Stuff

$\nabla|\mathbf{x}| = \frac{\mathbf{x}}{|\mathbf{x}|}$

$(\nabla|\mathbf{x}|)^2 = 1$

$\Delta|\mathbf{x}| = \frac{2}{|\mathbf{x}|}$

$G(\mathbf{x},t) = \frac{\delta(t-\mathbf{x}/c)}{4\pi|\mathbf{x}|}$

$\delta(g(\mathbf{x}')) = \sum_{g(x*)=0} \frac{\delta(\mathbf{x}'-x*)}{|g'(x*)|}$

$\delta(x^2-a^2) = \frac{1}{2|a|}(\delta(x+a)+\delta(x-a))$

$\int_{-\infty}^{\infty} f(x')\delta(g(x'))dx' = \sum_{g(x*)=0} \frac{f(x*)}{|g'(x*)|}$

$\nabla\frac{1}{|\mathbf{x}|} = -\frac{\mathbf{x}}{|\mathbf{x}|^3}$

$\Delta\frac{-1}{4\pi|\mathbf{x}|} = \delta^{(3)}(\mathbf{x}) \implies \Delta\frac{1}{|\mathbf{x}|} = -4\pi\delta^{(3)}(\mathbf{x})$

$\delta(x) = \int_{-\infty}^{\infty} e^{2\pi i x\xi}d\xi$

For a particle moving along world line, $\mathbf{x}(t)$, charge density is, $\rho(\mathbf{x},t) = e\delta(\mathbf{x}-\mathbf{x}(t))$, and current density is,

$\mathbf{j}(\mathbf{x},t) = \dot{\mathbf{x}}(t)\rho(\mathbf{x},t)$
$j^0(\mathbf{x},t) = c\rho(\mathbf{x},t)$

$\Box A^i(\mathbf{x},t) = \frac{4\pi}{c}j^i(\mathbf{x},t)$
$A^i(\mathbf{x},t) = \frac{1}{c}\int d^3\mathbf{x}'dt'\frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|/c)j^i(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}$
$= \frac{1}{c}\int d^3\mathbf{x}'\frac{j^i(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}$

In determining retarded time, use spacetime for arbitrary point $(\mathbf{x}_0,t_0)$, then use $c(t_0-t_r) = |\mathbf{x}_0-\mathbf{x}(t_r)|$ to find t_r. Then use in Lienard-Wiechert potentials:

$A^0(\mathbf{x},t) = \frac{ec}{c|\mathbf{x}-\mathbf{x}(t_r)|-\dot{\mathbf{x}}(t_r)\cdot(\mathbf{x}-\mathbf{x}(t_r))}$
$\mathbf{A}(\mathbf{x},t) = \frac{e\dot{\mathbf{x}}(t_r)}{c|\mathbf{x}-\mathbf{x}(t_r)|-\dot{\mathbf{x}}(t_r)\cdot(\mathbf{x}-\mathbf{x}(t_r))}$
$= \frac{\dot{\mathbf{x}}(t_r)}{c}A^0(\mathbf{x},t)$

$\mathbf{E} = -\frac{\partial A^0(\mathbf{x},t)}{\partial\mathbf{x}} - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t}$
$\mathbf{B} = \frac{\partial}{\partial\mathbf{x}}\times\mathbf{A}(\mathbf{x},t)$

$\frac{\partial t_r(\mathbf{x},t)}{\partial t} = \frac{c|\mathbf{x}-\mathbf{x}(t_r)|}{c|\mathbf{x}-\mathbf{x}(t_r)|-\dot{\mathbf{x}}(t_r)\cdot(\mathbf{x}-\mathbf{x}(t_r))}$
$= |\mathbf{x}-\mathbf{x}(t_r)|A^0(\mathbf{x},t)$
$\frac{\partial t_r(\mathbf{x},t)}{\partial\mathbf{x}} = \frac{\mathbf{x}-\mathbf{x}(t_r)}{c|\mathbf{x}-\mathbf{x}(t_r)|-\dot{\mathbf{x}}(t_r)\cdot(\mathbf{x}-\mathbf{x}(t_r))}$
$= \frac{(\mathbf{x}-\mathbf{x}(t_r))A^0(\mathbf{x},t)}{c}$

$\mathbf{u} = c\frac{\mathbf{x}-\mathbf{x}(t_r)}{|\mathbf{x}-\mathbf{x}(t_r)|} - \dot{\mathbf{x}}(t_r)$
$\mathbf{E} = e\frac{|\mathbf{x}-\mathbf{x}(t_r)|}{((\mathbf{x}-\mathbf{x}(t_r))\cdot\mathbf{u})^3}[(c^2-|\dot{\mathbf{x}}(t_r)|)\mathbf{u} + (\mathbf{x}-\mathbf{x}(t_r))\times(\mathbf{u}\times\ddot{\mathbf{x}}(t_r))]$
$\mathbf{B} = \frac{\mathbf{x}-\mathbf{x}(t_r)}{|\mathbf{x}-\mathbf{x}(t_r)|}\times\mathbf{E}(\mathbf{x},t)$

The first term in $\mathbf{E} \propto R^{-2}$ and the second term, $\propto R^{-1}$ as $R\to\infty$. Hence $\mathbf{S} \propto R^{-4}||R^{-2}$, and $\int d^2\boldsymbol{\sigma}\mathbf{S} \to 0$ for first term far away, but not for second term. Thus $\ddot{x}$ is the cause of energy flux at infity.

$P = \int d^2\boldsymbol{\sigma}\mathbf{S}$ is the radiated power.

For a particle in uniform motion, $\mathbf{x}(t) = \mathbf{v}t$ we have $|\mathbf{x}-\mathbf{v}t_r| = c(t-t_r)$, square and solve gives, $t_r = \frac{c^2t-\mathbf{x}\cdot\mathbf{v}-\sqrt{(c^2t-\mathbf{x}\cdot\mathbf{v})^2-(c^2-\mathbf{v}^2)(c^2t^2-\mathbf{x}^2)}}{c^2-\mathbf{v}^2}$ and $A^0(\mathbf{x},t) = \frac{e}{|\mathbf{R}|\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}$. Where $\mathbf{R} = \mathbf{x}-\mathbf{v}t$ is the vector between the particles current position and observer and θ the angle between motion and that. Strongest for $\theta = \pm\pi/2$. It gives, $\mathbf{E} = \frac{e\mathbf{R}}{R^3}\frac{1-\frac{v^2}{c^2}}{(1-\frac{v^2}{c^2}\sin^2\theta)^{3/2}}$ and $\mathbf{B} = \frac{\mathbf{v}}{c}\times\mathbf{E}$.

Radiation fields

$(\mathbf{B}\times\hat{\mathbf{r}})\times\hat{\mathbf{r}} = -\mathbf{B} + (\hat{\mathbf{r}}\cdot\mathbf{B})\hat{\mathbf{r}}$
$\hat{\mathbf{r}} = \frac{\mathbf{x}}{|\mathbf{x}|}$

Expand the $\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{|\mathbf{x}|(1+\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^2}+\dots)}$, and $|\mathbf{x}-\mathbf{x}'| = |\mathbf{x}|(1-\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^2}+\dots)$. Then, $j^i(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c) = j^i(\mathbf{x}',(t_0 = t-\frac{\mathbf{x}}{c}) + \frac{\mathbf{x}\cdot\mathbf{x}'}{c|\mathbf{x}|}+\dots)$ $\sim j^i(\mathbf{x}',t_0) + \dot{j}^i(\mathbf{x}',t_0)\frac{\mathbf{x}\cdot\mathbf{x}'}{x|\mathbf{x}|} + \ddot{j}^i(\mathbf{x}',t_0)O(\frac{\mathbf{x}\cdot\mathbf{x}'}{c|\mathbf{x}|}^2)$ then potential becomes,

$A^0(\mathbf{x},t) = \frac{1}{|\mathbf{x}|}Q(t_0) + \frac{\mathbf{x}\cdot\vec{d}(t_0)}{|\mathbf{x}|^3} + \frac{\mathbf{x}\cdot\dot{\vec{d}}(t_0)}{c|\mathbf{x}|^2} + \dots$
$\vec{d} = \frac{1}{c}\int d^3\mathbf{x}'\mathbf{x}'j^0(\mathbf{x}',t_0)$

$\vec{A}(\mathbf{x},t) = \frac{\dot{\vec{d}}(t_0)}{c|\mathbf{x}|} + \dots$

$\nabla\vec{j} = -\frac{\partial\rho}{\partial t}$

$|\mathbf{x}| \gg l$
$\lambda \gg l$

Dipole non-rel radiation, $\mathbf{E} = \frac{1}{c^2|\mathbf{x}|}[\ddot{\vec{d}}(t_0) + \hat{\vec{r}}(\ddot{\vec{d}}(t_0)\cdot\hat{\vec{r}})]$, where $\ddot{\vec{d}} = -\hat{z}qa\omega^2\sin\omega t_0$, and $\mathbf{B} = \hat{\vec{r}}\times\mathbf{E}$. Gives $\vec{S} = \hat{\vec{r}}\frac{\sin^2\theta|\ddot{\vec{d}}(t_0)|^2}{4\pi c^3|\mathbf{x}|^2}$ and θ is between $\hat{z}$ and $\hat{\vec{r}}$. Level curves of energy are two lobes, power radiated from antennas usually. Power goes like, $P = \frac{2}{3c^3}\ddot{\vec{d}}^2$ on average $< P >= \frac{q^2a^2\omega^4}{3c^3}$, with $\sin^2\omega t \to \frac{1}{2}$.

Waves

Vaccum equations in Coulomb: $A^0 = 0$
$\nabla\cdot\vec{A} = 0$
$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta)\vec{A} = 0$

Lorentz: $\partial_i A^i = 0$
$\Box A^i = 0$

$f(\mathbf{x}) = \int_{-\infty}^{\infty}\frac{d\vec{k}}{(2\pi)^3}\tilde{f}(\vec{k})e^{i\vec{k}\cdot\mathbf{x}}$
$\tilde{f}(\vec{k}) = \int_{-\infty}^{\infty}\frac{d\mathbf{x}}{(2\pi)^3}f(\mathbf{x})e^{i\mathbf{x}\cdot\vec{k}}$

$\delta(\vec{p}-\vec{k}) = \int_{-\infty}^{\infty}d\mathbf{x}e^{i(\vec{p}-\vec{k})\cdot\mathbf{x}}$

$\vec{A}(\mathbf{x},t) = \int_{-\infty}^{\infty}\frac{d\vec{k}}{(2\pi)^3}\tilde{A}(\vec{k},t)e^{i\vec{k}\cdot\mathbf{x}}$

$\nabla\cdot\vec{A} = 0 \implies \vec{k}\cdot\tilde{A} = 0$

$\hat{\vec{r}} = \frac{\mathbf{x}}{|\mathbf{x}|}$
$\vec{k} = \frac{\omega}{c}\hat{\vec{r}}$

$\vec{A} = \vec{A}_0e^{-i\omega(t-\frac{|\mathbf{x}|}{c})} = \vec{A}_0e^{i(\vec{k}\cdot\mathbf{x}-\omega t)}$
$\mathbf{E} = ik\vec{A} = \vec{\beta}e^{-i\alpha}e^{i(\vec{k}\cdot\mathbf{x}-\omega t)}$
$\mathbf{B} = i\vec{k}\times\vec{A} = \frac{\vec{k}}{|\vec{k}|}\times\mathbf{E}$

Most general, $\vec{A}(\mathbf{x},t) = \int\frac{d^3\vec{k}}{(2\pi)^3}[\vec{\beta}^*(-\vec{k})e^{i(\vec{k}\cdot\mathbf{x}+\omega t)} + \vec{\beta}(\vec{k})e^{i(\vec{k}\cdot\mathbf{x}-\omega t)}]$. Taking $\vec{\beta}(\vec{k}) = \vec{\beta}\delta^{(3)}(\vec{k}-\vec{p})(2\pi)^3$ we get $\vec{A}(\mathbf{x},t) = \vec{\beta}[e^{-i\vec{p}\cdot\mathbf{x}+i\omega t)} + e^{i\vec{p}\cdot\mathbf{x}-\omega t}] = \vec{\beta}\cos\omega t - \vec{p}\cdot\mathbf{x}$, with $\vec{p}\cdot\vec{\beta} = 0$ then $\mathbf{E} = \vec{\beta}$.

$F_{0\alpha} = E_\alpha$
$F_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma}B_\gamma$
$F^{ij}F_{ij} = \frac{1}{2}(B^2-E^2)$
$F^{ij}F^{kl}\epsilon_{ijkl} = \mathbf{E}\cdot\mathbf{B}$
$\int_V\nabla\cdot F dV = \oint_{\partial V}F\cdot ndS$
$\int_S\nabla\times F\cdot dS = \oint_{\partial S}F\cdot dr$
$S = -mc\int ds - \frac{e}{c}\int dsu^iA_i - \frac{1}{8\pi c}\int d^3\mathbf{x}(cdt)(B^2-E^2)$
$\frac{d}{dt}(\varepsilon_{em}+\varepsilon_{kin}) = -\oint_{\partial V}d^2\vec{\sigma}\cdot\vec{S}$
$\vec{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}$ = eng through perp area per time (energy flux)
$\frac{\mathbf{E}^2+\mathbf{B}^2}{8\pi}$ = eng density in EM
$\frac{d}{dt}\varepsilon_{kin} = \vec{j}\cdot\mathbf{E}$

$$\Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Lagrangian and Hamiltonian Equations
$H(p,q) = p\cdot v(p) - L$, where $p = \frac{\partial L}{\partial v}$
$\dot{q} = \frac{\partial H}{\partial p} = \{q,H\}$
$\dot{p} = -\frac{\partial H}{\partial q} = \{p,H\}$
$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$
$\{f,H\} = \sum\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}$

Maths
$Ax^2+Bx+C = A(x+\frac{B}{2A})^2 + (C-\frac{B^2}{4})$
$\sinh^-(\frac{x}{a}) = \ln\left(\frac{x}{a}+\frac{1}{a^2}\sqrt{x^2+a^2}\right)$
$= \int\frac{du}{\sqrt{u^2+a^2}}$
$\cosh^-(\frac{x}{a}) = \ln\left(\frac{x}{a}+\frac{1}{a^2}\sqrt{x^2-a^2}\right)$
$= \int\frac{du}{\sqrt{u^2-a^2}}$
$\tanh^-(\frac{x}{a}) = \frac{1}{2}\ln\left(\frac{a+x}{a-x}\right)$
$= \int\frac{adu}{a^2-u^2}$
$\pi = \int_a^b\frac{dx}{\sqrt{(b-x)(x-a)}}$
$\int\frac{du}{\sqrt{a^2-u^2}} = \sin^-(\frac{x}{a})$
$\int\frac{du}{a^2+u^2} = \frac{1}{a}\tan^-(\frac{u}{a})$
$\int\frac{du}{u^2(a^2-u^2)} = \frac{u\tanh^-(\frac{u}{a})-a}{a^3u}$
$\int\frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{(a-u)(a+u)}}{a^2u}$
$\int\frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u}$
$a\times b\times c = (a\cdot c)b - (a\cdot b)c$

Biancci Identities $\epsilon^{iklm}\partial_kF_{lm} = 0$
$i = 0 \implies \nabla\cdot\mathbf{B} = 0$
$i = \alpha \implies \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0$
$\frac{d}{dt}(m\gamma\vec{v}) = e(\mathbf{E}+\vec{v}\times\mathbf{B}/c)$
$\vec{p} = \varepsilon\frac{\vec{v}}{c}$
$E'_x = E_x$
$E'_y = \gamma(E_y-\beta B_z)$
$E'_z = \gamma(E_z+\beta B_y)$
$B'_x = B_x$
$B'_y = \gamma(B_y+\beta E_z)$
$B'_z = \gamma(B_z-\beta E_y)$
Lorentz two moving particles goes to: $F = e_1(1-\beta)E_2 + e_1/c^2V(V\cdot E_2)$