The Sensitivity Conjecture

Kunal Chawla

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Question

How does block sensitivity relate to sensitivity?

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Conjecture (Sensitivity Conjecture)

There exists $C > 0$ such that

$$bs(f) \leq s(f)^C$$

for all boolean functions f.

A related problem

Let Q^n be the graph defined as follows:

- The vertices of Q^n are elements of $(\mathbb{Z}/2\mathbb{Z})^n$
- Two elements $x, y \in \{\mathbb{Z}/2\mathbb{Z}\}^n$ are adjacent if there exists some i with $y = x + e_i$.

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Conjecture (Hypercube Sensitivity)

If H is an induced subgraph of Q^n with $|V(H)| \geq 2^{n-1} + 1$ vertices, then $\Delta(H) \geq \sqrt{n}$.

Hypercube Sensitivity implies Sensitivity

- Let $X_i : (\mathbb{Z}/2\mathbb{Z})^n \to \{-1, 1\}$ be given by

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- Let $\mu(x) = X_1(x) ... X_n(x)$.

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- Let $m = \deg(f)$. By permuting indices, and restricting to $(\mathbb{Z}/2\mathbb{Z})^m$, assume WLOG that $m = n$.

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- The Fourier coefficient $X_1 ... X_n$ is

$$\frac{1}{2^n} \sum_{x \in (\mathbb{Z}/2\mathbb{Z})^n} f(x)\mu(x) \neq 0.$$

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- Hence $|\{x : \mu(x)f(x) = 1\}| \neq |\{x : \mu(x)f(x) = -1\}|$.

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- By Hypercube Sensitivity, there exists $x \in E$ such that $x + e_i \in E$ for $\lceil \sqrt{n} \rceil$ values of i.

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- By Hypercube Sensitivity, there exists $x \in E$ such that $x + e_i \in E$ for $\lceil \sqrt{n} \rceil$ values of i.
- As $\mu(x + e_i) = -\mu(x)$ we have $f(x + e_i) \neq f(x)$, for at least $\sqrt{n}$ values of i.

□

Why is $\sqrt{n}$ sharp?

Theorem

There exists an induced subgraph $H \subset Q^n$ *such that* $|V(H)| = 2^{n-1} + 1$ *and* $\Delta(H) \leq \lceil \sqrt{n} \rceil$.

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$$V_j = \{x : x_i = 0 \text{ for } i \in I_j\}.$$

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- Let $I_1, \ldots, I_k$ be a partition of $\{1, \ldots, n\}$.Let

$$V_j = \{x : x_i = 0 \text{ for } i \in I_j\}.$$

- By the binomial theorem,

$$\sum_{x \in \bigcap_{j \in J} V_j} \mu(x) = 1_{n = \sum_{j \in J} |I_j|}.$$

Proof.

- By inclusion-exclusion,

$$\sum_{x \in (\mathbb{Z}/2\mathbb{Z})^n \setminus \cup_{j=1}^{k} V_j} \mu(x) = (-1)^k.$$

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- So

$$\sum_{x \in \cup_{j=1}^k V_j} (-1)^{k+1}\mu(x) + \sum_{x \notin \cup_{j=1}^k V_j} (-1)^k \mu(x) = 2.$$

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- Let $H = H_1 \cup H_2$, so that $|E| = 2^{n-1} + 1$.

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Proof.

- If $x \in H_2$ then $(-1)^k \mu(x + e_i) = 1$ for all i.

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- If $x \in H_1$ then $(-1)^{k+1} \mu(x + e_i) = 1$, and there is at most one$i$ such that $x + e_i \in V_j$. Hence $\deg(x) \le k$.
- This implies $\Delta(H) \le \max\{\max_j |I|_j, k\}$.
- Let I_j be a partition of $[n]$ into $\sim \sqrt{n}$ pieces of size $\sim \sqrt{n}$, so that $\Delta(H) \le \lceil \sqrt{n} \rceil$.

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We've seen

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- Why the $\sqrt{n}$ is sharp

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Lemma

Let H be a graph. If λ is an eigenvalue of the adjacency matrix $A(H)$, then $|\lambda| \leq \Delta(H)$.

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Let v be an eigenvector for λ and suppose that v_1 is the largest coordinate of v (in absolute value). Then

$$|\lambda v_1| = \left|\sum_{k=1}^{m} A_{1,k} v_k\right| \leq \sum_{k=1}^{m} |A_{1,k}||v_1| \leq \Delta(H)|v_1|.$$

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We have actually proven the following:

Lemma

Let B be a matrix such that $|B| \leq A(H)$ entrywise. Then if λ is an eigenvalue of B, we have $\lambda \leq \Delta(H)$.

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Construct an auxiliary object with nice spectral properties.

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Question

How can we estimate the eigenvalues of a pseudo-adjacency matrix of H?

Theorem (Cauchy's interlace Theorem)

Suppose that A is a symmetric $n \times n$ matrix, and B is a principal $m \times m$ submatrix, where $m < n$. If the eigenvalues of A are $\lambda_1 \leq ... \leq \lambda_n$ and the eigenvalues of B are $\beta_1 \leq ... \leq \beta_m$, then

$$\lambda_i \leq \beta_i \leq \lambda_{i+n-m}.$$

Lemma (Min-Max principle)

Suppose A is an $n \times n$ symmetric matrix, and let $R_A(x) = \frac{\langle Ax, x \rangle}{\langle x, x \rangle}$ be the Rayleigh quotient. If $\lambda_1 \leq ... \leq \lambda_k \leq ... \lambda_n$ are the eigenvalues of A, then

$$\lambda_k = \min_U \{\max_x \{R_A(x) | x \in U \text{ and } x \neq 0 | dim(U) = k\}\},$$

and

$$\lambda_k = \max_U \{\min_x \{R_A(x) | x \in U \text{ and } x \neq 0 | dim(U) = n - k + 1\}\}.$$

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Proof.

Let $A = \begin{bmatrix} B & X^T \\ X & Z \end{bmatrix}$, and let $\{v_k, ..., v_n\}$ be the last $n - k + 1$ eigenvectors of A, and $\{w_1, ..., w_m\}$ the eigenvectors of B.

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Let $V = \text{span}\{v_k, ..., v_n\}$, $W = \text{span}(w_1, ..., w_k)$. Lift $\widehat{W} = \begin{pmatrix} W \\ 0 \end{pmatrix}$.

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Hence $R_A(\hat{w}) = R_B(w)$

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The other direction is similar, with $V = \text{span}(v_1, \ldots, v_{k+n-m})$ and $W = \text{span}(w_k, \ldots, w_m)$. □

Proof of the main theorem

Consider the following sequence of matrices given by

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

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Lemma

Replacing every -1 with 1, we get the adjacency matrix of Q^n.

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Lemma

$A_n^2 = nI$

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Lemma

The eigenvalues of A_n are $\pm\sqrt{n}$ with multiplicity 2^{n-1}.

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Proof.

Let A_H be the induced principle submatrix of A. As $|H| \geq 2^{n-1} + 1$, then by Cauchy's interlacing theorem, we have

$$\Delta H \geq \lambda_1(A_H) \geq \lambda_{2^{n-1}}(A_n) = \sqrt{n}.$$

□

In Summary

We

1. Translated a problem into a combinatorial statement about graphs.
2. Encountered a variational representation of eigenvalues.
3. Used spectral methods to lower bound the maximum degree of a graph.