Topological Recurrence and Van Der Waerden's Theorem

Kunal Chawla

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Theorem (Poincare Recurrence)

Let $(X, \mathcal{F}, \mu)$ be a probability space and $T : X \to X$ a measure preserving transformation. Then for any set U of positive measure, we have $\mu(T^{-n}U \cap U) > 0$ for infinitely many n.

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Let (X, F, T) be a (compact) topological dynamical system and $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Then for some open set U_α we have

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Proposition

Van Der Waerdens' theorem and Multiple Recurrence are equivalent.

Proof.

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Theorem (Multiple Recurrence)

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For every $k \geq 1$ *these exists an open set* U_α *which contains an arithmetic progression* $x, T^r, T^{2r}x, \ldots, T^{(k-1)r}x$ *for some* $x \in X$ *and* $r > 0$*.*

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Starting the Induction

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Proof.

Induct on k. The case $k = 1$ is trivial. □

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Theorem (Minimal Recurrence)

Let (X, F, T) be a minimal system and U a non-empty open set, and $k \geq 1$. Then U contains an arithmetic progression $x, T^r, ..., T^{(k-1)r}x$ for some $x \in X$ and $r \geq 1$.

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The Technical Part of The Induction

Lemma

For any $J \geq 0$ there exists a sequence $x_0, \ldots, x_J$ of points in X, a sequence $U_{\alpha_0}, \ldots, U_{\alpha_J}$ of sets in the open cover, and $r_1, \ldots, r_J$ such that $T^{i(r_{a+1}+\ldots+r_b)} x_b \in U_{\alpha_a}$ for all $0 \leq a \leq b \leq J$ and $1 \leq i \leq k-1$.

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Proof.

Induct on J. Suppose for $J \geq 1$ we have constructed $x_0, \ldots, x_{J-1}$, $U_{\alpha_0}, \ldots, U_{\alpha_{J-1}}$, and $r_1, \ldots, r_{J-1}$.

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$$T^{i(r_{a+1}+\ldots+r_J)}x_J = T^{i(r_{a+1}+\ldots+r_{J-1})}\left(T^{(i-1)r_J}y\right) \in T^{i(r_{a+1}+\ldots+r_{J-1})}(V).$$

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As $T^{i(r_{a+1}+\ldots+r_{J-1})}x_{J-1} \in U_{\alpha_a}$, then we can pick V sufficiently small. □

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End of The Main Induction.

Let J be the number of sets in the open cover $\{U_\alpha\}_{\alpha \in A}$.

Lemma

For any $J \geq 0$ *there exists a sequence* $x_0, ..., x_J$ *of points in* X*, a sequence* $U_{\alpha_0}, ..., U_{\alpha_J}$ *of sets in the open cover, and* $r_1, ..., r_J$ *such that* $T^{i(r_{a+1}+\cdots+r_b)}x_b \in U_{\alpha_a}$ *for all* $0 \leq a \leq b \leq J$ *and* $1 \leq i \leq k-1$*.*

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$$x, T_r x, ..., T^{(k-1)r}x.$$

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To recap:

1. Rephrased the combinatorial statement in terms of topological recurrence.
2. Used structure of topological dynamical systems to make the proof easier.

Theorem (Szemerèdi's Theorem)

Let $k \geq 1$ and let A be a set of integers of positive upper density, i.e. $\limsup_{N\to\infty} \frac{1}{2N+1}|A \cap \{-N, ..., N\}| > 0$. *Then A contains an arithmetic progression* $n, n+r, ..., n+(k-1)r$.

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We use a 'compactness and contradiction argument'. Consider $2^{\mathbb{Z}}$ with T the left-shift, and suppose that A has positive upper density. Viewing $A \in 2^{\mathbb{Z}}$, consider the orbit closure $\overline{T^{\mathbb{Z}}A}$ as a subsystem.

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Suppose for contradiction that for some k, $A \cap T^r A \cap \ldots \cap T^{(k-1)r} A = \varnothing$ for all $r > 0$.
Consider the set $E = \{B \in 2^{\mathbb{Z}} | B \text{ contains } 0\}$, so that $E \cap T^r E \cap \ldots \cap T^{(k-1)r} E = \varnothing$.

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For each N, consider the measure μ_N assigning a mass of $\frac{1}{2N+1}$ to $T^{-n}A$, for $-N \leq n \leq N$, so that $\mu_N(E) = \frac{1}{2N+1}|A \cap \{-N, ..., N\}|$.

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Recall (by Curtis) that the set of Borel probability measures is sequentially compact in the weak-* topology, meaning that for some subsequence μ_{N_k} we have some μ such that $\int f d\mu_{N_k} \to \int f d\mu$ for all continuous f.

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As $\{\mu_N\}_{n \in \mathbb{N}}$ is asymptotically T-invariant, then μ is T-invariant and $\mu(E) > 0$. □

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In Conclusion

We've

1. Shown a correspondence between Ramsey-type statements and recurrence theorems
2. Given a topological proof of Van Der Waerden's theorem
3. Shown how to translate Szemeredi's theorem into an ergodic theoretical statement.

Theorem (Green-Tao)

The primes have arbitrarily large arithmetic progressions.