A Poisson Formula for Semi-Simple Lie Groups

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December 12, 2021

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1 Random Walks, Harmonic Functions, and Boundaries

Over the past semester we've studied random walks on groups of nonpositive curvature. One interesteing relation is that between asymptotic properties of random walks and the existence of bounded harmonic functions. For example, transience for any markov chain is equivalent to the existence of a positive non-constant subharmonic function [Woe94].

First consider the case of the free group F_2 acting on its Cayley graph Γ. Here our space has a natural compactification $\Gamma \cup \partial\Gamma$, where $\partial\Gamma$ is the Gromov boundary. The action of F_2 extends continuously to the boundary.

One can show that a simple random walk will converge almost surely to the Gromov boundary. Since cancellations are unlikely, the distance of sample path to the origin is bounded below by a sum of i.i.d random variables with positive expectation. Hence by the Strong Law of Large Numbers a random walk will almost surely leave any bounded set for all time. Since the Cayley graph is a tree, this implies any random walk will converge to a point in the Gromov boundary. This provides us with a *hitting measure* ν on the Gromov boundary. This measure is defined as $\nu(A) = \mathbb{P}(\lim_{n\to\infty} \omega_n \in A)$.

Letting μ be the measure driving the simple random walk, we have that

$$\begin{aligned}
\mu * \nu(A) &= \frac{1}{4}\sum_{i=1}^{4} \mathbb{P}(\lim_{n\to\infty} \omega_n \in g_i^{-1}A) \\
&= \frac{1}{4}\sum_{i=1}^{4} \mathbb{P}(\lim_{n\to\infty} g\omega_n \in A) \\
&= \mathbb{P}(\lim_{n\to\infty} \omega_{n+1} \in A) \\
&= \nu(A).
\end{aligned}$$

So that $\mu * \nu = \nu$. We say that ν is *μ-stationary.*

Transience of the random walk already provides us with a large number of harmonic functions. Fix any measurable $A \subset \partial\Gamma$ with positive hitting measure, and then $\varphi_A(g)$ be the probability that a random walk starting at g lands in A. That is to say, we set $\varphi_A(g) = g_*\nu(A)$. Since ν is μ-stationary, then φ_A is μ-harmonic. Likewise, given any bounded function f on Γ we can produce a μ-harmonic function on F_2 the same way, by defining $\varphi(g) = \mathbb{E}f(\lim_{n\to\infty} \omega_n)$.

It is a natural question whether every bounded harmonic function arises in this way. Can the space of bounded harmonic functions be identified with the space of bounded harmonic functions on some geometric 'boundary'? To answer the question, we borrow some techniques from PDEs. Recall the Poisson Representation Formula for bounded harmonic functions on the disk.

Theorem 1. *Let $h(z) : \mathbb{D} \to \mathbb{R}$ be a bounded harmonic function. Then there exists a bounded* *function* $\overline{h}:\mathbb{D}\to\mathbb{R}$ *such that*

$$h(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \overline{h}(e^{i\theta})\frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}d\varphi.$$

Let $SL(2,\mathbb{R})$ act on the $\mathbb{D}$ by fractional linear transformations, and set $\hat{h}(g)=\overline{h}(g(0))$. Let m be the Lebesgue probability measure on S^1. Then the Poisson formula states

$$\begin{aligned}\hat{h}(g) &= \int_{S^1}\overline{h}(\xi)\frac{dgm}{dm}dm(\xi)\\ &= \int_{S^1}\overline{h}(\xi)dgm(\xi)\\ &= \int_{S^1}\overline{h}(g\xi)dm(\xi)\end{aligned}$$

2 The Maximal Boundary

A common theme in geometric group theory is to study proximal systems. A metric G-space X is proximal if for any two points $x,y\in X$ we have $\liminf_{n\to\infty} d(T^n x, T^n y)=0$. Proximality is a powerful tool, underlying results like linear escape of random walk [Gou21]s, and the ping-pong lemma [CM17]. The idea behind the boundary is to examine a Lie group action which is *strongly proximal*. A G-space is strongly proximal if for every measure π there exists some sequence $g_n\in G$ such that $g_n\pi$ converges to a point measure.

From here onwards, we will follow Furstenberg's seminal paper [Fur63].

Definition 1. *Let G be a Lie group and M a compact G-space. We say that M is a* boundary *of G if the action of G is transitive and strongly proximal.*

Definition 2. *A closed subgroup $H\subset G$ is a B*-subgroup *if G/H is a boundary of G.*

The action of G on M induces an action on the space of probability measures on M. We can use this action to deduce information about the possible boundaries M.

Definition 3. *A group G has the* fixed-point property *if whenever G acts on a compact convex subset of a locally convex topological vector space, then G has a fixed point.*

We quote the following lemma from Furstenberg:

Lemma 1. *If G is solvable or compact, then G has the fixed-point property.*

Proposition 1. *If G has the fixed-point property, then the only boundaries are singletons.*

Proof. Let M be a boundary of G. As M is compact, then the set of probability measures is a compact convex set of measures on M. So we let π be a probability measure fixed by G. As M is a strongly proximal G-space, then we can pick a sequence g_n so that $g_n\pi$ converges to a point measure. As π is fixed by G, then π must have been a point measure, hence G has a global fixed point on M. As the action of G is transitive, then M must be a singleton. □

We use the action of G on the space of measures in order to construct a maximal boundary, corresponding to a minimal B-subgroup. From now on, let G be a semi-simple Lie group with finite center. We have an Iwasawa decomposition $G = KS$, where K is a maximal compact subgroup and S is a solvable normal subgroup. In the case $G = SL(2, \mathbb{R})$, then $K = SO(2)$ and S is the group of unimodular upper triangular matrices.

Our main tool will in constructing the minimal B-subgroup is be the following theorem:

Theorem 2. *There is a compact subgroup $K_0 \subset K$ and a point $p_0 \in G/S$ so that the measure $\pi_0 = m_{K_0} p_0$ is a limit of measures $\mu * m$, and π_0 is fixed by S.*

Here m_G is the Haar measure on G.

Proof. The group G acts on the set of measures Q on $M := G/S$. Let $\sigma : G \to G/S$ be the natural quotient map. Since S is solvable, we can find a measure $\pi \in Q$ fixed by S. In the Iwasawa decomposition $G = KS$ we can uniquely write $g = ks$. Therefore $\rho : K \to G/S$ is a homeomorphism. Then we can find a unique measure $\overline{\pi}$ on K with $\sigma(\overline{\pi}) = \pi$. We view $\overline{\pi}$ as a measure on G. Since $\sigma(s\overline{\pi}) = s\sigma(\overline{\pi}) = s\pi = \pi = \sigma(\overline{\pi})$, then $s\overline{\pi} * \pi = \overline{\pi} * \pi$. Then $\pi_1 := \overline{\pi} * \pi$ is invariant under S. We can recursively define $\pi_n = \overline{\pi} * \pi_{n-1}$. Let K_0 be the closed subgroup generated by the support of π_0. Then the measures $\nu_n = \frac{1}{n} \sum_{k=0}^{n} \overline{\pi}_k$ are supported on K_0. By compactness the sequence ν_n has a converging subsequence that satisfies $\overline{\pi} * \nu = \nu$. Hence $\nu = m_{K_0}$. This proves the theorem. □

In the case $G = SL(2, \mathbb{R})$, we have K_0 is the trivial group, $G/S = \mathbb{D}$, and π_0 is a point mass at ∞.

Now we let $H(G)$ be the subgroup of G that fixes π_0. We want to say that $H(G)$ is the minimal B-subgroup of G. We first show that $G/H(G)$ is a boundary.

Theorem 3. *Let G be a semi-simple Lie group with finite center. Then $H(G) = K_0 S$, $H(G)$ has the fixed point property, $H(G)$ is unbounded and is a B-subgroup of G. Also, any maximal comapct subgroup of G is transitive on $B(G)$.*

Proof. Clearly $H(G)$ contains K_0 and S. By our Iwasawa decomposition we can write $H = (H \cap K)S$. Then if $k \in H(G) \cap K$ we have $k\pi_0 = \pi_0$. In other words, $km_{K_0} = m_{K_0}$, so that $k \in K_0$. Hence $H(G) = K_0 S$.

Since $H(G)$ is a subgroup of G, then we can make the identification $H(G)/S \cong \sigma(H(G)) \subset M$. We have $\sigma(H(G)) = K_0 p_0$, which is the support of π_0. Since π_0 is invariant under $H(G)$, we have a measure on $H(G)/S$ invariant under $H(G)$. As S is solvable then it has the fixed point property, so existence of an invariant measure on $H(G)/S$ implies that $H(G)$ has the fixed point property. Hence $H(G)$ is unbounded.

Now we show that $G/H(G)$ is a boundary. Since S is a subgroup of $H(G)$ we have a natural equivariant map $\tau : G/S \to G/H(G)$. Pick a sequence of measures $\mu_n * m$ converging to π_0. As τ is equivariant we can write $\mu_n \tau m = \tau \mu_n * m \to \tau\pi_0$. Pick as $\pi_0 = M_{K_0} p_0$ and $K_0 \subset H(G)$ then $\tau(K_0 p_0)$ is a single point, so that $\tau\pi_0$ is a point measure on $G/H(G)$.

To prove the last claim, observe that K acts transitively on G/S. Since τ is surjective and equivariant then K acts transitively on $G/H(G)$. □

Now we show that $B(G) := G/H(G)$ is a maximal boundary

Theorem 4. *If B is any boundary of G, then there is an equivariant map from $B(G)$ onto B.*

Proof. Since $H(G)$ has the fixed point property, it leaves invariant some measure π on B. Since B is a boundary, then there exists a sequence $g_n \in G$ so that $g_n\pi$ converges to a point measure δ. As $H(G)$ contains S then all measures $g_n\pi$ are of the form $k\pi$ for some $k \in K$. As K is compact then this set is closed, so that $g_n \to k\pi$. As this is a point measure, then π is a point measure. Hence H must fix a point $x \in B$, so that $H(G)$ is contained in the stabilizer H_x of x. Hence we have an equivariant map $G/H(G) \to G/H_x = B$. □

3 Examples

First let us carry out the development for $SL(2, \mathbb{R})$. We claim that the maximal boundary is precisely the space of 1-dimensional linear subspaces of $\mathbb{R}^2$. Clearly $SL(2, \mathbb{R})$ acts transitively on P^1, which is compact. We claim that the action is strongly proximal.

Let $A_n = \begin{bmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{bmatrix} \begin{bmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{bmatrix}$. Then let $B_n = A_1 ... A_n$. Then for every interval of $I \subset \mathbb{D}$ and every neighbourhood I of $(0, 1)$, we have B_n maps I into U for n sufficiently large. Therefore P^1 is a boundary for $SL(2, \mathbb{R})$.

In our development, we can set $K = SO(2)$ and $S = AN$, where

$$A = \{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \lambda \in \mathbb{R} \}, N = \{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, t \in \mathbb{R} \}.$$

Then the normalizer of S is contained in a minimal B-subgroup. However, the normalizer of S is precisely the stabilizer of some point in P^1. Hence $H(G) = S$ and so $SL(2, \mathbb{R})/S = P^1$ is a maximal boundary for $SL(2, \mathbb{R})$.

One can continue this development for $SL(m, \mathbb{R})$. As in the two-dimensional case, we have the Iwasawa decomposition $K = SO(m)$ and $S = AN$, where A is the set of diagonal matrices with determinant 1, and N is the set of upper-triangular matrices with 1's on the diagonal. Let $\mathcal{F}_m$ be the space of flags. That is to say, $m-1$-tuples $(V_1, ..., V_{m-1})$ where each $V_k \subset \mathbb{R}^m$ is a k-dimensional subspace and $V_k \subset V_{k+1}$. This is also compact, and $SO(m)$ acts transitively. Likewise, if we pick a diagonal matrix with entres $g_1 > g_2 > ... > g_n$, then by a similar argument we can see that $\mathcal{F}_m$ is a strongly proximal $SL(m, \mathbb{R})$-space. Likewise, the normalizer of S is precisely the set of upper-triangular matrices, which preserve the standard flag given by $V_i = \text{span}(e_1, ..., e_i)$. Hence $\mathcal{F}_m$ is the maximal boundary for $SL(m, \mathbb{R})$.

Likewise, the maximal boundary for $SL(2, \mathbb{C})$ is the space of 1-dimensional complex lines. However, we can identify $SL(2, \mathbb{C})$ with the group of $4x4$ unimodular matrices preserving the Lorenz form $x^2 + y^2 + z^2 - t^2$. In other words, it leaves invariant the cone $x^2 + y^2 + z^2 = t^2$, which can be identified with the 2-sphere $x^2 + y^2 + z^2 = 1$. This gives an action of $SL(2, \mathbb{C})$ on S^2 which is the maximal boundary.

4 An Abstract Poisson Boundary

Using work of Gelfand, one can quickly construct a compact space which admits a Poisson representation formula.

Definition 4. *We say that a function $f(g) \to \mathbb{R}$ is* left uniformly continuous *(l.u.c) if it is bounded, and if whenever $g_n \to e$ we have $f(g_n g) \to f(g)$ uniformly in g.*

Let $\mathcal{H}$ be the family of l.u.c μ-harmonic functions. These will correspond to continuous boundary functions. Now let $\Omega = (G^{\mathbb{N}}, \mu^{\mathbb{N}})$ be the product space, and X_i the projection of the ith coordinate.

Definition 5. *Let $\mathcal{A}$ be the set of functions $z(g, \omega) : G \times \Omega \to \mathbb{R}$ defined as such that*

$$z(g, \omega) = \lim_{n\to\infty} f(gX_1X_2...X_n),$$

where f is l.u.c. and the limit exists almost surely for each g.

Then $\mathcal{A}$ forms an algebra of functions on $G \times \Omega$, and we have a right G-action given by $z^g(g', \omega) = z(gg', \omega)$. Since f is l.u.c. then the G-action is continuous.

We have a natural map $\alpha : \mathcal{A} \to \mathcal{H}$ given by setting $f(g) := \mathbb{E}z(g, \omega)$. That f is l.u.c follows because $z(g, \omega)$ is defined in terms of an l.u.c. function, and that f is μ-harmonic follows from the definition of $z(g, \omega)$.

We have another map $\beta : \mathcal{H} \to \mathcal{A}$ defined as follows: for $g \in G$ let $W_n(g) = f(gX_1...X_n)$. As f is μ-harmonic then $\mathbb{E}f(gX_n) = f(g)$. Therefore $W_n(g)$ is a bounded martingale, so that $\lim_{n\to\infty} W_n$ converges almost surely and defines a bounded function $z(g, \omega)$. We let $\beta(f) := z$. By the DCT for conditional expectation we get $\mathbb{E}\beta(f) = f$, so that $\alpha(\beta(f))$. Likewise, we have $\beta(\alpha(f))$. We can equip both $\mathcal{A}$ and $\mathcal{H}$ with the L^∞ norm, so that α and β are in fact isometric isomorphisms. The purpose of this transformation $\mathcal{H} \to \mathcal{A}$ is that the latter is an algebra. Indeed, it is a commutative C^*-algebra, so we can pull back the multiplication and turn $\mathcal{H}$ into a commutative C^*-algebra as well. By Gelfand's representation theorem, $\mathcal{H}$ is isomorphic as a C^*-algebra to the space of continuous functions with compact support on some topological spcae Π. As $\mathcal{H}$ contains the constants, then Π is compact. We call this space the *Poisson space* of (G, μ).

Recall that the points of Π are algebra homomorphisms of $\mathcal{H}$ onto $\mathbb{C}$. Clearly G acts on $\mathcal{H}$ to the right by $f^g(g') = f(gg')$. Hence G acts continuously on Π. We easily obtain a Poisson representation formula. Let $C(\Pi)$ be the algebra of continuous complex valued functions on Π. If $\varphi \in C(\Pi)$ corresponds to a function $f \in \mathcal{H}$, let $L(\phi)$ denote $f(e)$. Observe that L is a positive linear functions that sends 1 to 1, so that by the Riesz Representation Theorem we have $L(\phi) = \nu(\phi)$ for some ν. We get

$$f(g) = f^g(e) = L(\varphi^g) = \int_\Pi \varphi^g(p)d\nu = \int_\Pi \varphi(gp)d\nu(p).$$

We obtain the following:

Theorem 5. *There exists a probability measure ν on the Poisson space such that the formula*

$$f(g) = \int_{\Pi} \varphi(gp)d\nu(p)$$

gives a one-to-one correspondence between l.u.c. μ-harmonic functions and continuous functions on Π.

5 The Poisson Boundary and the Maximal Boundary

So far we've constructed two types of boundaries. The first is an abstract space that comes equipped with a Poisson representation formula, but is utterly devoid of geometry. The second is entirely geometric, but a priori does not admit a Poisson representation formula. How are these two boundaries related?

Denote the Poisson space by Π. By similar techniques as in section 4, one can show the following:

Theorem 6. *Let K be a maximal compact subgroup of G. Then K acts transitively on Π. Moreover, Π maps equivariantly onto any boundary B of G. Letting ν be the stationary measure on Π, we have $X_1X_2...X_n\nu$ converges almost surely to a point measure.*

The relationship between the two boundaries is the following.

Theorem 7. *The Poisson space is a finite covering of the maximal boundary B.*

Recall that $H(G) = K_0S$ where K_0 is compact (and so has finitely many connected components), and S is connected. Hence $H(G)$ only has finitely many connected components. Letting $H_0(G)$ be the connected component of the identity of $H(G)$, the group $H(G)/H_0(G)$ is finite. For each Poisson space Π, the stabilizer H_π of a point will be contained in $H(G)$. As $H(G)/H_\pi$ is finite, then H_π contains $H_0(G)$. We deduce the following:

Theorem 8. *A semi-simple group G with finite center possesses a finite number of Poisson spaces, each of which is a finite covering spces of the maximal boundary $B(G)$.*

For the Poisson space Π we have a Poisson representation formula for l.u.c. harmonic functions. By an approximation argument we get the following:

Theorem 9. *Let G be a semi-simple Lie group with finite center, and let $B(G)$ be its maximal boundary. There then is a finite collection Π_i of covering spaces with the following property. if μ is any absolutely continuous probaiblity measure on G, then the family of bounded solutions to*

$$f(g) = \int f(gg')d\mu(g')$$

is in one-to-one correspondence with the set of bounded measurable functions on some Π_i, given by

$$f(g) = \int_{\Pi_i} \hat{f}(p)dg\nu(p)$$

for some measure ν on Π_i.

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