Infinite Sequences and Series

Tests for Convergence and Divergence – A Summary

Theorems on Algebraic Operations on Series: Let $\sum a_n$ and $\sum b_n$ be any two series.

1. If $\sum a_n$ and $\sum b_n$ both converge, then $\sum (a_n \pm b_n)$ must **converge.**
2. If $\sum a_n$ converges, and C is a real number, then $\sum Ca_n$ must **converge.**

   If $\sum a_n$ diverges, and C is a real number, then $\sum Ca_n$ must **diverge.**
3. If one of $\sum a_n$ or $\sum b_n$ converges and the other diverges, then $\sum (a_n \pm b_n)$ must **diverge.**

TESTS FOR CONVERGENCE/DIVERGENCE

Geometric Series:

Let a and r be real numbers. A series of the form $a + ar + ar^2 + ar^3 + ... + ar^n + ... = \sum_{n=0}^{\infty} ar^n$ is called a geometric series. Geometric series **converges** to $\frac{a}{1-r}$ if $|r| < 1$, and **diverges** if $|r| \geq 1$.

Harmonic Series and *p*-Series:

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the harmonic series and it **diverges**.

The p-series has the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and it **converges** if $p > 1$ and **diverges** if $p \leq 1$.

The *n*-th Term Test (The Test for Divergence):

Let $\sum a_n$ be any series. If $\lim_{n \to \infty} a_n \neq 0$ then $\sum a_n$ must **diverge**.

(*note: if* $\lim_{n \to \infty} a_n = 0$, *no conclusion can be made*)

The Integral Test:

Let f be continuous and decreasing on $[k, \infty)$ such that $f(x) \geq 0$ on $[k, \infty)$. Let $a_n = f(n)$, then:

$\sum_{n=k}^{\infty} a_n$ **converges** iff $\int_k^{\infty} f(x)dx$ converges.

$\sum_{n=k}^{\infty} a_n$ **diverges** iff $\int_k^{\infty} f(x)dx$ diverges.

The Direct Comparison Test:

Let $\sum a_n$ and $\sum b_n$ be any two series such that $a_n \geq 0$ and $b_n \geq 0$ for all (large) n.

If $a_n \leq b_n$ for all (large) n and if $\sum b_n$ converges, then $\sum a_n$ must **converge**.

If $a_n \geq b_n$ for all (large) n and if $\sum b_n$ diverges, then $\sum a_n$ must **diverge**.

Remember it as: if the *smaller series diverges, the larger series must diverge*, and if the *larger series converges, the smaller series must converge*.

(*note: no conclusion can be made if the "smaller" series converges or the "larger" series diverges: in this case, try using the Limit Comparison Test*)

The Limit Comparison Test:

Let $\sum a_n$ and $\sum b_n$ be any two positive series. If $\lim_{n->\infty} \frac{a_n}{b_n} = C$, where C is a finite number ≠ 0, then:

$\sum a_n$ **converges** iff $\sum b_n$ converges.

$\sum a_n$ **diverges** iff $\sum b_n$ diverges.

To choose an appropriate $\sum b_n$, look at the behaviour of $\sum a_n$ for large n, take the highest power of n in the numerator and denominator (ignoring coefficients) and simplify:

For example, if $\sum a_n = \frac{5n^3 - n + 2}{2n^5 - 3n^2 + n - 1}$, then at large n, the n^3 and n^5 terms "dominates",

resulting in a $\sum b_n = \frac{n^3}{n^5} = \frac{1}{n^2}$ (note the omission of the coefficients).

(*note: it happens that putting b_n in the denominator usually makes the algebra easier – but ultimately it doesn't matter if you're taking the* $\lim_{n->\infty} \frac{a_n}{b_n}$ *or* $\lim_{n->\infty} \frac{b_n}{a_n}$)

The Alternating Series Test:

Def/ An alternating series is a series whose terms alternate in sign.

Ex. $-1^2 + 2^2 - 3^2 + 4^2 - \ldots$

Let $\sum a_n$ be any alternating series. If $|a_n| \geq |a_{n+1}|$ for all n, and if $\lim_{n->\infty} |a_n| = 0$, then the series must **converge**.

Remember this by: an alternate series only converges if its nth term converges to zero, and it's terms are non-increasing (ie. ignoring minus signs, each term is smaller than or same as its predecessor).

(*note: this test only tells if the alternating series converges – it tells you NOTHING about the positive-term series; also if the series fails the "non-increasing" condition of this test, no conclusion can be made about convergence or divergence of the series*)

Tip: ALWAYS check the nth term first, because the series *and* its positive term series **diverge** if the nth term is not zero. Also, to tell whether a function decreases or increases, you can use the First Derivative Test.

Absolute and Conditional Convergence

Let $\sum a_n$ be any series. $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges. If $\sum a_n$ converges absolutely, then the series $\sum a_n$ itself must converge.

Let $\sum a_n$ be any series. $\sum a_n$ **converges conditionally** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

The Root Test

Let $\sum a_n$ be any series. Suppose $\lim_{n->\infty} \sqrt[n]{|a_n|} = L$, then:

If $0 \leq L < 1$, $\sum a_n$ **converges absolutely**.

If $L > 1$ (including $L = +\infty$), $\sum a_n$ **diverges**.

If L = 1, no conclusion can be made.

The Ratio Test

Let $\sum a_n$ be any series of non-zero terms.

Suppose $\lim_{n->\infty} |\frac{a_{n+1}}{a_n}| = L$, then:

If $0 \leq L < 1$, $\sum a_n$ **converges absolutely**.

If $L > 1$ (including $L = +\infty$), $\sum a_n$ **diverges**.

If L = 1, no conclusion can be made.

*This test is very useful if a_n contains a factorial or if you are dealing with power series.

- Prepared by LM