end{pmatrix} \tag{22}$$

(23)

Now, using our rules for the lowering and raising of tensor objects, $g_{mn} = \frac{\partial x^a}{\partial x^m}\frac{\partial x^b}{\partial x^n} g_{ab}$ we can convert the flat space-time metric in euclidean coordinates to the spherical coordinate system. Note here that using the covariant metric, that is with indices on the bottom, then we must have the euclidean coordinates in terms of the spherical coordinates.

Applying $g_{mn} = A^a_m A^b_n g_{ab}$ we get,

$$g_{mn} = A^a_m A^b_n g_{ab} \tag{24}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ 0 & -\rho\sin\theta\sin\phi & \rho\cos\theta\sin\phi & 0 \\ 0 & \rho\cos\theta\cos\phi & \rho\sin\theta\cos\phi & -\rho\sin\phi \end{pmatrix}^2 \tag{25}$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{26}$$

Calculation of the square of the jacobian yields the columns,

$$A^a_m A^b_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A^a_m A^b_1 =$$
$$\begin{pmatrix} (0) \\ (\cos\theta\sin^2\phi - \rho\sin^2\theta\sin^2\phi \\ +\rho\cos\theta\cos^2\phi) \\ (-\rho\sin\theta\cos\theta\sin^2\phi - \rho^2\sin\theta\cos\theta\sin\phi) \\ (\rho\cos^2\theta\sin\phi\cos\phi - \rho^2\sin^2\theta\sin\phi\cos\phi \\ -\rho^2\cos\theta\sin\phi\cos\phi) \end{pmatrix}$$

$$= \begin{pmatrix} (0) \\ (\cos\theta - \rho\sin^2\theta)\sin^2\phi + \rho\cos\theta\cos^2\phi) \\ (-\rho\sin\theta\cos\theta(\sin^2\phi + \rho\sin\phi) \\ (((\cos\theta - \rho)\rho\cos\theta \\ -\rho^2\sin^2\theta)\sin\phi\cos\phi) \end{pmatrix}$$

$$A^a_m A^b_2 = \begin{pmatrix} (0) \\ (\sin\theta\cos\theta\sin^2\phi + \rho\sin\theta\cos\theta\sin^2\phi \\ +\rho\sin\theta\cos\theta) \\ (-\rho\sin^2\theta\sin^2\phi + \rho^2\cos^2\theta\sin^2\phi) \\ (\rho\sin\theta\cos\theta\sin\phi\cos\phi + \rho^2\sin\theta\cos\theta\sin\phi\cos\phi \\ -\rho^2\sin\theta\sin\phi\cos\phi) \end{pmatrix}$$

$$= \begin{pmatrix} (0) \\ (\sin\theta\cos\theta(\sin^2\phi + \rho\sin^2\phi + \rho)) \\ (\rho(-\sin^2\theta + \rho\cos^2\theta)\sin^2\phi) \\ (\rho(\sin\theta\cos\theta + \rho\sin\theta\cos\theta - \rho\sin\theta)\sin\phi\cos\phi) \end{pmatrix}$$

$$A^a_m A^b_3 = \begin{pmatrix} (0) \\ (\cos\theta\sin\phi\cos\phi - \rho\sin\phi\cos\phi) \\ (-\rho\sin\theta\sin\phi\cos\phi) \\ (\rho\cos\theta\cos^2\phi + \rho^2\sin^2\phi) \end{pmatrix}$$

$$= \begin{pmatrix} (0) \\ ((\cos\theta - \rho)\sin\phi\cos\phi) \\ (-\rho\sin\theta\sin\phi\cos\phi) \\ (\rho\cos\theta\cos^2\phi + \rho^2\sin^2\phi) \end{pmatrix}$$

Now we can calculate the full metric. All off diagonals will be zero because the Minkowski metric is diagonal.

$$g_{00} = A^a_0 A^b_0 g_{ab} = 1$$

$$\begin{aligned} g_{11} = A^a_1 A^b_1 g_{ab} = 0 \\ &- (((\cos\theta - \rho\sin^2\theta)\sin^2\phi + \rho\cos\theta\cos^2\phi)^2 \\ &+ (\rho\sin\theta\cos\theta(\sin^2\phi + \rho\sin\phi))^2 \\ &+ (((\cos\theta - \rho)\rho\cos\theta - \rho^2\sin^2\theta)\sin\phi\cos\phi)^2) \\ &= 1 \end{aligned}$$

$$\begin{aligned} g_{22} = A^a_2 A^b_2 g_{ab} = 0 \\ &- ((\sin\theta\cos\theta(\sin^2\phi + \rho\sin^2\phi + \rho))^2 \\ &+ (\rho(-\sin^2\theta + \rho\cos^2\theta)\sin^2\phi)^2 \\ &+ (\rho(\sin\theta\cos\theta + \rho\sin\theta\cos\theta - \rho\sin\theta)\sin\phi\cos\phi)^2) \\ &= \rho^2\sin^2\phi \end{aligned}$$

$$\begin{aligned} g_{33} = A^a_3 A^b_3 g_{ab} = 0 \\ &- (((\cos\theta - \rho)\sin\phi\cos\phi)^2 \\ &+ (-\rho\sin\theta\sin\phi\cos\phi)^2 \\ &+ (\rho\cos\theta\cos^2\phi + \rho^2\sin^2\phi)^2) \\ &= \rho^2 \end{aligned}$$