# Proof of the Kramers-Kronig Relation

Guangran Kevin Zhu

Created on: November 23, 2012
Modified on: May 27, 2013

The Kramers-Kronig relation, in essence, states the relations between the real and imaginary part of the frequency response of a time-causal system. Precisely speaking, let $u(t)$ be the unit step function[1], and $g(t)$ be some arbitrary function. The causal channel $h(t)$ is defined with the impulse response of $h(t) = u(t)g(t)$. Let the frequency response of the channel be $H(\omega) = R(\omega) + jX(\omega)$. Then the real part and imaginary part of $H(\omega)$ satisfy the Kramers-Kronig relations, which are

$$R(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{X(\omega_0)}{\omega_0 - \omega} d\omega_0 \quad , \tag{1}$$

$$X(\omega) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{R(\omega_0)}{\omega_0 - \omega} d\omega_0 \quad . \tag{2}$$

The improper integration is in the sense of the Cauchy principle value. Here, we give a proof without any reliance on the theory of complex variables.

Proof: The unitary Fourier transform pair is define as

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int f(t)e^{-j\omega t}dt \quad ,$$

$$f(t) = \frac{1}{\sqrt{2\pi}}\int F(\omega)e^{j\omega t}dt \quad .$$

Under this definition, the Fourier transform of $u(t)$ is

$$U(\omega) = \sqrt{\frac{\pi}{2}}\left[\delta(\omega) + \frac{1}{\pi j\omega}\right] \quad . \tag{3}$$

The Fourier transform of $g(t)$ is $G(\omega)$, which is expressed in terms of its real part and imaginary part as $G(\omega) = \alpha(\omega) + j\beta(\omega)$. Then, the frequency response of the channel becomes

$$\begin{aligned} H(\omega) =& U(\omega) \otimes G(\omega) \\ =& \sqrt{\frac{\pi}{2}}\left[\alpha(\omega) + \frac{1}{\pi}\int \frac{\beta(\omega_0)}{\omega_0 - \omega}d\omega_0\right] + j\sqrt{\frac{\pi}{2}}\left[\beta(\omega) - \frac{1}{\pi}\int \frac{\alpha(\omega_0)}{\omega_0 - \omega}d\omega_0\right] \quad . \end{aligned} \tag{4}$$

Since $h(t) = g(t)$ only in the interval $t \in (0, +\infty)$, the function $g(t)$ defined over the interval $t \in (-\infty, +\infty)$ can be considered either odd or even [2].

If $g(t)$ is odd, $\alpha(\omega) = 0$. The real part and the imaginary part of $H(\omega)$ become

$$R(\omega) = \frac{1}{\sqrt{2\pi}}\int \frac{\beta(\omega_0)}{\omega_0 - \omega}d\omega_0 \quad , \tag{5}$$

$$X(\omega) = \sqrt{\frac{\pi}{2}}\beta(\omega) \quad , \tag{6}$$

Substituting (5) to (6) gives the result in (1).

Similarly, when $g(t)$ is even, $\beta(\omega) = 0$. The real part and the imaginary part of $H(\omega)$ become

$$R(\omega) = \sqrt{\frac{\pi}{2}}\alpha(\omega) \quad , \tag{7}$$

$$X(\omega) = -\frac{1}{\sqrt{2\pi}}\int \frac{\alpha(\omega_0)}{\omega_0 - \omega}d\omega_0 \quad , \tag{8}$$

Substituting (5) to (6) gives the result in (2).

□

[1] It is interpreted as a functional here.

[2] Any function can be expressed as the sum of an even function and an odd function.