Spinning Top Mechanics

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1 Introduction

Observe figure 1. $x_1 y_1 z_1$ are the coordinates of space in which the top lives. $x\,y\,z$ are the coordinates attached to the top's principle axes. Define the line of nodes, ON, as the intersection of the $x\,y$ plane and $x_1 y_1$ planes. Thus ON must be perpendicular to z and z_1 always. Since z_1 is fixed, a change in ϕ corresponds to a rotation of z about z_1, and ON rotates z_1, hence it measures the precession of the top about z. ON will remain fixed for a change in θ, hence we say θ measures the lean of the top withr espect to z_1. Finally, ON will remain fixed for a change in ψ and it corresponds to a rotation of the top about z, hence we call it the spin angle. These are summarized,

$$\phi \rightarrow \quad \text{precession,} \tag{1}$$
$$\theta \rightarrow \quad \text{lean,} \tag{2}$$
$$\psi \rightarrow \quad \text{spin.} \tag{3}$$

Note that N is defined to point in the direction such that (z_1, z, N) is right handed, or that $z_1 \times z$ points in the direction of N. We can then write the angular velocities of these angles as,

$$\omega_\phi = \dot{\phi}\,\hat{z}_1, \tag{4}$$
$$\omega_\theta = \dot{\theta}\,\hat{n}, \tag{5}$$
$$\omega_\psi = \dot{\theta}\,\hat{z}. \tag{6}$$

We wish to write these in the $x\,y\,z$ system by pro-

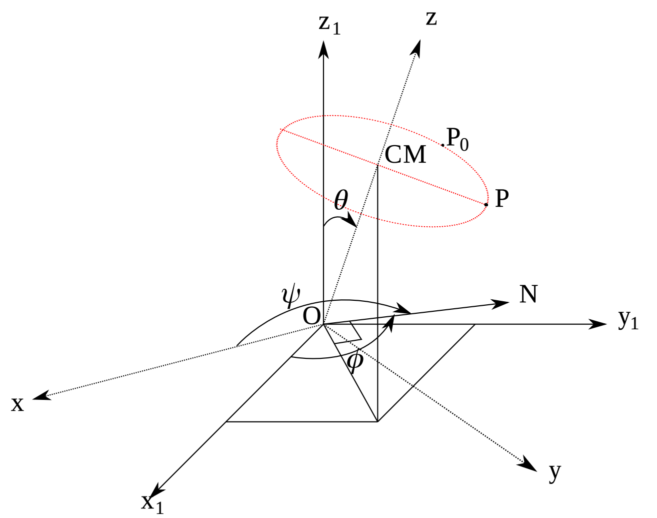

Figure 1: Shown is the schematics of a top. ON is called the line of nodes where $O\,x\,y$ and $O\,x_1 y_1$ intersect.

jecting the above equations 4–6 onto the $x\,y\,z$ basis. Looking at figure 2, we have that,

$$\begin{aligned}\hat{z}_1 &= \cos(\frac{\pi}{2} - \psi)\sin(\theta)\hat{x} + \sin(\frac{\pi}{2} - \psi)\sin(\theta)\hat{y} + \cos(\theta)\hat{z} \\ &= \sin(\psi)\sin(\theta)\hat{x} + \cos(\psi)\sin(\theta)\hat{y} + \cos(\theta)\hat{z}\end{aligned} \tag{7}$$

$$\begin{aligned}\hat{n} &= \cos(-\psi)\hat{x} + \sin(-\psi)\hat{y} \\ &= \cos(\psi)\hat{x} - \sin(\psi)\hat{y}.\end{aligned} \tag{8}$$

Using equations 7 and 8 in equations 4–6 we get that,

$$\omega_\phi \cdot \hat{x} = \dot{\phi}\sin(\psi)\sin(\theta) \tag{9}$$
$$\omega_\phi \cdot \hat{y} = \dot{\phi}\cos(\psi)\sin(\theta) \tag{10}$$
$$\omega_\phi \cdot \hat{z} = \dot{\phi}\cos(\theta) \tag{11}$$

$$\omega_\theta \cdot \hat{x} = \dot{\theta}\cos(\psi) \quad (12)$$
$$\omega_\theta \cdot \hat{y} = -\dot{\theta}\sin(\psi) \quad (13)$$
$$\omega_\theta \cdot \hat{z} = 0 \quad (14)$$

$$\omega_\psi \cdot \hat{x} = \omega_\psi \cdot \hat{y} = 0 \quad (15)$$
$$\omega_\psi \cdot \hat{z} = \dot{\psi} \quad (16)$$

Hence in the $x\,y\,z$ basis, which is fixed to the rotating top, we have the angular velocities to be,

$$\omega_x = \dot{\phi}\sin(\psi)\sin(\theta) + \dot{\theta}\cos(\psi) \quad (17)$$
$$\omega_y = \dot{\phi}\cos(\psi)\sin(\theta) - \dot{\theta}\sin(\psi) \quad (18)$$
$$\omega_z = \dot{\phi}\cos(\theta) + \dot{\psi} \quad (19)$$

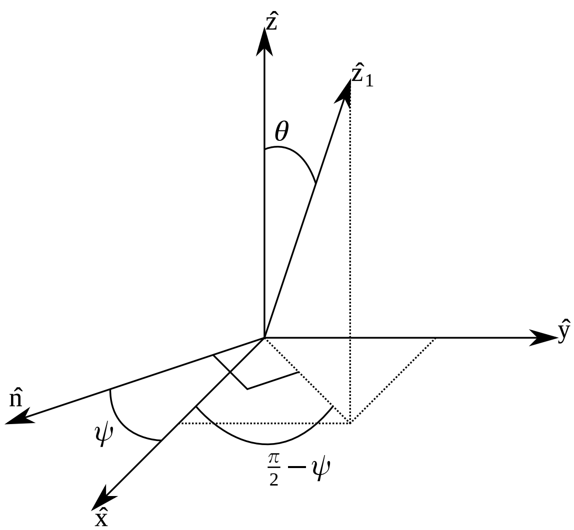

Figure 2: Shown is a different perspective of figure 1. $\hat{n}$ is the unit direction of ON.