Plane Scattering

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We consider the problem of a plane wave scattered by a n-layer slab. Section 1 gives the problem description. Section 2 gives the analytic solution. Section 3 shows some examples taken from published works and the reproduced solutions.

1 Problem Statement

Consider the problem of a plane wave polarized in the $+x$ direction and propagating in the $+z$ direction. This plane wave impinges on a n-layer dielectric slab as shown in Fig. 1. The plane wave undergoes multiple transmissions and reflections.

The forward propagating wave in the i-th layer has non-zero electric and magnetic fields, which are given by

$$\vec{E}^{fwd}_{(i)} = E^{fwd}_{(i)} e^{-jk_{(i)}z} \vec{a}_x \tag{1a}$$

$$\vec{H}^{fwd}_{(i)} = \frac{1}{\eta_{(i)}} E^{fwd}_{(i)} e^{-jk_{(i)}z} \vec{a}_y \tag{1b}$$

The backward propagating wave in the i-th layer also has non-zero electric and magnetic fields, which are given by

$$\vec{E}^{bwd}_{(i)} = E^{bwd}_{(i)} e^{+jk_{(i)}z} \vec{a}_x \tag{2a}$$

$$\vec{H}^{bwd}_{(i)} = -\frac{1}{\eta_{(i)}} E^{bwd}_{(i)} e^{jk_{(i)}z} \vec{a}_y \tag{2b}$$

The objective is to find the coefficient $E^{fwd}_{(i)}$ of the forward propagating wave and the coefficient $E^{bwd}_{(i)}$ of the backward propagating wave in all layers.

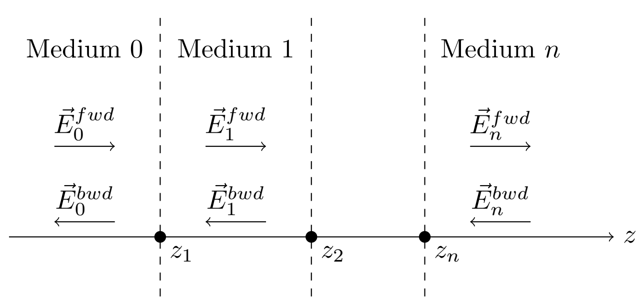

Figure 1: Forward and backward propagating waves in a multilayer slab.

2 Analytic Solution

The total electric field in the i-th layer is given by $\vec{E}_{(i)}^{fwd} + \vec{E}_{(i)}^{bwd}$ and the total magnetic field in the i-th layer is given by $\vec{H}_{(i)}^{fwd} + \vec{H}_{(i)}^{bwd}$.

The tangential electric field and the tangential magnetic field at the interface located at z_i must be continuous. These boundary conditions are translated into

$$\vec{E}_{(i-1)}^{fwd}(z_{(i)}) + \vec{E}_{(i-1)}^{bwd}(z_{(i)}) = \vec{E}_{(i)}^{fwd}(z_i) + \vec{E}_{(i)}^{bwd}(z_i) \tag{3a}$$

$$\vec{H}_{(i-1)}^{fwd}(z_{(i)}) + \vec{H}_{(i-1)}^{bwd}(z_{(i)}) = \vec{H}_{(i)}^{fwd}(z_i) + \vec{H}_{(i)}^{bwd}(z_i) \tag{3b}$$

Substituting (1), (2) into (3), we get the following relation

$$\underbrace{\begin{bmatrix} e^{-jk_{(i-1)}z_{(i)}} & e^{+jk_{(i-1)}z_{(i)}} \\ \frac{1}{\eta_{(i-1)}}e^{-jk_{(i-1)}z_{(i)}} & -\frac{1}{\eta_{(i-1)}}e^{+jk_{(i-1)}z_{(i)}} \end{bmatrix}}_{A_{(i)}} \begin{bmatrix} E_{(i-1)}^{fwd} \\ E_{(i-1)}^{bwd} \end{bmatrix} = \underbrace{\begin{bmatrix} e^{-jk_{(i)}z_{(i)}} & e^{+jk_{(i)}z_{(i)}} \\ \frac{1}{\eta_{(i)}}e^{-jk_{(i)}z_{(i)}} & -\frac{1}{\eta_{(i)}}e^{+jk_{(i)}z_{(i)}} \end{bmatrix}}_{B_{(i)}} \begin{bmatrix} E_{(i)}^{fwd} \\ E_{(i)}^{bwd} \end{bmatrix} \tag{4}$$

Applying the boundary conditions to all interfaces at z_1 ... z_n, we have

$$T \begin{bmatrix} E_{(0)}^{fwd} \\ E_{(0)}^{bwd} \end{bmatrix} = \begin{bmatrix} E_{(n)}^{fwd} \\ E_{(n)}^{bwd} \end{bmatrix} \tag{5}$$

where T is a 2×2 matrix given by $B_{(n)}^{-1}A_{(n)} \cdots B_{(1)}^{-1}A_{(1)}$.

Since $E_{(0)}^{fwd} = 1$ in Medium 0 and $E_{(n)}^{bwd} = 0$ in Medium n, $E_{(0)}^{bwd}$ and $E_{(n)}^{fwd}$ in (5) are given by

$$E_{(0)}^{bwd} = -\frac{T_{21}}{T_{22}} \tag{6}$$

$$E_{(n)}^{fwd} = T_{11} - \frac{T_{12}T_{21}}{T_{22}} \tag{7}$$

where T_{11}, T_{12}, T_{21} and T_{22} are the elements of T. Coefficients of the forward and backward propagating wave can be found by iterating from the coefficients in Medium 0 or in Medium n through (4).

3 Validation

Fig. 2 shows the responses of a single-section, two-section binomial, and two-section Tschebyscheff quarter-wavelength transformers. The responses are consistent with those in [1, Fig. 5-19].

Fig. 3 shows the reflections of dielectric mirrors, which are consistent with those reported in [2, Fig. 6.3.2].

References

[1] C. A. Balanis, *Advanced Engineering Electromagnetics.* New York, NY: John Wiley & Sons, Inc., 1989.

[2] S. J. Orfanidis, *Electromagnetic Waves and Antennas.* Online, 2016. [Online]. Available: http://eceweb1.rutgers.edu/~orfanidi/ewa/

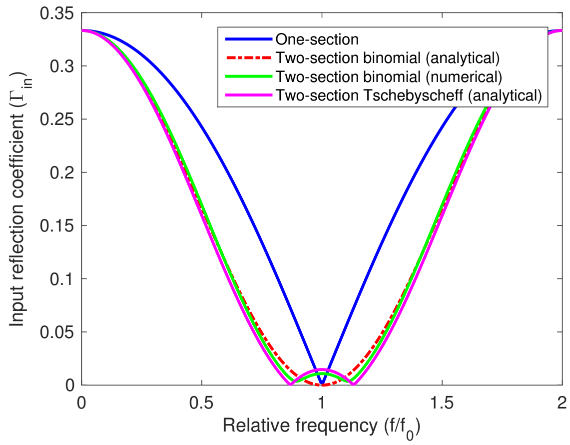

Figure 2: Responses of a single-section, two-section binomial, and two-section Tschebyscheff quarter-wavelength transformers.

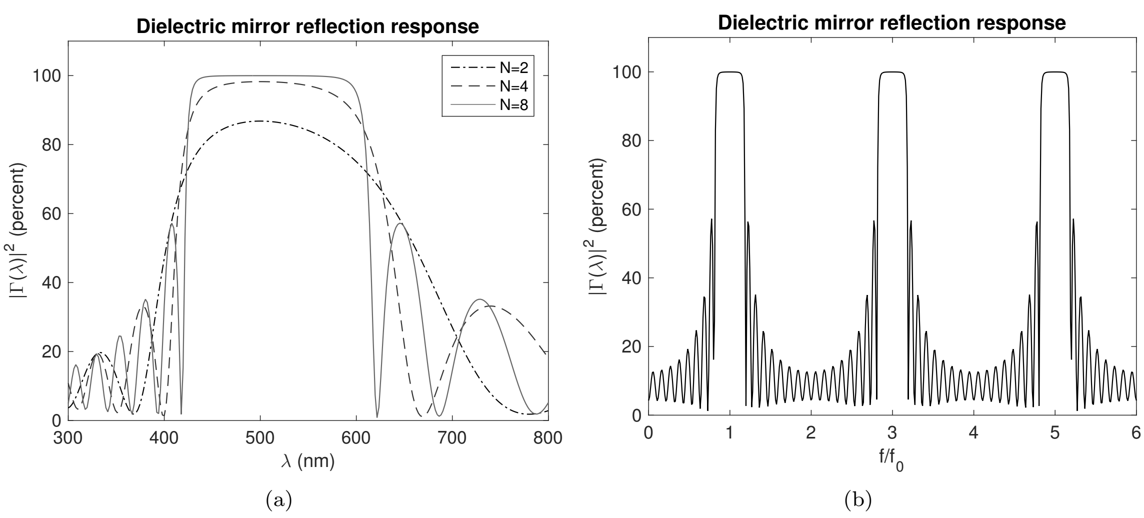

Figure 3: Dielectric mirror with quarter-wavelength layers.