Scattering of a Multi-layer Sphere

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1 Problem Statement and Solution Overview

The problem of a plane wave scattered by an L-layer medium is a boundary value problem in solving partial differential equations [1, Ch. 3.7].
Fig. 1 shows a plane wave polarized in the $+x$ direction and propagating in the $+z$ direction. The plane wave is scattered by an L-layer sphere. The centre of the sphere coincides with the origin of the spherical coordinate system.
Solving such problems includes the following steps:

1. Selecting a coordinate system so that the boundaries between layers conform to some of the principal axes of the coordinate system.
2. Selecting some potentials, which convert the vector Maxwell equations to the scalar Helmholtz equation.
3. Solving the Helmholtz equation using the method of separation of variables, which decomposes it into three linear ordinary differential equations.
4. Applying the homogeneous boundary condition to solve for the eigen functions of the ordinary differential equations.
5. Expressing the incident plane wave in terms of potentials, which are in series forms of the products of the eigen functions.
6. Expressing the scattered potentials and the potentials inside each layer in the series forms with some unknown coefficients.
7. Applying the conditions at the boundaries between layers, which leads to a system of linear equations.
8. Solving this linear system gives the coefficients of the potentials.
9. Expressing the fields in terms of the potentials.

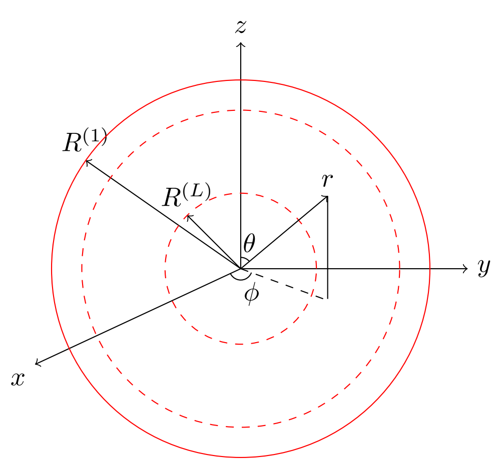


Figure 1: An x-polarized plane wave, propagating in the z direction, impinges upon an L-layer sphere. Starting from the inner core, each layer is characterized by its radius, permittivity, and permeability.

2 Potentials and Construction of Solutions

The purpose of using potentials instead of the electric and magnetic field vectors is that the Maxwell equations expressed in terms of potentials become scalar equations. This reduces the number of unknowns to be calculated.[1] Here, we follow the approach in [4, 129] to our problem, in which the magnetic and electric vector potentials are introduced to the scattering problem of a dielectric sphere. The magnetic vector potential $\vec{A}$ and the electric vector potential $\vec{F}$ are defined so that

$$\begin{aligned} \nabla \times \vec{A} &= \quad \mu \vec{H} \ , \\ \nabla \times \vec{F} &= - \epsilon \vec{E} \ , \end{aligned}$$

where ϵ and μ denote the permittivity and permeability. We assume $e^{j\omega t}$ to be the time-harmonic factor.[2] The Maxwell equations can be equivalently expressed in terms of the vector potentials as

$$\nabla \times \nabla \times \vec{A} - k^2 \vec{A} = j\omega\mu\epsilon \nabla \Phi^e \ , \tag{1a}$$

$$\nabla \times \nabla \times \vec{F} - k^2 \vec{F} = j\omega\mu\epsilon \nabla \Phi^m \ , \tag{1b}$$

where k is the wave number and $k = \omega\sqrt{\mu\epsilon}$, and ω is the angular frequency. Φ^m and Φ^e are some arbitrary scalar field quantities that will be discussed shortly. The electric and magnetic field can be expressed in terms of the potentials as

$$\vec{E} = -\frac{1}{\epsilon} \nabla \times \vec{F} + \frac{1}{j\omega\mu\epsilon} \nabla \times \nabla \times \vec{A} \ , \tag{2a}$$

$$\vec{H} = \quad \frac{1}{\mu} \nabla \times \vec{A} + \frac{1}{j\omega\mu\epsilon} \nabla \times \nabla \times \vec{F} \ . \tag{2b}$$

A convenient choice of the vector potentials in the scattering problems in a spherical coordinate system is to let $\vec{A} = A_r \vec{a}_r$ and $\vec{F} = F_r \vec{a}_r$. Substituting this choice to (1a) leads to

$$-\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial A_r}{\partial\theta} - \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 A_r}{\partial\phi^2} - k^2 A_r = -j\omega\mu\epsilon \frac{\partial \Phi^e}{\partial r} \ , \tag{3a}$$

$$\frac{\partial^2 A_r}{\partial r \partial\theta} = -j\omega\mu\epsilon \frac{\partial \Phi^e}{\partial\theta} \ , \tag{3b}$$

$$\frac{\partial^2 A_r}{\partial r \partial\phi} = -j\omega\mu\epsilon \frac{\partial \Phi^e}{\partial\phi} \ . \tag{3c}$$

Because the choice of Φ^e is arbitrary, we set

$$\frac{\partial A_r}{\partial r} = -j\omega\mu\epsilon \Phi^e \ , \tag{4}$$

[1]The choice of the potentials is not unique as their divergence can be freely defined. The Debye potential is used for scattering problems. The Hertzian potential is often for radiation problems [2, 3].

[2]Some texts [5, 6] use $e^{-j\omega t}$ as the time-harmonic factor due to the preference to represent a wave propagating in the z direction as e^{jkz}. To cross check solutions based on these two conventions, it should be noted that the field is a physical quantity. The same wave propagating in the $+z$ direction can be expressed as $\Re\{Ae^{jkz}e^{-j\omega t}\}$ or $\Re\{\bar{A}e^{-jkz}e^{j\omega t}\}$, where $\bar{(\cdot)}$ denotes the conjugate. Thus, the phasors expressed in these two different conventions are conjugate pairs.

to satisfy both (3b) and (3c), and (3a) becomes the scalar Helmholtz equation of $\frac{A_r}{r}$, which is discussed in the next section. Eq. (2) becomes

$$E_r = \frac{1}{j\omega\mu\epsilon}\left[\frac{\partial^2}{\partial r^2} + k^2\right] A_r \ , \tag{5a}$$

$$E_\theta = \frac{1}{j\omega\mu\epsilon}\frac{1}{r}\frac{\partial^2}{\partial r \partial\theta} A_r - \frac{1}{\epsilon}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} F_r \ , \tag{5b}$$

$$E_\phi = \frac{1}{j\omega\mu\epsilon}\frac{1}{r\sin\theta}\frac{\partial^2}{\partial r \partial\phi} A_r + \frac{1}{\epsilon}\frac{1}{r}\frac{\partial}{\partial\theta} F_r \ , \tag{5c}$$

$$H_r = \frac{1}{j\omega\mu\epsilon}\left[\frac{\partial^2}{\partial r^2} + k^2\right] F_r \ , \tag{5d}$$

$$H_\theta = \frac{1}{\mu}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} A_r + \frac{1}{j\omega\mu\epsilon}\frac{1}{r}\frac{\partial^2}{\partial r \partial\theta} F_r \ , \tag{5e}$$

$$H_\phi = -\frac{1}{\mu}\frac{1}{r}\frac{\partial}{\partial\theta} A_r + \frac{1}{j\omega\mu\epsilon}\frac{1}{r\sin\theta}\frac{\partial^2}{\partial r \partial\phi} F_r \ . \tag{5f}$$

3 Helmholtz's Equation in a Spherical Coordinate System

The Helmholtz equation of a scalar quantity Φ, in the spherical coordinate system, is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} + k^2\Phi = 0 \ . \tag{6}$$

It is a linear partial differential equation and is solved by the method of separation of variables. Let $\Phi(r,\theta,\phi) = A(r)B(\theta)C(\phi)$. Substituting it to (6), we obtain three linear ordinary differential equations, which are

$$\frac{d}{dr}\left(r^2\frac{dA}{dr}\right) + [(kr)^2 - n(n+1)]A = 0 \ , \tag{7a}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dB}{d\theta}\right) + [n(n+1) - \frac{m^2}{\sin^2\theta}]B = 0 \ , \tag{7b}$$

$$\frac{d^2C}{d\phi^2} + m^2C = 0 \ , \tag{7c}$$

where n and m are non-negative integers and $m \le n$. The operators associated with these differential equations are of the Sturm-Liouville type [7, ch. 5]; they are self-adjoint operators and their eigen functions are orthogonal to each other. The eigen functions form a complete set; any piecewise smooth function can be represented by a linear combination of these eigen functions.[3]
If we let $x = kr$, eq. (7a) becomes the spherical Bessel equation. Its eigen functions are known as the family of the spherical Bessel functions, denoted as $b_n(x)$. Within this family, the ordinary spherical Bessel function is denoted as $j_n(x)$, representing the standing wave. The spherical Hankel function of the first and second kind are denoted as $h_n^{(1)}(x)$ and $h_n^{(2)}(x)$ representing the inward and outward propagating waves. If we let $x = \cos\theta$, eq. (7b) is known as the associated Legendre equation and one of its eigen functions is the associated Legendre polynomial denoted as $P_n^m(x)$. When $m = 0$, we have the ordinary Legendre polynomial denoted as $P_n(x)$. Eq. (7c) is known as the Harmonic equation and its eigen functions are $e^{\pm jm\phi}$. Any solution to (6) can be expressed as the infinite sum of the products of these eigen functions, expressed as

$$\sum_n \sum_m c_{nm} b_n(x)|_{x=kr} P_n^m(x)|_{x=\cos\theta} e^{\pm jm\phi} \ ,$$

where c_{nm} denotes some coefficient.

[3]This explains why so many functions are supported by the wave equation to propagate freely in space.

4 Wave Transformation

A plane electromagnetic wave polarized in the x direction propagating in the z direction is expressed in terms of the eigen functions in the Cartesian coordinate system.[4] The electric part of the plane wave is expressed as

$$\vec{E}^{inc} = e^{-jkz}\vec{a}_x \ , \tag{8}$$

where the magnitude is 1 V/m. The magnetic part of the plane wave is expressed as

$$\vec{H}^{inc} = \frac{E_x^{inc}}{\eta}e^{-jkz}\vec{a}_y \ . \tag{9}$$

Expressing electric part of the plane wave in the form of the product of the eigen functions gives

$$\vec{E}^{inc} = \sum_{n=0}^{\infty} j^{-n}(2n+1)j_n(x)|_{x=kr}P_n(x)|_{x=\cos\theta}\ \vec{a}_x \tag{10a}$$

$$= E_r\vec{a}_r + E_\theta\vec{a}_\theta + E_\phi\vec{a}_\phi \tag{10b}$$

$$= \sum_{n=0}^{\infty} j^{-n}(2n+1)j_n(x)|_{x=kr}P_n(x)|_{x=\cos\theta}(\sin\theta\cos\phi\vec{a}_r + \cos\theta\cos\phi\vec{a}_\theta - \sin\phi\vec{a}_\phi) \ . \tag{10c}$$

The conversion from (8) to (10a) is known as the wave transformation and (10c) is the expression in the spherical coordinate system.
Since the component in the direction of $\vec{a}_r$ in (10c) solely depends on A_r shown in (5a), substituting the component in the direction of $\vec{a}_r$ in (10c) to (5a) gives

$$A_r^{inc} = \frac{\cos\phi}{\omega}\sum_{n=1}^{\infty} j^{-n}\frac{2n+1}{n(n+1)}\hat{J}_n(x)|_{x=kr}P_n^1(x)|_{x=\cos\theta} \ , \tag{11}$$

where we define $\hat{B}_n(x) = xb_n(x)$.[5] Thus, $\hat{J}_n(x) = xj_n(x)$, $\hat{H}_n^{(1)}(x) = xh_n^{(1)}(x)$, and $\hat{H}_n^{(2)}(x) = xh_n^{(2)}(x)$.
Similarly, expressing the magnetic part of the plane wave in the form of the product of the eigen functions gives

$$F_r^{inc} = \frac{\sin\phi}{\omega\eta}\sum_{n=1}^{\infty} j^{-n}\frac{2n+1}{n(n+1)}\hat{J}_n(x)|_{x=kr}P_n^1(x)|_{x=\cos\theta} \ . \tag{12}$$

The fields due to A_r^{inc} are known as the TM fields. The fields due to F_r^{inc} are known as the TE fields.

5 Expressions of the Potentials

We use the superscript $(\cdot)^{(l)}$ to identify the quantities in the l^{th} layer of the multi-layer sphere. The superscript $(\cdot)^{(0)}$ denotes the background. The scattered potentials, A_r^{scatt} and F_r^{scatt}, propagate outward and

[4] It is easy to show that e^{-jkz} is an eigen function of the Helmholtz equation expressed in the Cartesian coordinate system, which is $\frac{\partial^2\Phi}{\partial z^2} + k^2\Phi = 0$.

[5] These functions are known as the Riccati-Bessel functions in [8, pp. 283]. However, a different definition of Riccati-Besssel functions exists in the literature [9,]. They are defined as

$$\begin{aligned} S_n(x) =& xj_n(x) \ , \\ C_n(x) =& -xy_n(x) \ , \\ \xi(x) =& xh_n^{(1)}(x) \ , \\ \zeta(x) =& xh_n^{(2)}(x) \ . \end{aligned}$$

There is a sign difference in the relation to $y_n(x)$.

take the following form

$$A_r^{scatt} = \frac{\cos\phi}{\omega}\sum_{n=1}^{\infty} a_n^{(0)}\hat{H}_n^{(2)}(x)|_{x=k^{(0)}r}P_n^1(x)|_{x=\cos\theta} \ , \tag{13a}$$

$$F_r^{scatt} = \frac{\sin\phi}{\omega\eta^{(0)}}\sum_{n=1}^{\infty} b_n^{(0)}\hat{H}_n^{(2)}(x)|_{x=k^{(0)}r}P_n^1(x)|_{x=\cos\theta} \ . \tag{13b}$$

The potentials inside the l^{th} layer, $A_r^{(l)}$ and $F_r^{(l)}$, contain both the outward and inward propagating wave and take the following form

$$A_r^{(l)} = \frac{\cos\phi}{\omega}\sum_{n=1}^{\infty}\left[a_n^{(l)}\hat{H}_n^{(2)}(x)|_{x=k^{(l)}r} + c_n^{(l)}\hat{H}_n^{(1)}(x)|_{x=k^{(l)}r}\right]P_n^1(x)|_{x=\cos\theta} \ , \tag{14a}$$

$$F_r^{(l)} = \frac{\sin\phi}{\omega\eta^{(l)}}\sum_{n=1}^{\infty}\left[b_n^{(l)}\hat{H}_n^{(2)}(x)|_{x=k^{(l)}r} + d_n^{(l)}\hat{H}_n^{(1)}(x)|_{x=k^{(l)}r}\right]P_n^1(x)|_{x=\cos\theta} \ . \tag{14b}$$

The potentials inside the L^{th} layer, i.e. the inner core, $A_r^{(L)}$ and $F_r^{(L)}$, contain the standing wave and take the following form

$$A_r^{(L)} = \frac{\cos\phi}{\omega}\sum_{n=1}^{\infty} a_n^{(L)}\hat{J}_n(x)|_{x=k^{(L)}r}P_n^1(x)|_{x=\cos\theta} \ , \tag{15a}$$

$$F_r^{(L)} = \frac{\sin\phi}{\omega\eta^{(L)}}\sum_{n=1}^{\infty} b_n^{(L)}\hat{J}_n(x)|_{x=k^{(L)}r}P_n^1(x)|_{x=\cos\theta} \ . \tag{15b}$$

Substituting the potentials to (5) gives the expressions of the electric fields and magnetic field. The scattered electric fields are given as [10, pp. 654]

$$E_r^{scat} = -j\cos\phi\sum_{n=1}^{\infty} a_n^{(0)}\left[\hat{H}_n^{(2)\prime\prime}(x)|_{x=k^{(0)}r} + \hat{H}_n^{(2)}(x)|_{x=k^{(0)}r}\right]P_n^1(x)|_{x=\cos\phi} \ , \tag{16a}$$

$$E_\theta^{scat} = \frac{\cos\phi}{k^{(0)}r}\sum_{n=1}^{\infty}\left[ja_n^{(0)}\hat{H}_n^{(2)\prime}(x)|_{x=k^{(0)}r}\sin\theta P_n^{1\prime}(x)|_{x=\cos\theta} - b_n^{(0)}\hat{H}_n^{(2)}(x)|_{x=k^{(0)}r}\frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta}\right] \ , \tag{16b}$$

$$E_\phi^{scat} = \frac{\sin\phi}{k^{(0)}r}\sum_{n=1}^{\infty}\left[ja_n^{(0)}\hat{H}_n^{(2)\prime}(x)|_{x=k^{(0)}r}\frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta} - b_n^{(0)}\hat{H}_n^{(2)}(x)|_{x=k^{(0)}r}\sin\theta P_n^{1\prime}(x)|_{x=\cos\theta}\right] \ , \tag{16c}$$

$$H_r^{scat} = \frac{-j\sin\phi}{\eta^{(0)}}\sum_{n=1}^{\infty} b_n^{(0)}\left[\hat{H}_n^{(2)\prime\prime}(x)|_{x=k^{(0)}r} + \hat{H}_n^{(2)}(x)|_{x=k^{(0)}r}\right]P_n^1(x)|_{x=\cos\phi} \ , \tag{16d}$$

$$H_\theta^{scat} = \frac{\sin\phi}{\eta^{(0)}k^{(0)}r}\sum_{n=1}^{\infty}\left[jb_n^{(0)}\hat{H}_n^{(2)\prime}(x)|_{x=k^{(0)}r}\sin\theta P_n^{1\prime}(x)|_{x=\cos\theta} - a_n^{(0)}\hat{H}_n^{(2)}(x)|_{x=k^{(0)}r}\frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta}\right] \ , \tag{16e}$$

$$H_\phi^{scat} = \frac{-\cos\phi}{\eta^{(0)}k^{(0)}r}\sum_{n=1}^{\infty}\left[jb_n^{(0)}\hat{H}_n^{(2)\prime}(x)|_{x=k^{(0)}r}\frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta} - a_n^{(0)}\hat{H}_n^{(2)}(x)|_{x=k^{(0)}r}\sin\theta P_n^{1\prime}(x)|_{x=\cos\theta}\right] \ . \tag{16f}$$

Substituting the potentials to (5) gives the expressions of the electric fields and magnetic field in the l^{th}

layer.

$$
\begin{aligned}
E_r^{(l)} =& -j\cos\phi \sum_{n=1}^{\infty} a_n^{(l)} \left[\hat{H}_n^{(2)\prime\prime}(x)|_{x=k^{(l)}r} + \hat{H}_n^{(2)}(x)|_{x=k^{(l)}r}\right] P_n^1(x)|_{x=\cos\phi} \\
&+ c_n^{(l)} \left[\hat{H}_n^{(1)\prime\prime}(x)|_{x=k^{(l)}r} + \hat{H}_n^{(1)}(x)|_{x=k^{(l)}r}\right] P_n^1(x)|_{x=\cos\phi} \\
E_\theta^{(l)} =& \frac{\cos\phi}{k^{(l)}r} \sum_{n=1}^{\infty} \left[ja_n^{(l)} \hat{H}_n^{(2)\prime}(x)|_{x=k^{(l)}r} \sin\theta P_n^{1\prime}(x)|_{x=\cos\theta} - b_n^{(l)} \hat{H}_n^{(2)}(x)|_{x=k^{(l)}r} \frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta}\right. \\
&\left. + jc_n^{(l)} \hat{H}_n^{(1)\prime}(x)|_{x=k^{(l)}r} \sin\theta P_n^{1\prime}(x)|_{x=\cos\theta} - d_n^{(l)} \hat{H}_n^{(1)}(x)|_{x=k^{(l)}r} \frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta}\right] \\
E_\phi^{(l)} =& \frac{\sin\phi}{k^{(l)}r} \sum_{n=1}^{\infty} \left[ja_n^{(l)} \hat{H}_n^{(2)\prime}(x)|_{x=k^{(l)}r} \frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta} - b_n^{(l)} \hat{H}_n^{(2)}(x)|_{x=k^{(l)}r} \sin\theta P_n^{1\prime}(x)|_{x=\cos\theta}\right. \\
&\left. + jc_n^{(l)} \hat{H}_n^{(1)\prime}(x)|_{x=k^{(l)}r} \frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta} - d_n^{(l)} \hat{H}_n^{(1)}(x)|_{x=k^{(l)}r} \sin\theta P_n^{1\prime}(x)|_{x=\cos\theta}\right] \\
H_r^{(l)} =& \frac{-j\sin\phi}{\eta^{(l)}} \sum_{n=1}^{\infty} b_n^{(l)} \left[\hat{H}_n^{(2)\prime\prime}(x)|_{x=k^{(l)}r} + \hat{H}_n^{(2)}(x)|_{x=k^{(l)}r}\right] P_n^1(x)|_{x=\cos\phi} \\
&+ d_n^{(l)} \left[\hat{H}_n^{(1)\prime\prime}(x)|_{x=k^{(l)}r} + \hat{H}_n^{(1)}(x)|_{x=k^{(l)}r}\right] P_n^1(x)|_{x=\cos\phi} \\
H_\theta^{(l)} =& \frac{\sin\phi}{\eta^{(l)}k^{(l)}r} \sum_{n=1}^{\infty} \left[jb_n^{(l)} \hat{H}_n^{(2)\prime}(x)|_{x=k^{(l)}r} \sin\theta P_n^{1\prime}(x)|_{x=\cos\theta} - a_n^{(l)} \hat{H}_n^{(2)}(x)|_{x=k^{(l)}r} \frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta}\right. \\
&\left. + jd_n^{(l)} \hat{H}_n^{(1)\prime}(x)|_{x=k^{(l)}r} \sin\theta P_n^{1\prime}(x)|_{x=\cos\theta} - c_n^{(l)} \hat{H}_n^{(1)}(x)|_{x=k^{(l)}r} \frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta}\right] \\
H_\phi^{(l)} =& \frac{-\cos\phi}{\eta^{(l)}k^{(l)}r} \sum_{n=1}^{\infty} \left[jb_n^{(l)} \hat{H}_n^{(2)\prime}(x)|_{x=k^{(l)}r} \frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta} - a_n^{(l)} \hat{H}_n^{(2)}(x)|_{x=k^{(l)}r} \sin\theta P_n^{1\prime}(x)|_{x=\cos\theta}\right. \\
&\left. + jd_n^{(l)} \hat{H}_n^{(1)\prime}(x)|_{x=k^{(l)}r} \frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta} - c_n^{(l)} \hat{H}_n^{(1)}(x)|_{x=k^{(l)}r} \sin\theta P_n^{1\prime}(x)|_{x=\cos\theta}\right]
\end{aligned}
$$

Substituting the potentials to (5) gives the expressions of the electric fields and magnetic field in the L^{th} layer.

$$
\begin{aligned}
E_r^{(L)} =& -j\cos\phi \sum_{n=1}^{\infty} a_n^{(L)} \left[\hat{H}_n^{(2)\prime\prime}(x)|_{x=k^{(L)}r} + \hat{H}_n^{(2)}(x)|_{x=k^{(L)}r}\right] P_n^1(x)|_{x=\cos\theta} \quad , \\
E_\theta^{(L)} =& \frac{\cos\phi}{k^{(L)}r} \sum_{n=1}^{\infty} \left[ja_n^{(L)} \hat{H}_n^{(2)\prime\prime}(x)|_{x=k^{(L)}r} \sin\theta P_n^{1\prime}(x)|_{x=\cos\theta} - b_n^{(L)} \hat{H}_n^{(2)}(x)|_{x=k^{(L)}r} \frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta}\right] \quad , \\
E_\phi^{(L)} =& \frac{\sin\phi}{k^{(L)}r} \sum_{n=1}^{\infty} \left[ja_n^{(L)} \hat{H}_n^{(2)\prime\prime}(x)|_{x=k^{(L)}r} \frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta} - b_n^{(L)} \hat{H}_n^{(2)}(x)|_{x=k^{(L)}r} \sin\theta P_n^{1\prime}(x)|_{x=\cos\theta}\right] \quad , \\
H_r^{(L)} =& \frac{-j\sin\phi}{\eta^{(L)}} \sum_{n=1}^{\infty} b_n^{(L)} \left[\hat{H}_n^{(2)\prime\prime}(x)|_{x=k^{(L)}r} + \hat{H}_n^{(2)}(x)|_{x=k^{(L)}r}\right] P_n^1(x)|_{x=\cos\theta} \quad , \\
H_\theta^{(L)} =& \frac{\sin\phi}{\eta^{(L)}k^{(L)}r} \sum_{n=1}^{\infty} \left[jb_n^{(L)} \hat{H}_n^{(2)\prime\prime}(x)|_{x=k^{(L)}r} \sin\theta P_n^{1\prime}(x)|_{x=\cos\theta} - a_n^{(L)} \hat{H}_n^{(2)}(x)|_{x=k^{(L)}r} \frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta}\right] \quad , \\
H_\phi^{(L)} =& \frac{-\cos\phi}{\eta^{(L)}k^{(L)}r} \sum_{n=1}^{\infty} \left[jb_n^{(L)} \hat{H}_n^{(2)\prime\prime}(x)|_{x=k^{(L)}r} \frac{P_n^1(x)|_{x=\cos\theta}}{\sin\theta} - a_n^{(L)} \hat{H}_n^{(2)}(x)|_{x=k^{(L)}r} \sin\theta P_n^{1\prime}(x)|_{x=\cos\theta}\right] \quad .
\end{aligned}
$$

6 Boundary Conditions

The boundary conditions at the interface between the l^{th} layer and the $(l+1)^{th}$ layer are the continuities of the tangential components of the electric fields and magnetic fields. They are stated as

$$E_\theta^{(l)}(r^{(l+1)}) = E_\theta^{(l+1)}(r^{(l+1)}) \ , \tag{19a}$$

$$H_\theta^{(l)}(r^{(l+1)}) = H_\theta^{(l+1)}(r^{(l+1)}) \ . \tag{19b}$$

$$E_\phi^{(l)}(r^{(l+1)}) = E_\phi^{(l+1)}(r^{(l+1)}) \ , \tag{19c}$$

$$H_\phi^{(l)}(r^{(l+1)}) = H_\phi^{(l+1)}(r^{(l+1)}) \ . \tag{19d}$$

The electric and magnetic fields are dervied from the potentials from (5). Applying the boundary conditions in (19) to the electric and magnetic fields in the background and the first layer gives

$$a_n^{(0)} \begin{bmatrix} k^{(1)} \hat{H}_n^{(2)}{}'(x)|_{x=k^{(0)}r^{(1)}} \\ \mu^{(1)} \hat{H}_n^{(2)}(x)|_{x=k^{(0)}r^{(1)}} \end{bmatrix} + \begin{bmatrix} k^{(1)} j^{-n} \frac{2n+1}{n(n+1)} \hat{J}_n{}'(x)|_{x=k^{(0)}r^{(1)}} \\ \mu^{(1)} j^{-n} \frac{2n+1}{n(n+1)} \hat{J}_n(x)|_{x=k^{(0)}r^{(1)}} \end{bmatrix} = \begin{bmatrix} k^{(0)} \hat{H}_n^{(2)}{}'(x)|_{x=k^{(1)}r^{(1)}} & k^{(0)} \hat{H}_n^{(1)}{}'(x)|_{x=k^{(1)}r^{(1)}} \\ \mu^{(0)} \hat{H}_n^{(2)}(x)|_{x=k^{(1)}r^{(1)}} & \mu^{(0)} \hat{H}_n^{(1)}(x)|_{x=k^{(1)}r^{(1)}} \end{bmatrix} \begin{bmatrix} a_n^{(1)} \\ c_n^{(1)} \end{bmatrix} \ , \tag{20}$$

$$b_n^{(0)} \begin{bmatrix} k^{(1)} \hat{H}_n^{(2)}(x)|_{x=k^{(0)}r^{(1)}} \\ \mu^{(1)} \hat{H}_n^{(2)}{}'(x)|_{x=k^{(0)}r^{(1)}} \end{bmatrix} + \begin{bmatrix} k^{(1)} j^{-n} \frac{2n+1}{n(n+1)} \hat{J}_n(x)|_{x=k^{(0)}r^{(1)}} \\ \mu^{(1)} j^{-n} \frac{2n+1}{n(n+1)} \hat{J}_n{}'(x)|_{x=k^{(0)}r^{(1)}} \end{bmatrix} = \begin{bmatrix} k^{(0)} \hat{H}_n^{(2)}(x)|_{x=k^{(1)}r^{(1)}} & k^{(0)} \hat{H}_n^{(1)}(x)|_{x=k^{(1)}r^{(1)}} \\ \mu^{(0)} \hat{H}_n^{(2)}{}'(x)|_{x=k^{(1)}r^{(1)}} & \mu^{(0)} \hat{H}_n^{(1)}{}'(x)|_{x=k^{(1)}r^{(1)}} \end{bmatrix} \begin{bmatrix} b_n^{(1)} \\ d_n^{(1)} \end{bmatrix} \ . \tag{21}$$

Applying the boundary conditions in (19) to the electric and magnetic fields in the l^{th} layer and the $(l+1)^{th}$ layer gives

$$\begin{bmatrix} k^{(l+1)} \hat{H}_n^{(2)}{}'(x)|_{x=k^{(l)}r^{(l+1)}} & k^{(l+1)} \hat{H}_n^{(1)}{}'(x)|_{x=k^{(l)}r^{(l+1)}} \\ \mu^{(l+1)} \hat{H}_n^{(2)}(x)|_{x=k^{l}r^{(l+1)}} & \mu^{(l+1)} \hat{H}_n^{(1)}(x)|_{x=k^{l}r^{(l+1)}} \end{bmatrix} \begin{bmatrix} a_n^{(l)} \\ c_n^{(l)} \end{bmatrix} = \begin{bmatrix} k^{(l)} \hat{H}_n^{(2)}{}'(x)|_{x=k^{(l+1)}r^{(l+1)}} & k^{(l)} \hat{H}_n^{(1)}{}'(x)|_{x=k^{(l+1)}r^{(l+1)}} \\ \mu^{(l)} \hat{H}_n^{(2)}(x)|_{x=k^{(l+1)}r^{(l+1)}} & \mu^{(l)} \hat{H}_n^{(1)}(x)|_{x=k^{(l+1)}r^{(l+1)}} \end{bmatrix} \begin{bmatrix} a_n^{(l+1)} \\ c_n^{(l+1)} \end{bmatrix} \ , \tag{22}$$

$$\begin{bmatrix} k^{(l+1)} \hat{H}_n^{(2)}(x)|_{x=k^{(l)}r^{(l+1)}} & k^{(l+1)} \hat{H}_n^{(1)}(x)|_{x=k^{(l)}r^{(l+1)}} \\ \mu^{(l+1)} \hat{H}_n^{(2)}{}'(x)|_{x=k^{(l)}r^{(l+1)}} & \mu^{(l+1)} \hat{H}_n^{(1)}{}'(x)|_{x=k^{(l)}r^{(l+1)}} \end{bmatrix} \begin{bmatrix} b_n^{(l)} \\ d_n^{(l)} \end{bmatrix} = \begin{bmatrix} k^{(l)} \hat{H}_n^{(2)}(x)|_{x=k^{(l+1)}r^{(l+1)}} & k^{(l)} \hat{H}_n^{(1)}(x)|_{x=k^{(l+1)}r^{(l+1)}} \\ \mu^{(l)} \hat{H}_n^{(2)}{}'(x)|_{x=k^{(l+1)}r^{(l+1)}} & \mu^{(l)} \hat{H}_n^{(1)}{}'(x)|_{x=k^{(l+1)}r^{(l+1)}} \end{bmatrix} \begin{bmatrix} b_n^{(l+1)} \\ d_n^{(l+1)} \end{bmatrix} \ . \tag{23}$$

Applying the boundary condition in (19) to the electric and magnetic fields in the $(L-1)^{th}$ layer and the L^{th} layer gives

$$\begin{bmatrix} k^{(L)} \hat{H}_n^{(2)}{}'(x)|_{x=k^{(L-1)}r^{(L)}} & k^{(L)} \hat{H}_n^{(1)}{}'(x)|_{x=k^{(L-1)}r^{(L)}} \\ \mu^{(L)} \hat{H}_n^{(2)}(x)|_{x=k^{(L-1)}r^{(L)}} & \mu^{(L)} \hat{H}_n^{(1)}(x)|_{x=k^{(L-1)}r^{(L)}} \end{bmatrix} \begin{bmatrix} a_n^{(L-1)} \\ c_n^{(L-1)} \end{bmatrix} = a_n^{(L)} \begin{bmatrix} k^{(L-1)} \hat{J}_n{}'(x)|_{x=k^{(L)}r^{(L)}} \\ \mu^{(L-1)} \hat{J}(x)|_{x=k^{(L)}r^{(L)}} \end{bmatrix} \ , \tag{24}$$

$$\begin{bmatrix} k^{(L)} \hat{H}_n^{(2)}(x)|_{x=k^{(L-1)}r^{(L)}} & k^{(L)} \hat{H}_n^{(1)}(x)|_{x=k^{(L-1)}r^{(L)}} \\ \mu^{(L)} \hat{H}_n^{(2)}{}'(x)|_{x=k^{(L-1)}r^{(L)}} & \mu^{(L)} \hat{H}_n^{(1)}{}'(x)|_{x=k^{(L-1)}r^{(L)}} \end{bmatrix} \begin{bmatrix} b_n^{(L-1)} \\ d_n^{(L-1)} \end{bmatrix} = b_n^{(L)} \begin{bmatrix} k^{(L-1)} \hat{J}_n(x)|_{x=k^{(L)}r^{(L)}} \\ \mu^{(L-1)} \hat{J}_n{}'(x)|_{x=k^{(L)}r^{(L)}} \end{bmatrix} \ . \tag{25}$$

7 Implementation Issues

The infinite sum in (16) is approximated by a finite sum. The number of terms, $N_{\max}$, is empirically determined by $N_{\max} = \max(N_{\text{stop}}, |m^{(l)}x^{(l)}|, |m^{(l)}x^{(l-1)}|) + 15$, $l = 1, 2, ...L$, where $m^{(l)}$ is the relative refractive index $m^{(l)} = n^{(l)}/n^{(l-1)}$ and $x^{(l)}$ is related to the size of each layer as $x^{(l)} = k^{(l-1)}r^{(l)}$ [11]. The value of N_{stop} is the integer closest to

$$N_{\text{stop}} = \begin{cases} x^{(L)} + 4(x^{(L)})^{\frac{1}{3}} + 1, & 0.02 \le x^{(L)} < 8 \ , \\ x^{(L)} + 4.05(x^{(L)})^{\frac{1}{3}} + 2, & 8 \le x^{(L)} < 4200 \ , \\ x^{(L)} + 4(x^{(L)})^{\frac{1}{3}} + 2, & 4200 \le x^{(L)} < 20000 \ . \end{cases}$$

The derivatives of $\hat{B}_n(x)$ in (16) are computed from their recurrent relations

$$\begin{aligned} \hat{B}_n{}'(x) =& \frac{1}{2}\left[\hat{B}_{n-1}(x) + \frac{1}{x}\hat{B}_n(x) - \hat{B}_{n+1}(x)\right] \ , \\ \hat{B}_n{}''(x) =& \frac{1}{2}\left[\hat{B}_{n-1}{}'(x) + \frac{1}{x}\hat{B}_n{}'(x) - \frac{1}{x^2}\hat{B}_n(x) - \hat{B}_{n+1}{}'(x)\right] \ , \end{aligned}$$

which are easily derived from the recurrent relations of spherical Bessel functions. The associated Legendre polynomial is defined as[6]

$$P_n^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \ .$$

In this study, we are interested in the scattered field at $\theta = \pi$. Directly substituting $\theta = \pi$ into (16b) and (16c) incurs numerical difficulty when evaluating $\frac{P_n^1(x)}{\sin\theta}|_{x=\cos\theta}$ and $\sin\theta P_n^1{}'(x)|_{x=\cos\theta}$. This numerical difficulty is resolved by the recurrent relations of the associated Legendre polynomial. It is shown that [10, pp. 656]

$$\frac{P_n^1(x)}{\sin\theta}|_{x=\cos\theta} =(-1)^n\frac{n(n+1)}{2}, \quad \theta = \pi, \tag{26a}$$

$$\sin\theta P_n^1{}'(x)|_{x=\cos\theta} =(-1)^n\frac{n(n+1)}{2}, \quad \theta = \pi. \tag{26b}$$

To prove (26a), we use one of the recurrent relation of the associated Legendre polynomial, which is

$$P_{n+1}^1(x) = \frac{2n+1}{n}xP_n^1(x) - \frac{n+1}{n}P_{n-1}^1 \ ,$$

where $x = \cos\theta$. and thus

$$\frac{P_{n+1}^1(x)}{\sin\theta} = \frac{2n+1}{n}x\frac{P_n^1(x)}{\sin\theta} - \frac{n+1}{n}\frac{P_{n-1}^1(x)}{\sin\theta}. \tag{27}$$

Let us define $c_n(x) = \frac{P_n^1(x)}{\sin\theta}$, then (27) can be rewritten as

$$c_{n+1}(x) = \frac{2n+1}{n}xc_n(x) - \frac{n+1}{n}c_{n-1}(x) \ . \tag{28}$$

Since

$$\begin{aligned} P_1^1(x)|_{x=\cos\theta} =& -\sin\theta \ , \\ P_2^1(x)|_{x=\cos\theta} =& -3\cos\theta\sin\theta \ , \\ P_3^1(x)|_{x=\cos\theta} =& -\frac{3}{2}(5\cos^2\theta - 1)\sin\theta \ , \end{aligned}$$

[6] This definition of the associated Legendre polynomial contains the term $(-1)^m$, known as the Condon-Shortly phase [12, pp. 772]. The definition including this term is adopted in [10, pp. 951], [5, pp. 108], and [4, pp. 468]. Care must be exercised when comparing results in the books that use different definitions.

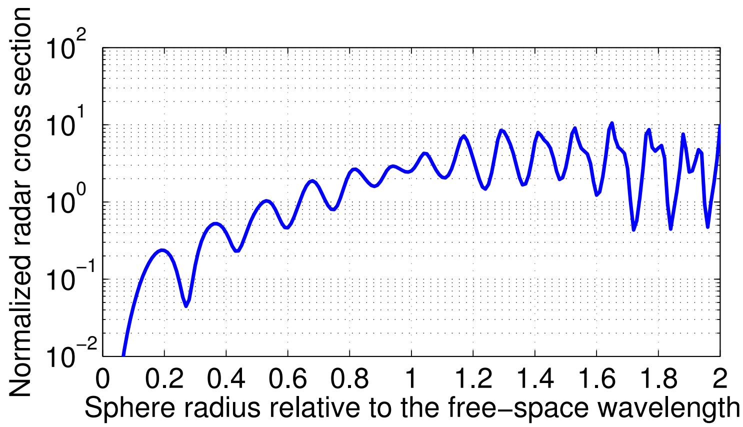


Figure 2: Normalized mono-static radar cross section of a dielectric sphere with a relative permittivity of 2.25 and a radius of one background wavelength [13].

then $c_1 = -1$, $c_2 = -3\cos\theta$ and $c_3 = -\frac{3}{2}(5\cos^2\theta - 1)$. Together with (28), we can calculate all terms of c_n for $n > 2$. When $\theta = \pi$ and $n = 2$, the first three terms, $c_1 = -1$, $c_2 = 3$, and $c_3 = -6$, satisfy (26a). Assume that (26a) is true when $n = k$, then from (28) we have

$$
\begin{aligned}
c_{k+1} =& -\frac{2k+1}{k}c_k - \frac{k+1}{k}c_{k-1} \\
=& -\frac{2k+1}{k}\frac{k(k+1)}{2}(-1)^k - \frac{k+1}{k}\frac{k(k-1)}{2}(-1)^{k-1} \\
=&(-1)^{k+1}\frac{(k+1)(k+2)}{2}
\end{aligned}
$$

which is (26a) at $n = k + 1$. □
The proof of (26b) relies on the recurrent relation

$$
\sin\theta P_n^{1\prime}(x) = (n+1)\frac{P_{n-1}^1(x)}{\sin\theta} - n\cos\theta\frac{P_n^1(x)}{\sin\theta}
$$

which can be rewritten as

$$
\sin\theta P_n^{1\prime}(x)| = (n+1)c_{n-1} - nxc_n \quad . \tag{29}
$$

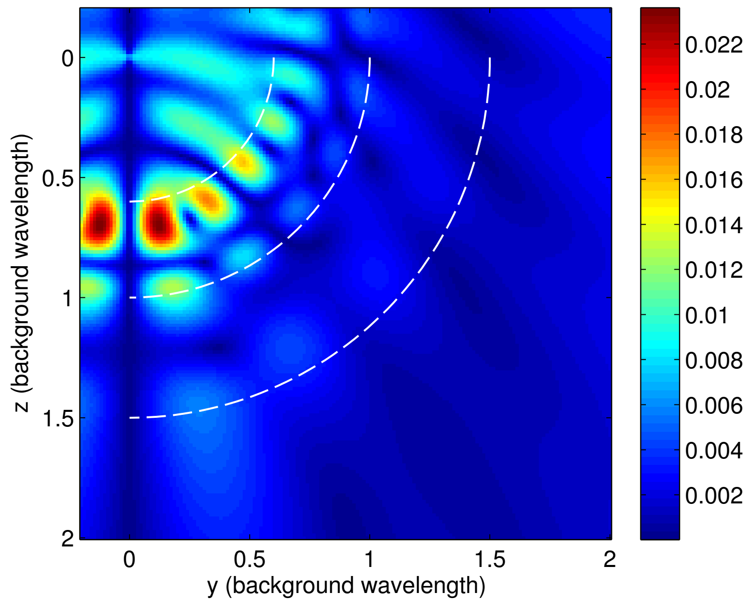
We can use c_n and (29) to calculate all terms of $\sin\theta P_n^{1\prime}(x)$ for $n > 1$ and $\sin\theta P_1^{1\prime}(x) = \cos\theta$. When $\theta = \pi$, (29) becomes

$$
\begin{aligned}
\sin\theta P_n^{1\prime}(\cos\theta) =&(n+1)\frac{n(n-1)}{2}(-1)^{n-1} + n\frac{n(n+1)}{2}(-1)^n \\
=&(-1)^n\frac{n(n+1)}{2}
\end{aligned}
$$

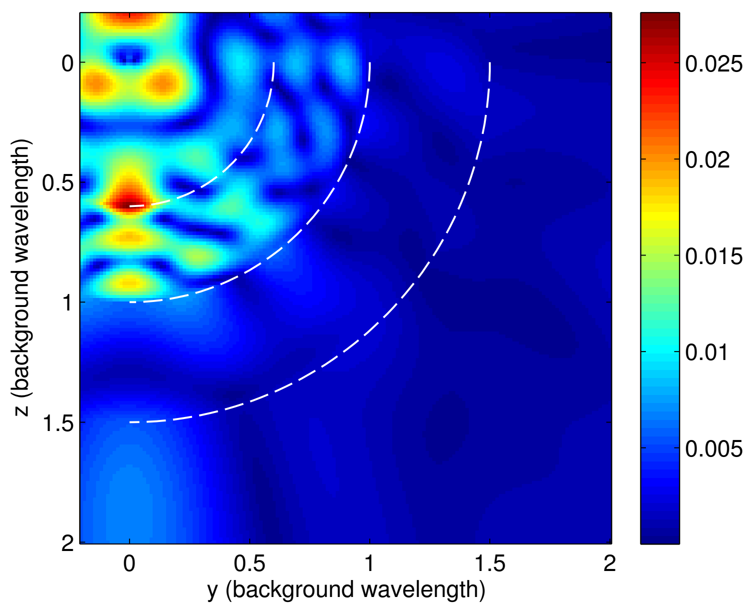
which is (26b).

8 Numerical Validation

Fig. 2 shows the normalized mono-static radar cross sections of a dielectric sphere with a relative permittivity of 2.25 and a radius of one background wavelength. The figure is consistent with the result reported in [13]. Fig. 3 demonstrates the scenario that an x-polarized plane wave propagating in the z direction is scattered by a three-layer dielectric sphere with the relative permittivity of 2, 8 and 4 and the radii of 1.5, 1 and 0.6 normalized to the background wavelength. The total field of $|H_r|$ and $|H_\theta|$ on the y-z plane passing through



(a)



(b)

Figure 3: Total field of (a) $|H_r|$ and (b) $|H_\theta|$ as the x-polarized plane wave interacts with a three-layer dielectric sphere with the relative permittivity of 2, 8 and 4 and the radii of 1.5, 1 and 0.6 of a background wavelength.

the origin are shown in Fig. 3. The continuity of these tangential components demonstrates the correctness in the solution to the linear systems and the implementation of (16).

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