Partial and Total Reflection of an Obliquely Incident Plane Wave

Guangran Kevin Zhu

Created on: October 21, 2012
Modified on: May 27, 2013

Consider the problem of an obliquely incident $\mathrm{TM_y}$ polarized plane wave reflected by a dielectric half space as shown in Fig. 1. The electric and magnetic field of the incident plane wave are given by

$$\vec{E}_i = e^{-jk_0(x\sin\theta_i + z\cos\theta_i)}\vec{a}_y \tag{1a}$$

$$\vec{H}_i = \frac{-1}{j\omega\mu_0}\nabla\times\vec{E}_i \tag{1b}$$

The reflected electric and magnetic field are given by

$$\vec{E}_r = \Gamma e^{-jk_0(x\sin\theta_i - z\cos\theta_i)}\vec{a}_y \tag{2a}$$

$$\vec{H}_r = \frac{-1}{j\omega\mu_0}\nabla\times\vec{E}_r \tag{2b}$$

where Γ is the angle-dependent reflection coefficient. The refracted electric and magnetic field are given by

$$\vec{E}_t = \tau e^{-jk_1(x\sin\theta_t + z\cos\theta_t)}\vec{a}_y \tag{3a}$$

$$\vec{H}_t = \frac{-1}{j\omega\mu_0}\nabla\times\vec{E}_t \tag{3b}$$

where τ is the angle-dependent reflection coefficient. Applying the Snell law of reflection, which is

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{\sqrt{\epsilon_0\mu_0}}{\sqrt{\epsilon_1\mu_1}} \tag{4}$$

we obtain the expressions of Γ and τ, which are given by

$$\Gamma = \frac{\eta_1\cos\theta_i - \eta_0\cos\theta_t}{\eta_1\cos\theta_i + \eta_0\cos\theta_t} \tag{5}$$

$$\tau = \frac{2\eta_1\cos\theta_i}{\eta_1\cos\theta_i + \eta_0\cos\theta_t} \tag{6}$$

Partial Reflection For non-magnetic materials, i.e. $\mu_0 = \mu_1 = 1$, partial reflection occurs when $\epsilon_0 < \epsilon_1$. Given the expression of the incident, reflected, and refracted wave in (1), (2), (15), we are ready to verify the Poynting theorem. The time-averaged Poynting vector is defined as

$$\vec{S} = \frac{1}{2}\left[\vec{E}\times\vec{H}^*\right] \tag{7}$$

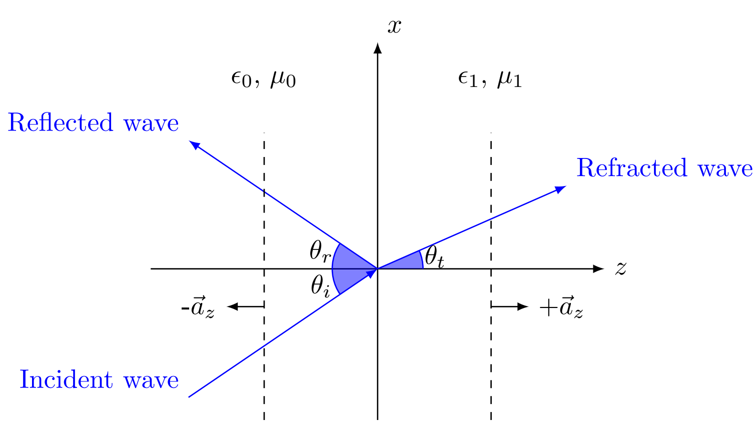

Figure 1: Plane wave partially reflected by a half space. The dashed lines mark the surface to verify the Poynting theorem.

The Poynting vectors of the incident, reflected, and refracted wave are given by

$$\vec{S}_i = \frac{\sin\theta_i}{2\eta_0}\vec{a}_x + \frac{\cos\theta_i}{2\eta_0}\vec{a_z} \tag{8}$$

$$\vec{S}_r = \frac{(\alpha\eta_0 - \eta_1\cos\theta_i)^2\sin\theta_i}{2\eta_0(\alpha\eta_0 + \eta_1\cos\theta_i)^2}\vec{a}_x - \frac{\cos\theta_i(\alpha\eta_0 - \eta_1\cos\theta_i)^2}{2\eta_0(\alpha\eta_0 + \eta_1\cos\theta_i)^2}\vec{a}_z \tag{9}$$

$$\vec{S}_t = \frac{\eta_1^2\mu_0\mu_1\cos\theta_i\sin(2\theta_i)}{\eta_0(\alpha\eta_0 + \eta_1\cos\theta_i)^2}\vec{a}_x + \frac{2\alpha\eta_1\cos\theta_i^2}{(\alpha\eta_0 + \eta_1\cos\theta_i)^2}\vec{a}_z \tag{10}$$

where $\alpha = \sqrt{1 - (\frac{\eta_1}{\eta_0}\sin\theta_i)^2}$ and $\alpha > 0$. It is observed that all the Poynting vectors are real-valued. There is real power flowing into the half space. The power is conserved in the z direction. This is verified by applying the Poynting theorem to the surfaces that are parallel to the dielectric interface and bound the interface in between. It can be shown that

$$\vec{S}_i \cdot (-\vec{a}_z) + \vec{S}_r \cdot (-\vec{a}_z) = \vec{S}_t \cdot \vec{a}_z \tag{11}$$

Total Reflection Total reflection occurs when $\epsilon_0 > \epsilon_1$ and $\theta_i > \theta_c$, where θ_c denotes the critical angle and $\theta_c = \arcsin(\sqrt{\epsilon_1/\epsilon_0})$. Then, $\cos\theta_t$ in (5) and (6) becomes an imaginary number, which is given by

$$\cos\theta_t = -j\sqrt{\left(\frac{\eta_1}{\eta_0}\sin\theta_i\right)^2 - 1} = -j\beta \tag{12}$$

The Poynting vectors of the incident, reflected, and refracted wave are given by

$$\vec{S}_i = \frac{\sin\theta_i}{2\eta_0}\vec{a}_x + \frac{\cos\theta_i}{2\eta_0}\vec{a}_z \tag{13}$$

$$\vec{S}_r = \frac{\sin\theta_i}{2\eta_0}\vec{a}_x - \frac{\cos\theta_i}{2\eta_0}\vec{a}_z \tag{14}$$

$$\vec{S}_t = 2e^{-\frac{2\beta\mu_1\omega}{\eta_1}z}\left(\frac{\eta_1^2\mu_0\mu_1\cos^2\theta_i\sin\theta_i}{\beta^2\eta_0^3 + \eta_0\eta_1^2\cos^2\theta_i}\vec{a}_x + i\frac{\beta\eta_1\cos^2\theta_i}{\beta^2\eta_0^2 + \eta_1^2\cos^2\theta_i}\vec{a}_z\right) \tag{15}$$

It is observed that the Poynting vector of the refracted wave in (15) attenuates exponentially with respect to z. The part of the Poynting vector of the refracted wave in the z direction is reactive. Therefore, there is no real power flowing into the half space. The other part of the Poynting vector of the refracted wave in the x direction carries real power. Since its amplitude attenuates in the z direction, most of the carried power is along the interface between the two media. This wave is also known as the surface wave. It is easy to see that the real power is conserved in the z direction, satisfying the Poynting theorem.

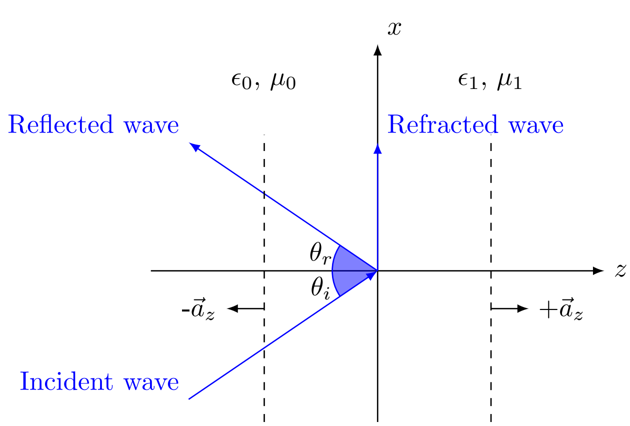

Figure 2: Plane wave totally reflected by a half space. The dashed lines mark the surface to verify the Poynting theorem.