Gravitational Radiation

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1 Introduction

Gravitational waves (GW)s are a resulting prediction of general relativity where accelerating objects give off radiation that propagate through the spacetime. To date they have yet to be directly observed, though there is strong indirect evidence of their existence inferred from the changes in binary neutron star system periods. Specifically, the PSR B1913+16 pulsar, or Hulse-Taylor binary system, exhibit a decay in orbital period very consistent with the theoretical prediction due to GW emission. See figure 1.

Detectors will be able to measure GWs in the future. They are predicted by all causal forms of general relativity, and it is only a matter of time and observational advancement. They will provide a unique window into our past and a key to future astronomy observations. Where light is obstructed by matter in our universe, GWs will propagate through passing us information from events, such as the big bang, that would be nearly impossible otherwise. The discovery of GWs will herald a new age in astronomy, opening a door for new experiments, and prompting funding from interested investors.

LISA, a five-million kilometre interferometer in space will be able to detect GWs from binary systems in the Milky Way, "or something else is wrong". There are so many such sources of GWs at various frequencies that LISA at first will appear to be measuring stochastic noise. LISA however will overcome this. She will have a way of combining her six data streams in a manner such that it becomes insensitive to GWs, allowing astronomers to isolate instrumental noise.

Some current ground interferometer-based detectors are: LIGO: which are three 4km detectors separated by 3000km; Virgo: a French-Italian effort, similar to LIGO, with higher low-frequency sensitivity; GEO600: a German-English effort, 600m, and using advanced interferometer techniques achieves multi-km sensitivity; TAMA300: based near Tokyo, is 300m long and more of an experimental testing facility; and, ACIGA: based in Australia, soon to be constructed, and will be a multi-km interferometer.

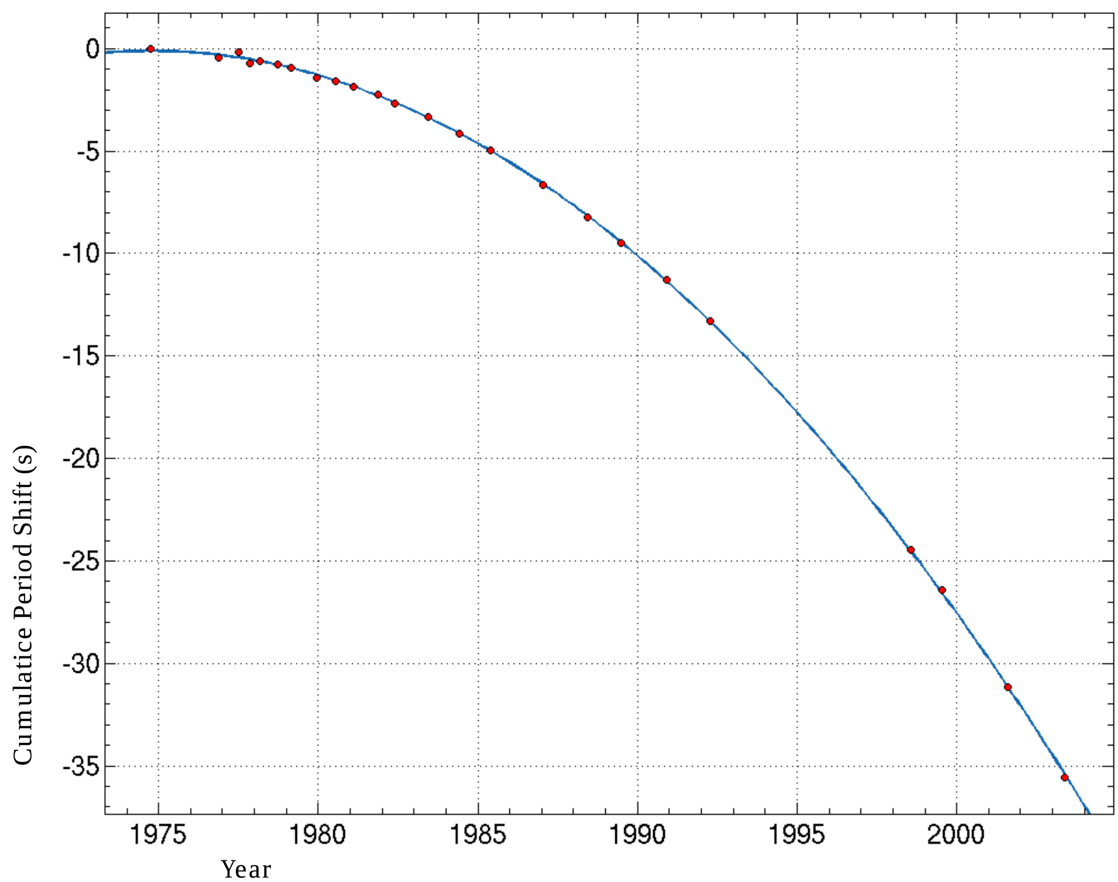

Figure 1: Shown is the decrease in the cumulative decay of the period of the PSR B1913+16 pulsar as it orbits a centre of mass with another star. The line indicates the general relativistic prediction of decay due to GWs.

There is actually quite the controversy over the history of the formulation of GWs. Indeed, in 1936 Einstein, who had recently written a paper with Nathan Rosen, wrote to Max Born,

> Together with a young collaborator, I arrived at the interesting result that gravitational waves do not exist, though they had been assumed a certainty to the first approximation. This shows that the non-linear general relativistic field equations can tell us more or, rather, limit us more than we have believed up to now.

It is surprising that Einstein would take this perspective so quickly since GWs were one of Einsteins first predictions of general relativity. Einstein submitted his paper to the Physical Review, and promptly received a critical referees report. John T. Tate, the editor, sent in response to Einstein, "I would be glad to have your reaction to the various comments and criticisms the referee has made.", probably with not a small amount of amusement. Einstein, instead of replying to the referee comments, wrote to Tate,

> Dear Sir,
>
> We (Mr. Rosen and I) had sent you our manuscript for publication and had not authorized you to show it to specialists before it is printed. I see no reason to address the - in any case erroneous - comments of your anonymous expert. On the basis of this incident I prefer to publish the paper elsewhere.
>
> respectfully,
>
> P.S. Mr. Rosen, who has left for the Soviet Union, has authorized me to represent him in this matter.

Of course Einstein's paper was not accepted by Tate who refused to throw the editorial procedure out the window. For more discussion on the history see a report by Daniel Kennefick [6].

GWs are a result of linearizing the Einstein field equations (EFE)s locally. We shall do precisely this in a gauge invariant manner, under the assumption of a near flat metric that asymptotically becomes flat. We shall then choose a gauge analogous with the Lorentz gauge of electromagnetic theory and derive the solution in a vacuum space time. We shall show in a gauge-invariant manner that the gravity waves are contained within a transverse-traceless (TT) piece of the metric perturbation. We will discuss sources of gravity waves from a slow-moving massive source where the local spacetime is not nearly flat, deriving the quadrapole formula. We will discuss the measurements made by a GW detector and derive a fractional stretching of the spacetime.

2 The Linearized Einstein Field Equation

We will now go through the derivation of linearized gravity, which can be found in any number of books on general relativity; James Hartle [5] for example. In linearized theory the metric tensor is locally written as,

$$g_{ab} = \eta_{ab} + h_{ab} \tag{1}$$

$$||h_{ab}|| \ll 1 \tag{2}$$

$$h_{ab} \to 0 \text{ as } r \to \infty, \tag{3}$$

where η_{ab} = diag(-1,1,1,1) is the Minkowski metric, and h_{ab} is called the metric perturbation. Condition 2 means that the magnitude of non-zero elements are much less than one. It implies that we have a weak gravitational field. Our coordinate system is constrained to quasi-Cartesian so as to preserve the form of η_{ab}. We can still make small changes in our coordinates that leave η_{ab} unchanged but change the functional form of the metric perturbation, h_{ab}. In this setting, the metric perturbation transforms as a tensor under Lorentz transformations but not under general coordinate transformations.

The smallness of the perturbation[1] means in all our derivations we need only keep terms linear in h_{ab} and its derivatives, and discard higher terms. As a consequence we can raise and lower indices with η_{ab}. We will see that h_{ab} contains our GWs and other radiation terms, which will turn out to be artifacts of the gauge freedom.

[1] We could rewrite our perturbation as $h_{ab} \to \epsilon h_{ab}$ and then discard all terms of order $O(\epsilon^2)$

A quick aside, since we are neglecting higher order terms, our theory is no longer covariant as stated above. Having said that, if we treat our approximate metric tensor as a true tensor we can show that if we consistently drop high order terms then we can treat this approximate tensor as a true tensor. Take an exact solution of the EFEs to be $\bar{g}_{ab}$, then the difference between the two, $\bar{g}_{ab} - g_{ab} \sim O(||h||^2_{ab})$. Treating the approximate tensor at a true tensor we go to a new frame and look at the difference, $\bar{g}'_{ab} - g'_{ab} = \frac{\partial x^\alpha}{\partial x'^a}\frac{\partial x^\beta}{\partial x'^b}(\bar{g}_{\alpha\beta} - g_{\alpha\beta}) \sim O(||h_{ab}||^2)$. So we see that by treating the approximate tensor as a true tensor we make at most a second order error, and if we consistently drop these higher order terms we can treat g_{ab} as a tensor to good measure.

We can now calculate the Christoffel symbols. We get,

$$\begin{aligned} \Gamma^a{}_{bc} &= \frac{1}{2} g^{ae}(g_{eb|c} + g_{ce|b} - g_{bc|e}) \\ &= \frac{1}{2}\eta^{ae}(h_{eb|c} + h_{ce|b} - h_{bc|e}) \\ &= \frac{1}{2}(h^a{}_{b|c} + h^a{}_{c|b} - h_{bc}{}^{|a}), \end{aligned} \tag{4}$$

We have used the convention that $A_{|a} \equiv \frac{\partial A}{\partial x^a} \equiv \partial_a A$ are all equivalent. We then work out the linearized Riemann curvature tensor,

$$\begin{aligned} R^a{}_{bcd} &= \Gamma^a{}_{bd|c} - \Gamma^a{}_{bc|d} + \Gamma^a{}_{ic}\Gamma^i{}_{bd} - \Gamma^a{}_{id}\Gamma^i{}_{bc} \\ &= \Gamma^a{}_{bd|c} - \Gamma^a{}_{bc|d} \\ &= \frac{1}{2}(h^a{}_{b|dc} + h^a{}_{d|bc} - h_{bd}{}^{|a}{}_{|c} - h^a{}_{b|cd} - h^a{}_{c|bd} + h_{bc}{}^{|a}{}_{|d} \\ &= \frac{1}{2}(h^a{}_{d|bc} - h_{bd}{}^{|a}{}_{|c} - h^a{}_{c|bd} + h_{bc}{}^{|a}{}_{|d}), \end{aligned} \tag{5}$$

where we dropped the quadratic Christoffel terms in the first line since they would contain second order perturbation terms. Next we contract on the first and third indices, and relabel to get the Ricci tensor,

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ &= \frac{1}{2}(h^c{}_{b|ac} - h_{ab}{}^{|c}{}_{|c} - h^c{}_{c|ab} + h_{ac}{}^{|c}{}_{|b}) \\ &= \frac{1}{2}(h^c{}_{b|ac} - \Box h_{ab} - h_{|ab} + h_{ac}{}^{|c}{}_{|b}), \end{aligned} \tag{6}$$

where $\Box = \partial_c \partial^c = \nabla^2 - \partial_t^2$ is the wave operator, and $h = h^c{}_c$ is the trace of h_{ab}. Contracting once more we get the Ricci scalar,

$$\begin{aligned} R = R_d{}^d &= \frac{1}{2}(h^{cd}{}_{|dc} - \Box h_d{}^d - h_{|d}{}^{|d} + h_{dc}{}^{|cd}) \\ &= \frac{1}{2}(h^{cd}{}_{|dc} - \Box h - \Box h + h_{dc}{}^{|cd}) \\ &= h_{cd}{}^{|dc} - \Box h \end{aligned} \tag{7}$$

We now assemble the Einstein tensor,

$$
\begin{aligned}
G_{ab} &= R_{ab} - \frac{1}{2} g_{ab} R \\
&= \frac{1}{2}\left(h^{c}{}_{b|ac} - \Box h_{ab} - h_{|ab} + h_{ac}{}^{|c}{}_{|b} - (\eta_{ab} + h_{ab})(h_{cd}{}^{|dc} - \Box h)\right) \\
&= \frac{1}{2}\left(h^{c}{}_{b|ac} - \Box h_{ab} - h_{|ab} + h_{ac}{}^{|c}{}_{|b} - \eta_{ab} h_{cd}{}^{|dc} + \eta_{ab}\Box h\right) \qquad (8)
\end{aligned}
$$

Equation 8 is ungainly, but with a simple redefinition of variables it can be simplified. Define,

$$
\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2}\eta_{ab} h. \qquad (9)
$$

This is called the trace reverse metric since, $\bar{h}^{c}_{c} = -h$. We then rearrange equation 9 to get $h_{ab} = \bar{h}_{ab} - \frac{1}{2}\eta_{ab}\bar{h}$ which we substitute into 8.

$$
\begin{aligned}
G_{ab} &= \frac{1}{2}\left((\bar{h}^{c}{}_{b} - \frac{1}{2}\eta^{c}{}_{b}\bar{h})_{|ac} - \Box(\bar{h}_{ab} - \frac{1}{2}\eta_{ab}\bar{h}) + \bar{h}_{|ab} + (\bar{h}_{ac} - \frac{1}{2}\eta_{ac}\bar{h})^{|c}{}_{|b} - \eta_{ab}(\bar{h}_{cd} - \frac{1}{2}\eta_{cd}\bar{h})^{|dc} - \eta_{ab}\Box\bar{h}\right) \\
&= \frac{1}{2}\left(\bar{h}^{c}{}_{b\,|ac} - \frac{1}{2}\eta^{c}{}_{b}\bar{h}_{|ac} - \Box\bar{h}_{ab} + \Box\frac{1}{2}\eta_{ab}\bar{h} + \bar{h}_{|ab} + \bar{h}_{ac}{}^{|c}{}_{|b} - \frac{1}{2}\eta_{ac}\bar{h}^{|c}{}_{|b} - \eta_{ab}\bar{h}_{cd}{}^{|dc} + \eta_{ab}\frac{1}{2}\eta_{cd}\bar{h}^{|dc} - \eta_{ab}\Box\bar{h}\right) \\
&= \frac{1}{2}\left(\bar{h}^{c}{}_{b\,|ac} - \Box\bar{h}_{ab} + \bar{h}_{ac}{}^{|c}{}_{|b} + \eta_{ab}\bar{h}_{cd}{}^{|dc}\right). \qquad (10)
\end{aligned}
$$

This looks like a nicer form to work with. Equation 10 with the momentum-energy tensor, T_{ab}, yields our EFEs[2],

$$
\begin{aligned}
G_{ab} &= 8\pi T_{ab} \\
\Longrightarrow \bar{h}^{c}{}_{b\,|ac} - \Box\bar{h}_{ab} + \bar{h}_{ac}{}^{|c}{}_{|b} + \eta_{ab}\bar{h}_{cd}{}^{|dc} &= 16\pi T_{ab} \qquad (11)
\end{aligned}
$$

Equation 11 is the general form of our linearized EFEs.

3 Gauge Transformations

Suppose in our spacetime, equipped with manifold and metric (M, g_{ab}), there is a diffeomorphism ϕ which takes $(M, g_{ab}) \to (M, \phi^{*} g_{ab})$. The physical spacetime remains unchanged by this diffeomorphism. Let us now consider a one-parameter family of spacetimes representing the same physical spacetimes. In particular say that the first order perturbation of the metric, $h_{ab} = \frac{d g_{ab}(\lambda)}{d\lambda}$ as $\lambda \to 0$, be physically equivalent after the diffeomorphism, $h'_{ab} = \frac{d\phi^{*}_{\lambda} g_{ab}(\lambda)}{d\lambda}$ as $\lambda \to 0$. That is, merely state the obvious, that $(M, g_{ab}(\lambda))$ and $(M, \phi^{*}_{\lambda} g_{ab}(\lambda))$ represent the same physical spacetimes.

Say the diffeomorphism, ϕ_{λ} is generated by a motion of the coordinates,

$$
x'^{a} = x^{a} + \lambda\chi^{a} \implies \frac{\partial x^{a}}{\partial x'^{b}} = \delta^{a}{}_{b} - \lambda\chi^{a}{}_{|b} \qquad (12)
$$

We now consider two points in the manifold $A, B \in M$. In applying our diffeomorphism we will have two

[2]Note the lack of the common κ scalar comes from a choice of units

changes to consider: The change of the physical components of the spacetime, and the intrinsic coordinate transformation-induced changes. We consider the value of the metric at B, $g_{B\,ab}$, via first-order Taylor expansion, to include both physical- and coordinate-induced changes for infinitesimal motions,

$$g_{B\,ab} = g_{A\,ab} + \lambda g_{A\,ab|c}\chi^{c}. \tag{13}$$

We then consider the metric upon dragging it via coordinate transformation 12. This dragging will inherently leave physical components of the metric untouched,

$$\begin{aligned} g_{A\to B\,ab} &= (\delta^{\alpha}{}_{a} - \lambda\chi^{\alpha}{}_{|a})(\delta^{\beta}{}_{b} - \lambda\chi^{\beta}{}_{|b})g_{A\,\alpha\beta} \\ &= g_{A\,ab} - \lambda(g_{A\,a\beta}\chi^{\beta}{}_{|b} + g_{A\,\alpha b}\chi^{\alpha}{}_{|a}) + O(\lambda^2). \end{aligned} \tag{14}$$

We then have that for an infinitesimal motion of the coordinate system, that the purely coordinate transformation-induced change to the metric is,

$$\lim_{\lambda\to 0}\frac{g_{B\,ab} - g_{A\to B\,ab}}{\lambda} = g_{A\,ab|c}\chi^{c} + g_{A\,a\beta}\chi^{\beta}{}_{|b} + g_{A\,\alpha b}\chi^{\alpha}{}_{|a} = \mathfrak{L}_{\chi}g_{ab}. \tag{15}$$

Equation 15 says that a small change in our coordinates, which is merely a gauge-transformation, introduces a gauge freedom given by the Lie derivative[3] of the metric tensor along the motion of the change. Thus our first-order perturbations are physically equivalent if,

$$\begin{aligned} h'_{ab} &= h_{ab} - \mathfrak{L}_{\chi}g_{ab}, \quad g_{ab} \equiv g_{ab}(\lambda = 0) \\ &= h_{ab} - g_{ab|c}\chi^{c} - g_{a\beta}\chi^{\beta}{}_{|b} - g_{\alpha b}\chi^{\alpha}{}_{|a} \\ &= h_{ab} - \chi_{a|b} - \chi_{b|a} \end{aligned} \tag{16}$$

where the second term in the second line vanishes for derivatives of g_{ab} with $\lambda = 0$. Equation 16 tells us how the metric perturbation changes under a gauge transformation. Note the similarity with a gauge transformation in electromagnetism, $A' = A - \nabla\chi$. Now we can continue with GWs. For more information on Manifolds, diffeomorphisms, and gauge transformations check out R. Wald [8].

4 Vacuum Solutions in Transverse-Traceless (TT) Gauge

Before we impose any gauge let's look at how, from equation 16, the trace-reversed metric perturbation changes,

$$\begin{aligned} \bar{h}'_{ab} &= h'_{ab} - \frac{1}{2}\eta_{ab}h' \\ &= (h_{ab} - \chi_{a|b} - \chi_{b|a}) - \frac{1}{2}\eta_{ab}(h - \chi^{c}{}_{|c} - \chi_{c}{}^{|c}) \\ &= \bar{h}_{ab} - 2\chi_{(a|b)} + \eta_{ab}\chi_{c}{}^{|c} \end{aligned} \tag{17}$$

We now choose a class of gauge commonly used when studying radiation. We choose the gauge condition,

[3] We then realize something that should feel intuitive: If we move our coordinates along a line of symmetry our Lie derivative of the metric vanishes, hence our world will remain unchanged, and we will not introduce any gauge freedom.

$\bar{h}_{ab}{}^{|a} = 0$, which is analogous with the Lorentz gauge in electromagnetism where the electromagnetic potential satisfies $A_a{}^{|a} = 0$. We suppose that our metric perturbation is not in Lorentz gauge. What conditions must χ^a satisfy in order to impose it? Applying our gauge to 17 we find,

$$\begin{aligned}\bar{h}'_{ab}{}^{|a} &= (\bar{h}_{ab} - 2\chi_{(a|b)} + \eta_{ab}\chi_c{}^{|c})^{|a} \\ &= \bar{h}_{ab}{}^{|a} - \chi_{a|b}{}^{|a} - \chi_{b|a}{}^{|a} + \eta_{ab}\chi_c{}^{|ca} \\ &= \bar{h}_{ab}{}^{|a} - \Box\chi_b\,,\end{aligned} \tag{18}$$

hence any metric perturbation can be put into Lorentz gauge by solving the inhomogeneous wave equation, $\bar{h}_{ab}{}^{|a} = \Box\chi_b$. This wave equation can always be solved given our boundary conditions of asymptotic flatness. Thus we take $\bar{h}_{ab}$ to be in Lorentz gauge. Selecting this gauge has brought our gauge freedom, given by 15, from 4 freely specifiable functions of 4 variables, to 4 functions of 4 variables that satisfy the homogeneous wave equation, $\Box\chi_b = 0$.

Returning to the Einstein tensor, equation 10, we find that in this gauge it reduces to,

$$G_{ab} = -\frac{1}{2}\Box\bar{h}_{ab}\,, \tag{19}$$

which in a vacuum spacetime yields the EFE,

$$\Box\bar{h}_{ab} = 0 \tag{20}$$

and has superposition of planar wave solutions (as in electromagnetism),

$$\bar{h}_{ab}(x^a) = \mathrm{Re}\int d^3k\; A_{ab}(k)e^{ik_a x^a}\,, \tag{21}$$

where $k^a = (\omega, k)$. The coefficients, A_{ab} depend only on the spacial wavevectors but are otherwise arbitrary. The Lorentz gauge imposes the constraint $k^a A_{ab} = 0$. These solutions are GWs. They are valid only locally where we took the metric to be nearly flat. Obviously, they are not so simple in the vicinity of highly distorted spacetimes, that is, they are not valid near GW generating objects.

We will now use h_{ab} as the trace reversed metric to drop the bar, which only serves to clutter.

We next impose several other constraints on the gauge freedom. We shall require that the metric perturbation is purely spatial,

$$h_{tb} = 0, \tag{22}$$

and traceless,

$$h_i^i = 0. \tag{23}$$

We are using $i, j, \ldots, n$ to be spatial indices. Taking the divergence of the spatial metric perturbation, by the Lorentz gauge, tells us that the spatial metric perturbation is transverse,

$$h_{ij}{}^{|j} = 0. \tag{24}$$

This is called the transverse traceless (TT) gauge. Does the TT gauge violate causality? Indeed, if one is familiar with electromagnetism it might appear so, since, following dogmatically with the analogy, the Coulomb potential in electromagnetism propagates instantaneously. However, if you knew this much about electromagnetism then you must know that in fact the current density can be decomposed into transverse and longitudinal components, and that the longitudinal one cancels out the instantaneous portion of the electric potential. Hence TT gauge in electromagnetism preserves causality. We don't wish to blindly follow this analogy though, so later on we will show a similar relation in our case.

Why would one choose the TT gauge? There are several reason. For one, it completely fixes the local gauge freedom. If we count up the constraints, which is an easy task for the reader, we count six, leaving four from our available ten for physical quantities. Thus, h^{TT}_{ab} contains only physical modes of radiation. It also reveals the familiar concept, that gravity waves are polarized, with two modes of polarization.

This fact can be seen by observing that $h^{TT}_{ab} = h^{TT}_{ab}(t-z)$ satisfies our wave equation $\Box h^{TT}_{ab}(t-z) = h^{TT}_{ab}{}''(t-z) - h^{TT}_{ab}{}''(t-z) = 0$. Our Lorentz condition then implies that $h^{TT}{}_{zb}{}^{|z} = 0$, which from our boundary conditions implies, $h^{TT}{}_{zb} = 0$. Hence the only non-zero components of the metric perturbations are the variations of x and y. Symmetry, and our traceless constraint, then imply,

$$h^{TT}_{xx} = -h^{TT}_{yy} = h_{+}(t-z) \tag{25}$$

$$h^{TT}_{xy} = h^{TT}_{yx} = h_{\times}(t-z) \tag{26}$$

which correspond to two waveforms of the GW.

Another reason why the TT gauge is attractive is because it yields a simple relation between the Riemann curvature tensor and the metric perturbation. From equation 5, and our TT constraints,

$$\begin{aligned} R_{itjt} &= \frac{1}{2}(h^{TT}{}_{it|tj} - h^{TT}{}_{tt|ij} - h^{TT}{}_{ij|tt} + h^{TT}{}_{tj|it}) \\ &= -\frac{1}{2} h^{TT}{}_{ij|tt} = -\frac{1}{2}\ddot{h}^{TT}{}_{ij} \end{aligned} \tag{27}$$

The other non-zero components of the Riemann tensor can be obtained from the Bianchi identities and Riemann symmetries.

For a globally vacuum spacetime one can always satisfy the TT gauge. We can see this by noting, from above, that the most general Lorentz gauge transformation obeys $\Box \chi^a = 0$, the general solution of which is,

$$\chi^a = \mathrm{Re} \int d^3k\, C^a(k) e^{i k_a x^a}. \tag{28}$$

Plugging relation 21 into equation 17, recalling that the coefficients in the integrand are functions of the spatial wavevectors only, we get,

$$A'_{ab} = A_{ab} - 2i C_{(a} k_{b)} + i \eta_{ab} C_c k^c. \tag{29}$$

Thus, if we can show that for every $k^a = (\omega, k)$ there exists a $C^a(k)$ which satisfies the TT constraints,

$$0 = \eta^{ab} A'_{ab} = \eta^{ab} A_{ab} + 2i C_c k^c \quad \text{Traceless,} \tag{30}$$

$$0 = A'_{tb} = A_{tb} - i C_t k_b - i C_b k_t + i \eta_{tb} C_c k^c \quad \text{Purely spatial,} \tag{31}$$

then we can always apply the TT gauge in a globally vacuum spacetime. Manipulating 30 we get $C_c k^c = \frac{i}{2}\eta^{ab} A_{ab}$, which we substitute into equation 31 to get,

$$0 = A_{tb} - i C_t k_b - i C_b k_t - \frac{1}{2}\delta^a{}_t A_{ab}$$

$$\implies \frac{1}{2} A_{tb} = i(C_t k_b + C_b k_t) \tag{32}$$

$$\implies C_t = \frac{A_{tt}}{4i\omega}. \tag{33}$$

Inserting relation 33 back into equation 32 we find,

$$\frac{1}{2} A_{tb} = \frac{A_{tt}}{4\omega} k_b + i C_b k_t$$

$$\implies C_k = \frac{2\omega A_{tk} - A_{tt} k_k}{4i\omega^2} \tag{34}$$

Equations 33 and 34 are solutions of the coefficients of the gauge transformation 28. È. Flanagan [4] gives solutions, with $l^a = (\omega, -k)$,

$$C_a = \frac{3A_{bc} l^b l^c}{8i\omega^4} k_a + \frac{\eta^{bc} A_{bc}}{4i\omega^4} l_a + \frac{1}{2i\omega^2} A_{ab} l^b \tag{35}$$

5 Global Solutions

It is not realistic to have a global vacuum spacetime, so we return to the problem of a non-zero stress-energy tensor. We revert to the point before establishing global TT gauge. Thus our metric perturbations now has gauge degrees of freedom, physical radiative degrees of freedom, and physical non-radiative degrees of freedom tied to the matter source. In general it is impossible to to solve the EFEs in TT gauge in this setting. We will show that it is possible to separate the metric perturbation into parts, each relating to one of those three types of freedom.

At this point we should clear the fog from our eyes, that we may have serendipitously acquired while working with the Lorentz gauge. In the Lorentz gauge we saw that each component of the metric perturbation obeyed a wave equation. In general, this need not be the case, and here we will now formulate a theory where physical observables are invariant of gauge. We will see that the physical radiative information is tied into a gauge invariant TT piece obeying a wave equation, while the other two can be formed into gauge invariant Poisson type equations.

The first analysis of this type seems to be by Sir Arthur Eddington [3] who was critical of Einstein's 1918 work, wherein he imposed radiative properties into the metric. Einstein concluded in that paper that only "transverse-transverse" pieces carried energy. We will follow the derivations of Bardeen [1], and the lecture notes of Bertschinger [2].

We start by decomposing the metric perturbation into several irreducible pieces. We still keep the condition of asymptotic flatness as $r \to \infty$. We define the quantities, ϕ, β_i, γ, H, ϵ_i, λ, and h^{TT}_{ij} which follow the equations,

$$h_{tt} = 2\phi, \tag{36}$$
$$h_{ti} = \beta_i + \gamma_{|i}, \tag{37}$$
$$h_{ij} = h^{TT}_{ij} + \frac{1}{3}H\,\delta_{ij} + \epsilon_{(j|i)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\lambda. \tag{38}$$

We also impose the constraints,

$$\beta_i{}^{|i} = 0 \quad \text{(transverse, 1 constraint)}, \tag{39}$$
$$\epsilon_i{}^{|i} = 0 \quad \text{(transverse, 1 constraint)}, \tag{40}$$
$$h^{TT}{}_{ij}{}^{|i} = 0 \quad \text{(transverse, 3 constraints)}, \tag{41}$$
$$\delta^{ij}h^{TT}_{ij} = 0 \quad \text{(traceless, 1 constraint)}, \tag{42}$$

along with the boundary conditions,

$$\gamma \to 0, \quad \epsilon_i \to 0, \quad \lambda \to 0, \quad \nabla^2\lambda \to 0 \tag{43}$$

as $r \to \infty$. Here $H = \delta^{ij}h_{ij}$ is the diagonal of the spatial part of the metric perturbation, and h^{TT} is in TT gauge containing our physical radiative degrees of freedom. In equation 37 β_i, and $\gamma_{|i}$ contain respectively transverse and longitudinal degrees of freedom. We examine the uniqueness of the decomposition for h_{ti} by taking the first derivative of equation 37,

$$\begin{aligned} h_{ti}{}^{|i} &= (\beta_i + \gamma_{|i})^{|i} \\ &= \nabla^2\gamma \quad \text{(applying constraint 39)}, \end{aligned} \tag{44}$$

and noting that by boundary condition 43 this has a unique solution. Similarly, we look at the second derivative of equation 38,

$$\begin{aligned} h_{ij}{}^{|ij} &= \left(h^{TT}_{ij} + \frac{1}{3}H\,\delta_{ij} + \epsilon_{(j|i)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\lambda\right)^{|ij} \\ &= h^{TT}{}_{ij}{}^{|ij} + \frac{1}{3}H\,\delta_{ij}{}^{|ij} + \frac{1}{3}H^{|ij}\delta_{ij} + \frac{1}{2}\epsilon_{j|i}{}^{|ij} + \frac{1}{2}\epsilon_{i|j}{}^{|ij} + \left(\nabla^2\nabla^2 - \frac{1}{3}\delta_{ij}{}^{|ij}\nabla^2\right)\lambda + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\lambda^{|ij} \\ &= \frac{1}{3}\nabla^2 H + \nabla^2\nabla^2\lambda + \nabla^2\nabla^2\lambda - \frac{1}{3}\nabla^2\nabla^2\lambda, \quad \text{(Using constraints 41, 40)} \\ \implies 5\nabla^2\nabla^2\lambda &= 3h_{ij}{}^{|ij} - \nabla^2 H, \end{aligned} \tag{45}$$

which, using our boundary condition 43, gives us a unique solution of λ. We can thus solve for ϵ_i by taking only the first derivative of equation 38, $h_{ij}{}^{|i} = \frac{1}{3}H_{|j} + \frac{1}{2}\nabla^2\epsilon_j + \nabla^2\lambda_{|j} - \frac{1}{3}\nabla^2\lambda_{|j}$, and solving $3\nabla^2\epsilon_j = 6h_{ij}{}^{|i} - 2H_{|j} - 4\nabla^2\lambda_{|j}$, which is again solvable by our boundary conditions 43.

Before we go any further let's count the number of free functions in our parametrization of the metric perturbation. We have 4 scalars (ϕ, γ, H, and γ), 6 rank-one tensor components (β_i, and ϵ_i), and 6 rank-two tensor components in our spatial, symmetric h^{TT}_{ij}. This gives us 16 free functions from equations 36–38, and 6 constraints in relations 39–42, or 10 free functions in total. This is rightly so as it is consistent with a four-dimensional, symmetric, rank-two tensor.

We would now like to investigate the gauge invariance of this unique decomposition. Let's parametrize our motion as $\chi_a = (\chi_t, \chi_i) = (A, B_i + C_{|i})$, with constraints, $B_i{}^{|i} = 0$ and $C \to 0$ as $r \to \infty$. Hence we see that B_i and C are, respectively, the transverse and longitudinal pieces of our gauge transformation. Following equation 16 our metric perturbation transforms like $h_{ab} \to h_{ab} - \chi_{a|b} - \chi_{b|a}$. We apply this to our decomposed form of our metric perturbation and find that each of the pieces transform like,

$$\phi \to \phi - \dot{A}, \tag{46}$$
$$\beta_i \to \beta_i - B_i, \tag{47}$$
$$\gamma \to \gamma - A - \dot{C}, \tag{48}$$
$$\epsilon_i \to \epsilon_i - 2B_i, \tag{49}$$
$$\lambda \to \lambda - 2C, \tag{50}$$
$$H \to H - 2\nabla^2 C, \tag{51}$$
$$h_{ij}^{TT} \to h_{ij}^{TT}. \tag{52}$$

Note to find the transformation rules for H and h_{ij}^{TT} one wacked both sides of the equation with a δ^{ij} and uses the constraints on B_i and ϵ_i. We see that we can rearrange these relations to form, in addition to h_{ij}^{TT}, three gauge invariant quantities,

$$\Phi \equiv -\phi + \dot{\gamma} - \frac{1}{2}\ddot{\lambda}, \tag{53}$$
$$\Theta \equiv \frac{1}{3}(H - \nabla^2 \lambda), \tag{54}$$
$$\Sigma_i \equiv \beta_i - \frac{1}{2}\dot{\epsilon}_i. \tag{55}$$

Of interest here is that in the Newtonian limit $\Phi = \frac{-1}{2}\Theta = \Phi_N$, the Newtonian potential, which implies that the Newtonian potential is gauge invariant. Let's quickly count the degrees of freedom: 1 from Φ, 1 from Θ, 3 from Σ_i, 6 from h_{ij}^{TT}, minus 1 from the constraint $\delta^{ij} h_{ij}^{TT} = 0$, minus 3 from $h^{TT}{}_{ij}{}^{|i} = 0$, gives us 7. We note that $\Sigma_i{}^{|i} = 0$ gives us 1 more constraint so we have in total 6 degrees of freedom. This is consistent with our notion that the general metric contains in total 10 degrees of freedom, 4 physical, and 6 gauge.

Before we look at the EFEs we do a similar decomposition on the stress-energy tensor, T_{ab}. We define the quantities, ρ, S_i, S, P, σ_{ij}, σ_i, and σ which follow a similar set of equations,

$$T_{tt} = \rho, \tag{56}$$
$$T_{ti} = S_i + S_{|i}, \tag{57}$$
$$T_{ij} = P\,\delta_{ij} + \sigma_{ij} + \sigma_{(j|i)} + \left(\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\sigma, \tag{58}$$

the constraints,

$$S_i{}^{|i} = 0 \quad 1 \text{ constraint}, \tag{59}$$
$$\sigma_i{}^{|i} = 0 \quad 1 \text{ constraint}, \tag{60}$$
$$\sigma_{ij}{}^{|i} = 0 \quad 3 \text{ constraints}, \tag{61}$$
$$\delta^{ij}\sigma_{ij} = 0 \quad 1 \text{ constraint}, \tag{62}$$

and the boundary conditions,

$$S \to 0, \quad \sigma_i \to 0, \quad \sigma \to 0, \quad \nabla^2\sigma \to 0 \tag{63}$$

as $r \to \infty$. Note that the scalars S, and σ are not the tensor scalar of their indicized counterparts. We can arbitrarily specify ρ, P, and S_i, and σ_{ij} and the energy conservation, $T_{ab}{}^{|a} = 0$, will give us S, σ, and σ_i, with the following equations,

$$\nabla^2 S = \dot{\rho}, \tag{64}$$
$$\nabla^2\sigma = -\frac{3}{2}P + \frac{3}{2}\dot{S}, \tag{65}$$
$$\nabla^2\sigma_i = 2\dot{S}_i. \tag{66}$$

We now return to the EFEs, given by equation 10, for the trace reversed metric perturbation, using equations 36–38, and writing them in terms of the gauge invariant observables.

$$G_{tt} = -\nabla^2\Theta, \tag{67}$$
$$G_{ti} = -\frac{1}{2}\nabla^2\Sigma_i - \dot{\Theta}_{|i}, \tag{68}$$
$$G_{ij} = -\frac{1}{2}\Box h_{ij}^{TT} - \dot{\Sigma}_{(j|i)} - \frac{1}{2}(2\Phi_{|ij} + \Theta_{|ij}) + \delta_{ij}\left(\frac{1}{2}\nabla^2(2\Phi + \Theta) - \ddot{\Theta}\right). \tag{69}$$

We equate these to the respective stress-energy components, and after quite a bit of substitution, and using the conservation relations 64–66, we arrive at,

$$\nabla^2\Theta = -8\pi\rho, \tag{70}$$
$$\nabla^2\Phi = 4\pi\left(\rho + 3P - 3\dot{S}\right), \tag{71}$$
$$\nabla^2\Sigma_i = -16\pi S_i, \tag{72}$$
$$\Box h_{ij}^{TT} = -16\pi\sigma_{ij}. \tag{73}$$

We immediately notice that only h_{ij}^{TT} obeys a wave equation while the other functions obey a Poisson-type equation. Notice that in a vacuum spacetime they become homogeneous equations; 5 Laplace equations, and 1 wave equation. Hence we see in a gauge-transparent manner that only the TT part of the metric perturbation carries physical radiation, and that picking a gauge that seems to make other components radiative actually does not affect this result.

(Misner, Thorne, and Wheeler) [7] show that far away the only time-varying portion of the metric is contained in h_{ij}^{TT}. Essentially, if we take an expansion in $1/r$ of the gauge invariant observables, we will find that at large distances the $O(1/r)$ term dominates, and that the coefficients of all except the h_{ij}^{TT} will simply be conserved linear momentum due to the conservation equations, $T_{ab}{}^{a} = 0$.

In order to calculate these gauge invariant observables we require knowledge of the metric perturbation globally, that is, they are non-local. This non-locality also brings into question the causality of the physical, non-radiative degrees of freedom. They are causal, which can be deduced by writing our gauge invariant Riemann tensor 5 in terms of the gauge invariant observables, and finding it takes the form of a wave-like equation.

Despite the fact we have non-local solutions, some detectors, LIGO for instance, can directly observe the Riemann tensor components,

$$R_{itjt} = -\frac{1}{2}\ddot{h}_{ij}^{TT} + \Phi_{|ij} + \dot{\Sigma}_{(i|j)} - \frac{1}{2}\ddot{\Theta}\delta_{ij}, \tag{74}$$

that is, combinations of the gauge invariant observables can be measured locally, but not individually. This makes this formalism less pragmatic, but it's still necessary, and interesting.

In general, we cannot split up the metric into radiative and non-radiative pieces. The short answer is that, locally, we cannot distinguish GWs from a time-varying gravitational field. A quick and dirty proof of this is that since we loose our boundary conditions, it is impossible to decompose vectors into purely longitudinal, and transverse components. For example, the vector $(x^2 - y^2)\partial_z$ is both longitudinal and transverse.

The main result from this section is that only the TT components contains physical, radiative degrees of freedom. Secondly, we have stated that the non-locality of this solution makes it impossible to calculate the metric perturbation through observations in finite region of space. There is a limit in which we can define GWs locally, in the case when the wavelengths associated with the GWs are much smaller than the length scales defining the background metric.

6 Generation of GWs from Slow-Motion Bodies

In this section we calculate the standard TT piece of the metric perturbation to get the quadrapole relation for the emitted radiation. Recall equation 11, in Lorentz gauge,

$$\Box h_{ab} = -16\pi T_{ab}. \tag{75}$$

This takes the form of a radiative field with source, $\Box f = s$, which can be solved via a Green's function which represents the field of a point source, or delta function source, $\Box G = \delta^4(x^a - x'^a)$. We have then that $f(x^a) = \int d^4x'^a G(x^a, x'^a) s(x'^a)$. For the wave operator, $\Box$, the Green's function is,

$$G(x^a, x'^a) = -\frac{\delta(t' - t_r)}{4\pi|x - x'|}. \tag{76}$$

$t_r = t - |x - x'|/c$ is called the retarded time, taking into account propagation of information. We apply this to equation 75 looking only at the spatial components since they contain the radiative information,

$$h_{ij} = 4\int d^3x' \frac{T_{ij}(t_r, x')}{|x - x'|} \tag{77}$$

$$\approx 4\int d^3x' \frac{T_{ij}(t - r/c, x')}{r} + O(1/r) \quad r = |x| \quad \text{at large distances} \tag{78}$$

$$= \frac{4}{r}\int d^3x' T_{ij}(t - r/c, x'). \tag{79}$$

We introduce fractional errors L/r, where L is the size of the source, which we neglect since the body is slow-moving in comparison with c. Equation 79 is a first order multipolar expansion and almost gives us the quadrapole expansion. We massage it as follows:

Break $T_{ab}{}^{|b} = 0$ into time and spatial components,

$$T_{tt}{}^{|t} + T_{ti}{}^{|i} = 0, \tag{80}$$

$$T_{ti}{}^{|t} + T_{ij}{}^{|j} = 0. \tag{81}$$

From this it follows that,

$$\begin{aligned} T_{ti}{}^{|ti} &= -T_{ij}{}^{|ji} \\ \text{LHS} &\rightarrow -(T_{tt}{}^{|t})^{|t} \\ \implies T_{tt}{}^{|tt} &= T_{ij}{}^{|ij}. \end{aligned} \tag{82}$$

Wack both sides with $x^k x^l$, and manipulate. Note, we can raise and lower the spatial indices at will with $\delta_i{}^j$.

$$\text{LHS:} \quad (T_{tt}\ x^k x^l)^{|tt}, \tag{83}$$

$$\text{RHS:} \quad (T_{ij} x^k x^l)^{|ij} - 2(T_{ki} x^l + T_{il} x^k)^{|i} + 2T^{kl}. \tag{84}$$

The reader is urged to carefully expand out equation 84 to see that it follows. This implies that,

$$T^{kl} = \frac{1}{2}(T_{tt}\ x^k x^l)^{|tt} - \frac{1}{2}(T_{ij} x^k x^l)^{|ij} + (T_{ki} x^l + T_{il} x^k)^{|i}. \tag{85}$$

We place this into our multipolar expansion, equation 79,

$$\implies h^{kl} = \frac{4}{r}\int d^3x' \frac{1}{2}(T_{tt}\ x^k x^l)^{|tt} - \frac{1}{2}(T_{ij} x^k x^l)^{|ij} + (T_{ki} x^l + T_{il} x^k)^{|i} \tag{86}$$

$$= \frac{2}{r}\int d^3x'(T_{tt} x^k x^l)^{|tt} \quad \text{Applying Gauss's Thrm. to the 2nd and 3rd terms,} \tag{87}$$

$$= \frac{2}{r}\frac{\partial^2}{\partial t^2}\int d^3x' \rho x^k x^l \quad \text{Noting } T_{tt} = \rho, \text{ the mass density.} \tag{88}$$

We define the second moment of the mass distribution as,

$$I^{ij} = \int d^3x' \rho x^i x^j \quad \text{with trace } I = I_i^i, \tag{89}$$

and then define the quadrapole moment tensor as,

$$\mathfrak{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I. \tag{90}$$

To finish the derivation we subtract out all non-TT pieces, that is, we remove all components parallel to the propagation direction $n = x/r$ and the trace, with the projection operator, $P_{ij} = \delta_{ij} - n_i n_j$.

$$h^T_{ij} = h_{kl} P^k_{\ i} P^l_{\ j} \quad \text{is now transverse,} \tag{91}$$

$$h^{TT}_{ij} = h_{kl} P^k_{\ i} P^l_{\ j} - \frac{1}{2} h_{kl} P^k_{\ i} P^{li} \quad \text{is now TT.} \tag{92}$$

Thus our final form of the quadrapole formula is, for a slow moving source,

$$h^{TT}_{ij}(t,x) = \frac{2}{r}\frac{\partial^2 \mathfrak{I}_{kl}}{\partial t^2}\left(P^k_{\ i} P^l_{\ j} - \frac{1}{2} P^k_{\ i} P^{li}\right). \tag{93}$$

This does not hold for bodies with fast internal velocities, like binary star systems for instance. In particular these do not work for systems with weak gravity, or where dynamics are dominated by self-gravity. This formula can be extended, and is shown in section 4.2 of Flanagan [4].

7 Fractional Stretching in Detectors

Consider the equations of motion for a particle following a geodesic,

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{\ bc} u^b u^c = 0, \tag{94}$$

written in terms of the proper time, τ. If we then consider the geodesic equation in terms of coordinate time, $t(\tau)$, instead of proper time, we can rewrite our geodesic equation using $\frac{dx^a}{d\tau} = \frac{dx^a}{dt}\frac{dt}{d\tau}$. We use the chain rule and rearrange to get,

$$\frac{d^2x^a}{dt^2} + \Gamma^a_{\ bc} \frac{dx^b}{dt}\frac{dx^c}{dt} = -\frac{dx^a}{dt}\left(\frac{\frac{d^2t}{d\tau^2}}{\left(\frac{dt}{d\tau}\right)^2}\right), \tag{95}$$

and in taking the non-relativistic limit, $x^i{}_{|t} \ll 1$, we can write the spatial equations of motion to good approximation as,

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{\ tt}, \tag{96}$$

by dropping all velocity terms.

We take our metric perturbation to be dominated by a GW in TT gauge. From our previous discussion we have that this Christoffel symbol is,

$$\Gamma^{i}{}_{tt} = \frac{1}{2}(2h^{TT\,i}{}_{t|t} - h^{TT}{}_{tt}{}^{|i}) \tag{97}$$

$$= 0 \quad \text{since } h^{TT}_{at} = 0 \text{ in TT}, \tag{98}$$

$$\implies \frac{d^2 x^i}{dt^2} = 0 \tag{99}$$

This does not mean that GWs have no effect. What it means is that the coordinates move with our GWs, or that the coordinates of the slowly, freely falling body do not change with GWs. This exemplifies why doing gauge invariant analysis is very important. Indeed, what happens is that proper separation changes.

Consider a particle on the x-axis at $z = 0$ with a GW in TT gauge propagating down the z-axis. The proper separation between two particles on the x-axis is then,

$$L = \int_0^{L_c} dx \sqrt{g_{xx}} \tag{100}$$

$$= \int_0^{L_c} dx \sqrt{1 + h^{TT}_{xx}(t, z = 0)} \tag{101}$$

$$\approx \int_0^{L_c} dx \left(1 + \frac{1}{2} h^{TT}_{xx}(t, z = 0)\right) \tag{102}$$

$$= L_c + L_c \frac{1}{2} h^{TT}_{xx}(t, z = 0), \tag{103}$$

where we used the fact that coordinate separation doesn't change in TT gauge. This implies a fractional change in proper length of,

$$\frac{\delta L}{L} \approx \frac{1}{2} h^{TT}_{xx}(t, z = 0) \tag{104}$$

This mean that h^{TT}_{xx} acts to cause strain. The magnitude, h, is often called the wave strain. This plays a key role in interferometers attempting to measure GWs. The accumulated phase change of a light ray which travels up and down an arm of length is given by,

$$\delta\phi = 4\pi \frac{\delta L}{\lambda}, \tag{105}$$

where δL is the change in mirror position relative to the beam-splitter, and λ is the photon wavelength.

One can derive a gauge invariant form of this relation using the geodesic deviation equation,

$$u^b \nabla_b (u^c \nabla_c L^a) = -R^a{}_{bcd}\,[z(\tau)]\, u^b L^c u^d, \tag{106}$$

where our geodesic is given by a curve $x^a(\tau) = z^a(\tau) + L^a(\tau)$. This analysis holds to first order in $L/\mathfrak{L}$, where $\mathfrak{L}$ is the length scale over which the curvature changes due to GWs. For ground based GW detectors L is a few kilometres and $\mathfrak{L} \sim 3000\,\text{km}$. For larger detectors like LISA, we must use different tools of analysis. Thus there is a trade off between GW sensitivity, and accuracy of this analysis. We can see from equation 105 that longer interferometers will exhibit greater changes in phase but at the cost of this analysis being rendered inaccurate due to significant $O((L/\mathfrak{L})^2)$ error.

8 Conclusion

We have derived linearized gravity and used it to show the existence of GWs in a vacuum spacetime. We further showed, in a gauge-invariant manner, that the GWs are contained within a TT piece of the metric perturbation. Since we did this globally, we discovered that we cannot use the gauge-invariant quantities in observations and must choose a gauge. We derived the quadrapole equation for the generation of GWs by a slow-moving source in TT gauge. Finally, we deduced the equation of fractional stretching of proper separation, and phase change in a GW interferometer detector.

In covering a basic groundwork on GWs, we realize that they form an interesting and wide ranging subject. There are likely many observational applications of GWs yet undiscovered. In any case, they hold a key to the future of astronomy.

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