Hyperbolicity of the Curve Complex: 3 Approaches

Kunal Chawla

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Contents

1 Introduction

Definition 1. *Let S be a surface with complexity $3g - 3 + p \geq 2$. The curve complex $\mathcal{C}(S)$ is the simplical complex whose 0-cells are isotopy classes of essential closed curves, and k-cells are sets of k isotopy classes which can be realized disjoint*

We also make use of the *arc and curve graph*, whose vertex set is the set of isotopy classes of arcs and curves in $(S, \partial S)$, with edges defined similarly. If S is a sphere with $p \leq 3$ punctures, a torus with at most one puncture, or a four-holed sphere, we say that S is sporadic. The curve graph for these are defined slightly differently, and we don't consider them in this report.

Definition 2. *Let X be a geodesic metric space. We say that X is δ-hyperbolic if there exists some $\delta \geq 0$ such that for all $x, y, z \in X$, the δ-neighbourhood of $[x, y] \cup [y, z]$ contains $[x, z]$.*

For my final paper I decided to compare and contrast three different proofs of hyperbolicity of the curve complex. The first is by Masur and Minsky in 1998, which relies on Teichmuller theory [MM98]. The second is by Matt Clay, Kasra Rafi, and Saul Schleimer [CRS15], who use a surgery procedure discovered by Hatcher [Hat91] together with the same 'guessing geodesics' lemma used by Masur-Minsky. With this argument, Clay-Rafi-Schleimer show uniform hyperbolicity of the curve complex. The third, by Sebastian Hensel, Piotr Przytycki, and Richard Webb [HPW13], uses a similar construction with a more direct proof of hyperbolicity. Not only do they show uniform hyperbolicity, but they extract an effective δ.

The mapping class group acts on the curve complex. From this action, we can deduce multiple interesting statements about the mapping class group :

- The mapping class group has infinite diameter
- pseudo-Anosovs act hyperbolically on $\mathcal{C}(S)$, while reducible and elliptic elements act reducibly.
- The distance formula for elements of $\mathcal{MCG}(S)$.

Similar programs have been executed in the study of $Out(F_n)$. Bestvina and Feighn showed that the free factor complex of the free group is

δ-hyperbolic. They did this y studying Outer space, the analogue of Teichmuller space for $Out(F_n)$. Handel and Mosher used Masur-Minsky's 'guessing geodesics' lemma to show that the splitting graph of the free group is hyperbolic. Instead of relying on another so-called 'geometric space', they worked directly on the splitting graph to show a property similar to exponential divergence [HM13].

2 Why the Main Theorem is Not Trivial

The geometry of the curve complex is unclear from just the definition. It is visiually clear when two curves are at distance 1 or 2, but it is harder to detect when two curves are very far apart in the curve graph. This raises the question: does the curve graph has infinite diameter?

Any space of bounded diameter is trivially δ-hyperbolic. To see why the main theorem is not trivial, we first show the following:

Theorem 1. *For S not sporadic,* $\operatorname{diam}\mathcal{C}(S) = \infty$.

The proof relies on the theory of laminations. A lamination on S is a closed subset foliated by geodesics. For more information, see the book of Bonahon [Bon99]. The key fact we use is that the space of geodesics laminations with the topology of Hausdorff convergence is compact.

Proof. Now let $\gamma_1, \gamma_2, ...$ be a sequence of geodesics which converge to some filling lamination $\overline{\lambda}$ which is not a geodesic (for instance, take γ transverse to the vertical foliation of some pseudo-Anosov φ and let $\gamma_i = \varphi^i(\gamma)$). We claim that $d_{\mathcal{C}}(\gamma_1, \gamma_n) \to \infty$, maybe after passing to some subsequence. Suppose for contradiction that $d_{\mathcal{C}}(\gamma_1, \gamma_n)$ is bounded. Pass to a subsequence again to suppose that $d_{\mathcal{C}}(\gamma_1, \gamma_n) = M < \infty$. Then for each γ_n we can find a sequence $\gamma_1 = \beta_0^{(n)}, ..., \beta_M^{(n)} = \gamma_n$ of disjoint curves witnessing $d_{\mathcal{C}}(\gamma_1, \gamma_n) = M$.

Now by compactness we have that $\beta_{M-1}^{(n)}$ has a subsequence which converges to some lamination $\overline{\mu}$. Since $\beta_{M-1}^{(n)}$ and $\beta_M^{(n)} = \gamma_n$ are disjoint, the limiting laminzations cannot have any transverse intersection. That is to say, the intersection $\overline{\mu} \cap \overline{\lambda}$ can only contain leaves. Since $\overline{\lambda}$ is filling then this intersection is nonempty, hence if λ is a minimal lamination in $\overline{\lambda}$ we have $\lambda \subset \overline{\mu}$.

Passing to a further subsequence and inducting, we have that $\{\gamma_1\}_{n\geq 1}$ converges to some lamination which is not a geodesic, a contradiction. $\square$

3 Guessing Geodesics

A well known fact about hyperbolic spaces is the that geodesics are contracting, in the sense that there exists some $C, D > 0$ depending only on δ such that for any geodesic γ, and for any two points $x, y \in X$ which are distance C away from γ satisfy $d(\pi(x), \pi(y)) \leq D$, where $\pi : X \to \gamma$ is the nearest point projection.

While Hensel-Przytycki-Webb directly show that all triangles are slim, Masur-Minsky and Rafi-Clay-Schleimer both use the contraction property to identify hyperbolicity. Masur-Minsky make the following definition

Definition 3. *A path $\gamma : I \to X$ is contracting if there exists a map $\pi : X \to I$ and constants $a, b, c \geq 0$ satisfying the following properties:*

1. *(Retraction) For any $t \in I$,* $\operatorname{diam}(\gamma([t, \pi(\gamma(t))])) \leq c$.

2. *(Lipschitz) If $d(x, y) \leq 1$ then* $\operatorname{diam}\gamma([\pi(x), \pi(y)]) \leq c$.

3. *(Contracting) If $d(x, \gamma(\pi(\pi(x)))) \geq a$ and $d(x, y) \leq b \cdot d(x, \gamma(\pi(x)))$ then* $\operatorname{diam}\gamma[\pi(x), \pi(y)] \leq c$.

The map π should be thought of as a coarse closest-point projection. We also take the map to I instead of $\operatorname{im}(\gamma)$, so that we don't have to care about the parametrization speed of γ.

In a hyperbolic space, every two points can be connected by a contracting path. To detect hyperbolicity, we should coarsen this condition. A family of paths is *coarsely transitive* if there exists $D \geq 0$ such that any two points x, y more than D distance apart can be connected by a path in the family. A family of paths is contracting if each path is contracting, for a uniform choice of a, b, c.

Masur and Minsky prove the following:

Theorem 2. *If X is a geodesic metric space which admits a coarsely transitive and contracting path family Γ, then X is hyperbolic. In addition, the paths in Γ are uniformly quasi-geodesic.*

Since the projection π maps to an interval, as opposed to a subset of the space X, Masur and Minsky use a modified definition of quasi-geodesics. Say that a path $\gamma : I \to X$ is a (K, C, s)-quasigeodesic if for all $x, y \in I$,

$$\operatorname{length}_s(\gamma[x, y]) \leq K d(\gamma(x), \gamma(y)) + C.$$

Here, length_s is ns where n is the smallest integer such that $[x, y]$ can be divided into n closed subintervals $J_1, ..., J_n$ with $\text{diam}(\gamma(J_i)) \leq s$.[1]

Now we say that X has *stability of quasi-geodesics* if for all $K \geq 1, C, s \geq 0$ there exists $R = R(K, C, s) > 0$ such that any (K, C, s)-quasigeodesic is in an R-neigbhourhood of the geodesic connecting its eendpoints.

The proof of the theorem is in two steps:

Lemma 1. *If X has a coarsely transitive path family Γ then X has stability of quasi-geodesics and the paths in Γ are uniform quasi-geodesics.*

Sketch of Proof. Assume WLOG that Γ is actually transitive, because for any curve of length $\leq D$ the constant map provides a contracting projection.

Let $\gamma : [0, M] \to X$ be a path in Γ, and let $\alpha : [0, L] \to X$ be a (K, C, s)-quasigeodesic connecting the endpoints of α. We want to show that α is in a uniform neighbourhood of γ. If α makes a long excursion away from γ, then by the contraction property we can circumvent this by travelling along the projection to γ. Since our R is allowed to depend on (K, C, s), this will show the claim.

Applying the same argument to a bona fide geodesic $[\gamma(0), \gamma(M)]$ we deduce that α is in a uniformly bounded neighbourhood of any geodesic connecting its endpoints. □

Lemma 2. *Stability of quasi-geodesics implies hyperbolicity.*

Proof. Let $x, y, z \in X$. We want to show that $[x, y]$ is in a uniform δ-neighbourhood of $[x, z] \cup [z, y]$. Let $z' \in [x, y]$ minimize distance to z. We claim that $[x, z'] \cup [z', z]$ is a $(3, 0, 0)-$quasigeodesic. Let $u \in [x, z']$ and $v \in [z', z]$, so that by choice of z' we have

$$d(u, v) \leq d(z', v)$$

and

$$d(u, v) \geq d(u, z') - d(z', v).$$

Then adding twice the first inequality to the second we obtain

$$3d(u, v) \geq d(z', v) + d(z', u).$$

By the stability of quasi-geodesics, $[x, z'] \cup [z', z]$ (in particular $[x, z']$) is in a uniformly bounded neighbourhood of $[x, z]$. Repeating the argument for all segments in the triangle shows that X is δ-hyperbolic. □

[1] This is for $s > 0$, for $s = 0$ we take the normal length.

4 Reduction to the Complex of Arcs and Curves

Both Clay-Rafi-Schleimer and HPW show that the complex of arcs and curves is hyperbolic, and then argue that $\mathcal{AC}(S)$ is quasi-isometric to $\mathcal{C}(S)$ in order to conclude. This is true because both are geodesic spaces, so that δ-hyperbolicitty is a quasi-isometry invariant.

To obtain (effective) uniform hyperbolicity, both groups needed to show that the quasi-isometry constants of the map between $\mathcal{AC}(S)$ and $\mathcal{C}(S)$ can be chosen uniformly. We take our exposition from HPW.

Suppoose that $\mathcal{C}(S)$ is connected, so that S is not a four-holed spehre or a once-punctured torus. Construct the map $r : \mathcal{AC}^{(0)}(S) \to \mathcal{C}^{(0)}(S)$ by assigning to an arc γ a boundary component of the regular neighbourhood of $\gamma \cup \partial S$. To each curve we assign $r(\gamma) = \gamma$. Clearly there is some ambiguity in the definition of r, bceause we had to make a choice for each arc. Because the different boundary components of $N(\gamma \cup \partial S)$ are disjoint, make whatever choice you'd like.

We claim that r is 2-lipschitz. If a and b are disjoint arcs that do not fill S, then $r(a)$ and $r(b)$ are disjoint. Now suppose that a, b are disjoint and do not fill S, so that S must be the two-holed torus. Then the endpoints of a, b must be on the same componenet, so that $r(a), r(b)$ intersect at most once. Therefore $(r(a) \cup r(b))^c$ is a two-holed disc, so that $d_{\mathcal{C}}(r(a), r(b)) \leq 2$.

Also for any arc a and curve b which are disjoint, the curves a and $r(b)$ are disjoint. Hence r is a $(2, 2)$-quasiisometry.

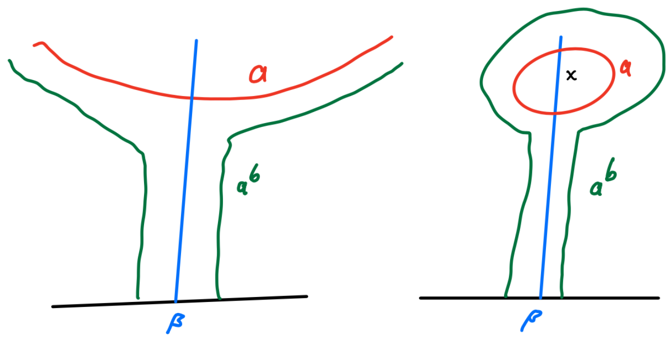
5 Surgery Sequences

Masur-Minsky's proof relies on Teichmuller theory to construct a coarsely transitive family of paths, Hensel-Przytycki-Webb and Clay-Rafi-Schleimer both work directly on the surface. They make use of 'surgery sequences' as constructed by Hatcher [Hat91]

Let a and b be arcs in minimal position, with chosen endpoints α and β. Let x be the first point of intersection. In a regular neighbourhood $N_S(\alpha \cup \beta)$, let N' be a component that contains the portion of b between β and x. Now let a^b be a component of the boundary of N'.

If A is a system of disjoint arcs, and b is a directed arc cutting A, then we can construct a new system A^b by taking the first arc a_1 in A intersecting b, and surgering a_1 along b.

Figure 1: Surgery of a along b



HPW use a slightly modified construction. Again let a and b be directed arcs in minimal position. Let $x \in a \cap b$ be an intersection point, and let $a' \subset a, b' \subset b$ be subarcs with endpoints α, x (resp. β, x). If $a' \cup b'$ is an embedded arc, we say that $a' \cup b'$ is the one-corner arc obtained from a^α, b^β. 'One-corner' tranlates to the Polish word 'jeden ròg', which is similar to 'jednorożec', the Polish word for 'unicorn'. Hence HPW call these 'unicorn' arcs.

We have a linear order on unicorn arcs by declaring $a' \cup b' \leq a'' \cup b''$ is $a'' \subset a'$ and $b' \subset b''$. Then we have a sequence $(a = c_0, ..., c_n = b)$ of unicorn arcs between a and b. Denote this path by $\mathcal{P}(a^\alpha, b^\beta)$.

Let us confirm that $\mathcal{P}(a^\alpha, b^\beta)$ is indeed a path in $\mathcal{AC}(S)$. Suppose that c is a unicorn path. If c is adjacent to b we are done, otherwise let x be the intersection point defining c and let b' be the subpath of c from x to β. Construct c' by travelling from x to b' along a until an intersection x', and traversing along b until reaching β. Then $c < c'$. The arc c' can be homotoped off c by traversing different sides of a, b in some sufficiently small regular neighbourhood.

While both groups use surgery sequences in a key way, they take different approaches. HPW use unicorn paths to construct slim triangles between points, and use this to directly show hyperbolicity. On the other hand, CRS use a slightly different approach. One characterization of δ-hyperbolicity is the exponential divergence property: for any two points x, y with distance $\geq R$ from some basepoint o, the length of the shortest path from x to y

which avoids the ball $B_R(o)$ grows exponentially in R. CRS find a collection of paths such that, for any two paths γ_1, γ_2 which have one endpoint close together, the paths grow exponentially closer.

5.1 Slim Unicorn Triangles

We outline HPW's proof. The key observation is that unicorn paths give rise to 1-slim triangles:

Lemma 3. *Let $a^\alpha, b^\beta, d^\delta$ be oriented arcs in minimal position. Then for every $c \in \mathcal{P}(a^\alpha, b^\beta)$ there exists some $c^* \in \mathcal{P}(a^\alpha, d^\delta) \cup \mathcal{P}(d^\delta, b^\beta)$ such that $d_{\mathcal{AC}}(c, c^*) \leq 1$.*

Proof. Suppose c intersects d, else we are done. Then let $d' \subset d$ be the subarc given by starting at δ and traversing along d until hitting c. Let $\sigma \in c \cap d'$ be this intersection point. One of the components of $c \setminus \{\sigma\}$ is contained in either a' or b', so suppose WLOG it is contained in a'. Then the curve $c^* = (c \cap a') \cup d'$ is in $\mathcal{P}(a^\alpha, d^\delta)$. By traversing slightly to the left or right of d, we see that $d_{\mathcal{AC}}(c, c^*) \leq 1$. □

To conclude that the curve graph is hyperbolic, they need to show that unicorn arcs are uniformly close to geodesics. First they show that subpaths of unicorn paths are unicorn paths up to a constant error (of 1). In particular, they show the following:

Lemma 4. *For every $0 \leq i < j \leq n$ either $\mathcal{P}(c_i^\alpha, c_j^\beta)$ is a subpath of $\mathcal{P}(a^\alpha, b^\beta)$ or $j = i + 2$ and c_i, c_j are adjacent in $\mathcal{AC}(S)$.*

The exception occurs because the unicorn arcs c_i, c_j may not be in minimal position.

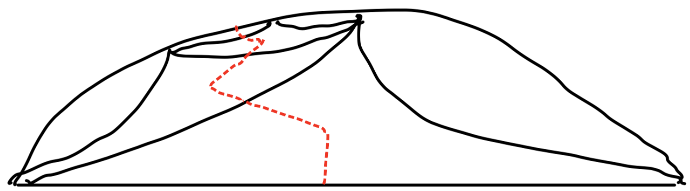
Now from a classical bisection argument and 1-slimness of unicorn triangles, we have the following:

Lemma 5. *Let $x_0, ..., x_m$ with $m \leq 2^k$ be a sequence of vertices in $\mathcal{AC}(S)$. Then for any $c \in \mathcal{P}(x_0, x_m)$ there exists i and$c^* \in \mathcal{P}(x_i, x_{i+1})$ such that $d_{\mathcal{AC}(S)}(c, c^*) \leq k$.*

With the lemma, they conclude the following:

Proposition 1. *Let γ be a geodesic between vertices a, b in $\mathcal{AC}(S)$. Then there exists $c \in \mathcal{P}(a, b)$ such that $d_{\mathcal{AC}}(c, \gamma) \leq 6$.*

Figure 2: The classical bisection argument



Proof. Let $c \in \dashv, \lfloor$ maximize distance to γ, and call this distance k. Now let $a'b'$ be the longest subpath containing c which is within $2k$ of c, then $a'b' \in \mathcal{P}(a,b)$. Then let a'', b'' be the closest points to a', b' in the geodesic γ. Applying our previous lemma and the triangle inequality gets us $d_{\mathcal{AC}(S)}(a'', b'') \leq 6k$. Concatenate $[a', a''] \cup [a'', b''] \cup [b'', b']$. The path given by this concatenation is of the form $x_0, ..., x_m$ for $m \leq 8k$. By our bisection lemma, $k \leq \log_2(8k) + 1$, so that $k \leq 6$. □

This estimate, together with the slim triangles estimate and the quasi-isometry $\mathcal{AC}(S) \to \mathcal{C}(S)$ gives effective hyperbolicity of the curve complex. In fact, it gives 17-hyperbolicity.

5.2 Projection Onto Surgery Sequences

CRS obtain hyperbolicity by finding a coarsely transitive collection of paths which are exponentially converging. We sketch their argument here.

Let A be a collection of isotopy classes of arcs. We say that A is a system if it forms an $|A|$-simplex in $\mathcal{AC}(S)$. That is to say, all curves and arcs in A are disjoint. For two systems A, B let $\text{inner}(A, B) = \min_{a \in A, b \in B} d_{\mathcal{AC}}(a, b)$. We identify a vertex γ with the system $\{\gamma\}$.

Now we define the surgery sequence starting at A with target γ inductively. Let $A_0 = A$ and $A_{i+1} = A_i^\gamma$. Observe that $|\text{inner}(A^\gamma, B) - \text{inner}(A, B)| \leq 1$ and $\text{inner}(A^\gamma, \gamma) \leq \text{inner}(A, \gamma)$.

Now we show that our collection of surgery sequences forms a coarsely disjoint and contracting collection of paths.

To see why they are coarsely transitive, fix $a, b \in \mathcal{AC}(S)$. Now pick an arc γ disjoint from b, and let $\{A_i\}_{i=0}^N$ be the surgery sequence starting at $\{a\}$

with target γ. Then $\mathrm{inner}(A_N, b) \leq \mathrm{inner}(A_N, \gamma) + \mathrm{inner}(\gamma, b) \leq 2$, so we are done.

Now we define our projection mappings. For a vertex $a \in \mathcal{AC}(S)$ we define the *footprint* $\varphi(a)$ onto the surgery sequence started at A with target γ as

$$\{i \in [0, N] | a \cap A_i = \varnothing\}.$$

Then this is an interval. Define the projection $\pi : \mathcal{AC}(S) \to [0, N]$ as follows: let $\{a_j\}$ be the surgery sequence starting at $\{a\}$ with target γ, and letting $\pi(a)$ be the least $m \in [0, N]$ such that there exists k with $A_m \cap a_j \neq \varnothing$, else we set $\pi(a) = N$.

The key lemma is that, for any two systems B and C, the surgery sequences have a common descendant within a constant distance of their union. This constant will not depend on the surface, which will give us uniform hyperbolicity. Using this, they check that our family of paths is contracting.

Lemma 6. *Suppose that A is a system and γ is a directed arc with* $\mathrm{inner}(A, \gamma) \geq 6$*. Let $B, C \subset A$ be subsystems. Let $\{A_i\}_{i=0}^{N}$ be the surgery sequence starting at A with target γ, and let $\{B_i\}, \{C_i\}$ be the induced surgery sequences. Then there exists $k \in [0, N]$ such that (i) $B_k \cap C_k \neq \varnothing$ and* $\mathrm{inner}(A_0, A_i) \leq 5$ *for all $i \in [0, k]$*

First they show that in the surgery sequence $\{A_i\}$ and $c \in [0, \mathrm{inner}(A, \gamma) - 1]$ that there exists an index i such that $\mathrm{inner}(A, A_i) = c$. This follows from the triangle inequality $\mathrm{inner}(A, \gamma) \leq \mathrm{inner}(A, A_N) + \mathrm{inner}(A_N, \gamma)$ and since $|\mathrm{inner}(A^\gamma, \gamma) - \mathrm{inner}(A, \gamma)| \leq 1$.

Now let ℓ be minimal such that $\mathrm{inner}(A, A_\ell) = 3$, which exists as $\mathrm{inner}(A, \gamma) \geq 6$. Then $\mathrm{inner}(A, A_{\ell-1}) = 2$. Then b must hit every element of A_ℓ. By picking a hyperbolic metric on the surface and straightening everything to geodesics, isotope A_ℓ to be pairwise disjoint and in minimal position with γ.

Suppose that $B_\ell \cap C_\ell = \varnothing$, else we are done. As $\mathrm{inner}(A, A_\ell) = 3$ then both B_ℓ and C_ℓ must be filling, so let B', C' be minimal filling subsystems. By applying an euler characteristic argument there exists a common arc in $B_{\ell+1} \cap C_{\ell+1}$.

6 The ‘Geometric’ approach of Masur and Minsky

6.1 Teichmuller Geometry

Masur and Minsky took a radically different approach from those of some more modern proofs. They rely on the auxiliary geometric space - the Teichmuller space. They use geodesics on the Teichmuller space to guess geodesics in the curve graph. Again, fix S to be a non-sporadic surface, and denote by $\mathcal{T}(S)$ the Teichmuller space of x.

For a point $x \in \mathcal{T}(S)$ and a curve $\alpha \subset S$, we recall the extremal length $\mathrm{Ext}_x(\alpha)$. Now for two hyperbolic structures $x, y \in \mathcal{T}(S)$, the Teichmuller distance $d_{\mathcal{T}}(x, y)$ is the log of the dilatation of the Teichmuller map $(S, x) \to (S, y)$. By writing out the definition of extremal length, we observe Kerchoff’s characterization:

$$d_{\mathcal{T}}(x, y) = \frac{1}{2} \log \sup_{\alpha \in \mathcal{C}(S)} \frac{\mathrm{Ext}_y(\alpha)}{\mathrm{Ext}_x(\alpha)}.$$

We recall a few more facts related to Teichmuller space. A holomorphic quadratical different on S is a form that can locally be written $q(z) = \varphi(z)dz^2$ for a holomorphic φ. Then wherever q is nonzero, we can write $q = d\eta^2$ for holomorphic η, which locally gives us two foliations, given by the horizontal and vertices axes in $\mathbb{C}$. These are globally well-defined, and we call them the *horizontal* and *vertical* foliations.

Now we can use q to define a Teichmuller geodesic as follows: for $t \in \mathbb{R}$ let $L_q(t)$ be the conformal structure given by scaling the vertical foliation by e^t and the horizontal foliation by e^{-t}. The map $L_q(t) : \mathbb{R} \to \mathcal{T}(s)$ is a geodesic by Kerchoff’s’ characterization.

6.2 A Nice Family of Paths

Given a conformal structure (S, x), we can define the map $x \to \mathcal{C}(S)$ by sending x to a curve which minimizes extremal length. Our goal will be to take a Teichmuller geodesic, and use this projection to construct a path in $\mathcal{C}(S)$.

To do this, we first relate distance in the curve graph to intersection numbers, and then to extremal lengths.

Lemma 7. *If α, β are curves, then $d_{\mathcal{C}}(\alpha, \beta) \leq 2i(\alpha, \beta) + 1$.*

Proof. Put α and β in minimal position. If $i(\alpha, \beta) = 0$ we are done. If $i(\alpha, \beta) = 1$ then let γ be a component of the boundary of a regular neighbourhood of $\alpha \cup \beta$. Since S is not sporadic, then γ is a nontrivial curve disjoint from α and β, so that $d_{\mathcal{C}}(\alpha, \beta) \leq 2$.

Now we induct on $i(\alpha, \beta) \geq 2$. We essentially use a surgery argument. Take two points of $\alpha \cap \beta$ which are adjacent in α (the subsegment between them in α has interior disjoint from β). There are two ways to perofmr surgery, by replacing a segemtn of β with a segment of α. These two surgery procedures produce nontrivial curves β_1, β_2 with $i(\alpha, \beta_j) \leq i(, \beta) - 1$. Now neither curve can be peripheral, so that $d(\alpha, \beta) \leq 2 + d(\alpha, \beta_j)$. Induction does the job. □

Using the relation $\mathrm{Ext}_x(\alpha) \, \mathrm{Ext}_x(\beta) \geq i(\alpha, \beta)^2$ and the lemma we end up with the following statement:

Lemma 8. *Let $\Phi : \mathcal{T}(S) \to \mathcal{C}(S)$ be the map sending a structure x to the collection of curves of shortest extremal length. Then there exists $c = c(S)$ such that $\operatorname{diam} \Phi(x) \leq c$ for all $x \in \mathcal{T}(S)$.*

Therefore the map Φ Now denote by L_q a Teichmuller geodesic as described in in the previous section. Consider the map $F : \mathbb{R} \to \mathcal{C}(S)$ given by sending t to one of the curves in $\Phi(L_q(t))$.

We claim that the collection of such paths $\{F_q\}$ is coarsely transitive. Let α and β be curves with $d_{\mathcal{C}}(\alpha, \beta) \geq 3$. Then the union $\alpha \cup \beta$ must be filling. Hence there exists a holomorphic quadratic differential with α, β as its horizontal and vertical foliations. Then $F_q(\infty)$ and $F_q(-\infty)$ are α and β, respectively.

Now we define our projection $\pi : \mathcal{C}(S) \to \mathbb{R}$. To a curve β, let $|\beta|_{q,h}, |\beta|_{v,h}$ be the q-lengths of the projection of β to the horizontal and vertical foliations. We say that β is balanced with respect to q if $|\beta^*|_{q,h} = |\beta^*|_{q,v}$ for the q-geodesic representative β^*. Observe that β^* is geodesic with respect to q_t as well, and $|\beta^*|_{q_t,h}, |\beta^*|_{q_t,v}$ are equal to $e^t |\beta^*|_{q,h}, e^{-t} |\beta^*|_{q,v}$.

Now if β is not vertical or horizontal with respect to q, there exists t such that β is balanced with respect to q_t. Let $\pi_q(\beta) = t$. If β is vertical or horizontal, let $\pi_q(\beta)$ be ∞ or $-\infty$, respectively.

To finish the proof, we will need to show the following:

Theorem 3. *The path family $\{F_q\}$ satisfies the contraction property for the projections π_q.*

6.3 The Nested Train-Track Argument

In this section we will recall some of the theory of train-tracks. A train-track on S is an embedded 1-complex τ such that

- Each edge (branch) is smoothly embedded and has well-defined tangent vectors at endpoints
- At each vertex (switch) the corresponding vectors are tangent
- The valence of each switch is at lesat 3, unless the switch is in a closed curve component
- Every component R of $S \setminus \tau$ is not one of the following: an annulus, once-punctured disk, or unpunctured disk with $0, 1, 2$ cusps.

A transverse measure τ is a nonnegative function on the branches with satisfies the switch condition, that for any switch, the sums over incoming and outgoing branches are equal. Let $P(\tau)$ be the set of transverse measures supported on τ.

We say that a train-track is recurrent if every branch is contained in a closed train-route. Say that $\sigma < \tau$ if σ is a subtrack of τ, and say that ω is carried on τ if we can homotope S in a way that takes every route on σ to a route on τ.

Say that τ is maximal if it is not a proper subtrack of any other track, which is equivalent to required that all complementary regions are ideal triangles or punctured monogons.

When trying to understand the geometry of the curve graph, one key difficulty is detecting when two curves are far apart. Suppose α and β are disjoint curves. Suppose that α is carried on a maximal train-track τ and passes through every branch. Then β is also carried on τ (else it would intersect τ and hence α).

This is the key idea. If we let $\{\tau_j\}_{j=0}^n$ be a sequence of maximal train-tracks such that $P(\tau_j) \subset \text{int}P(\tau_{j-1})$, and β_j is a sequence of curves such that $\beta \in \int(P(\tau_{j-1})) \setminus P(\tau_j)$, then $d_{\mathcal{C}}(\beta_1, \beta_j) \geq j$.

Using this, Masur and Minsky show that if two curves are both far from another, and close to each other, then both are nested in extensions of the same track. As a result, theyir intersection numbers with other curves are very large compared to their intersection with each other. They make this quantitative in the following way:

Lemma 9. *Given $Q, k > 0$ there exists D, d such that for any $\alpha, \beta, \in \mathcal{C}(S)$, if (i) $d_{\mathcal{C}}(\alpha, \beta) \geq D$, (ii) $d_{\mathcal{C}}(\gamma, \beta) \leq d \cdot d_{\mathcal{C}}(\alpha, \beta)$, then*

$$\min_{\alpha'} i(\beta, \alpha') \cdot \min_{\alpha'} i(\gamma, \alpha') \geq Q \cdot i(\beta, \gamma)$$

where α' varies over the k-neighbourhood of α in $\mathcal{C}$.

Using the nesting property, Masur and Minsky show that the projections π_q are contracting.

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