MA122 - SERIES AND MULTIVARIABLE CALCULUS: TESTS FOR CONVERGENCE OF INFINITE SERIES

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- Geometric Series Test, p-Test, Divergence Test, Comparison Test, Limit Comparison Test, Integral Test, Convergence of Absolute Value, the Ratio Test and the Alternating Series Test.

As we know, finding out whether a series converges or diverges can be very difficult. In fact, there is no single "trick" that one can do to find out. However, we have many good "tricks" at hand, tests that can tell us if the series converges or diverges. Sometimes they succeed and we find out, or they fail and we have to try a new test.

Test 1 (Geometric Series Test). *A geometric series* $\sum_{n=0}^{\infty} ar^n = a+ar+ar^2+\ldots$ *converges if and only if* $|r|<1$. *If it converges, its sum equals* $\frac{a}{1-r}$.

(1) *When to use it.* This test can only be used with geometric series.
(2) *How to use it.* Check if the series is geometric: for example, divide consecutive terms $a_2/a_1, a_3/a_2, \ldots$, if the those quotients are equal then the series is geometric and r equals any of the quotients ($r = a_2/a_1$ for example). a is the first term of the series. If the series is geometric and $|r|<1$ then the series converges. Otherwise it diverges.

Test 2 (The p-Test). *A series of the form* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if* $p>1$ *and diverges if* $p \leq 1$.

(1) *When to use it.* This test can only be used with series of the form $\sum 1/n^p$.
(2) *How to use it.* Simple, bring the series to the form $\sum 1/n^p$.
    - If $p>1$ it converges.
    - If $p \leq 1$ it diverges.

Example 0.0.1. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

and $1/2<1$. On the other hand, the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges because $3/2>1$.

Test 3 (The Divergence Test). *Suppose* $\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+a_4+\ldots$ *is a series and* $\lim_{n\to\infty} a_n \neq 0$, *then the series diverges.*

(1) *When to use it.* This test can only be used to prove *divergence*. We use this test when the terms seem rather large, or when we have the intuition that $\lim a_n$ might not be 0.

(2) *How to use it.* Calculate $\lim_{n\to\infty} a_n$.
- If the result is NOT zero then the series diverges and we are done (and we win).
- If the result if zero the test is inconclusive. No luck. Next test.

Example 0.0.2. Study the following series for convergence:

$$\sum_{n=1}^{\infty} \frac{n+7}{13n+100}$$

Notice that:

$$\lim_{n\to\infty} \frac{n+7}{13n+100} = \frac{1}{13} \neq 0$$

Therefore, by the Divergence Test, the series diverges.

Test 4 (Comparison Test). *Suppose that* $0 \leq a_n \leq b_n$ *for all* n.

(1) *If* $\sum_{n=1}^{\infty} b_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

(2) *If* $\sum_{n=1}^{\infty} a_n$ *diverges, then* $\sum_{n=1}^{\infty} b_n$ *diverges.*

(1) *When to use it.* We use this method when the series reminds us of another series (usually one like $\sum 1/n^p$ for some p) which we know that converges/diverges. Then try to bound the given series by the series you know. The Limit Comparison Test is usually *easier* to use! (see below).

(2) *How to use it.* You are given a series. Your task is to bound the general term of the given series by another expression which you know converges (then you want to bound above) or diverges (bound below).

Example 0.0.3. Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$$

The series reminds me of $\sum_{n=1}^{\infty} 1/n^2$ which converges (use p-test, for example). Next I try to bound the given series above by $1/n^2$. In fact:

$$\frac{1}{n^2+n+1} < \frac{1}{n^2}$$

for all n. Therefore, by the Comparison Test (with $a_n = 1/(n^2+n+1)$ and $b_n = 1/n^2$), the series converges.

Test 5 (Limit Comparison Test). *Let* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *be two series of positive numbers. If the limit*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

exists and $L \neq 0$ is a non-zero finite number, then both series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge or both diverge.

(1) *When to use it.* We use this method when the series reminds us of another series (usually one like $\sum 1/n^p$ for some p) which we know that converges/diverges. The Limit Comparison Test is usually *easier* to use than the regular Comparison Test.
(2) *How to use it.* You are given a series $\sum_{n=1}^{\infty} a_n$ which reminds you of another series $\sum_{n=1}^{\infty} b_n$ which you know converges (or diverges). Calculate:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

- If L is not zero and not infinite then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.
- If $L = 0$ or $L = \infty$ then the test is inconclusive. Tough luck. Maybe you did not choose $\sum_{n=1}^{\infty} b_n$ correctly. Maybe it is time to move on to another test.

Example 0.0.4. Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$$

The series reminds me of $\sum_{n=1}^{\infty} 1/n^2$ which converges (use p-test, for example). Next we compute the limit:

$$\lim_{n\to\infty} \frac{\frac{1}{n^2+n+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1$$

Therefore, since $1 \neq 0$, by the Limit Comparison Test (with $a_n = 1/(n^2+n+1)$ and $b_n = 1/n^2$), the series converges.

Example 0.0.5. Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{n^3+n+1}{n^4+n+1}$$

The series is a little bit too complicated to find a similar and easier series right away. However, if we "forget" about the lower order terms of n:

$$\frac{n^3+n+1}{n^4+n+1} \sim \frac{n^3}{n^4} = \frac{1}{n}$$

and $\sum_{n=1}^{\infty} 1/n$ is the harmonic series and diverges by the p-test. Thus, we take $b_n = 1/n$ and compute:

$$\lim_{n\to\infty} \frac{\frac{n^3+n+1}{n^4+n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n(n^3+n+1)}{n^4+n+1} = \lim_{n\to\infty} \frac{n^4+n^2+n}{n^4+n+1} = \lim_{n\to\infty} \frac{1+1/n^2+1/n^3}{1+1/n^3+1/n^4} = 1$$

Therefore the series diverges like the harmonic does.

Test 6 (**Integral Test - this test is not required in this course**). *Suppose* $\sum_{i=1}^{\infty} a_i$ *is a series with* $a_i = f(i)$*, where* $f(x)$ *is a decreasing positive function. Then:*

- *If* $\lim_{n\to\infty} \int_1^n f(x)dx$ *is finite, then* $\sum a_i$ *converges.*
- *If* $\lim_{n\to\infty} \int_1^n f(x)dx$ *is infinite, then* $\sum a_i$ *diverges.*

(1) *When to use it.* We use this method when the general term in the series is a function that we know how to integrate (easily).
(2) *How to use it.* Calculate:

$$\lim_{n\to\infty} \int_1^n f(x)dx = L$$

- If L is not infinite then $\sum_{n=1}^{\infty} a_n$ converges.
- If $L = \infty$ then the series diverges.

Example 0.0.6. Does the series $\sum_{n=1}^{\infty} 2ne^{-n^2}$ converge? We calculate:

$$\lim_{n\to\infty} \int_1^n 2xe^{-x^2}dx = \lim_{n\to\infty} \left[-e^{-x^2}\right]_1^n = \lim_{n\to\infty} e - e^{-n^2} = e$$

Therefore the series converges.

0.0.1. *Tests for Series with positive and negative terms.*

Test 7 (**The Ratio Test**). *For a series* $\sum_{n=1}^{\infty} a_n$*, suppose that the sequence of ratios* $|a_{n+1}|/|a_n|$ *has a limit:*

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L.$$

- *If* $L < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*
- *If* $L > 1$ *or* $L = \infty$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*
- *If* $L = 1$ *the test tells us nothing about the series* $\sum_{n=1}^{\infty} a_n$*.*

(1) *When to use it.* This method is very useful when the general term contains powers (like 2^n) or factorials ($n!$).
(2) *How to use it.* Calculate:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$$

and follow the list above.

Example 0.0.7. Does the series following series converge?

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

Here $a_n = 2^n/(n!)$, thus $a_{n+1} = 2^{n+1}/((n+1)!)$. We calculate the limit:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2^{n+1}n!}{2^n(n+1)!} = \lim_{n\to\infty} \frac{2}{n+1} = 0$$

Always have in mind that $(n+1)! = (n+1)n!$. Since $0 < 1$, the series converges.

Test 8 (Test of convergence in absolute value). *Suppose $\sum_{n=1}^{\infty} a_n$ is a series and $\sum_{n=1}^{\infty} |a_n|$ converges. Then $\sum_{n=1}^{\infty} a_n$ converges as well.*

(1) *When to use it.* We use this method for series with some negative terms, and such that when we forget the sign of the terms we recognize the series.
(2) *How to use it.* Calculate $|a_n|$ and check $\sum_{n=1}^{\infty} |a_n|$ for convergence.
    - If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.
    - If $\sum_{n=1}^{\infty} |a_n|$ diverges then the test is inconclusive. Try the ratio test or the alternating series test.

Example 0.0.8. Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converge? Here $a_n = \frac{(-1)^n}{n^2}$ so $|a_n| = 1/n^2$. Moreover we know that $\sum_{n=1}^{\infty} 1/n^2$ converges, by the p-test. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges as well.

Example 0.0.9. Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge? Here $a_n = \frac{(-1)^n}{n}$ so $|a_n| = 1/n$. Moreover we know that $\sum_{n=1}^{\infty} 1/n$ diverges, by the p-test. Thus, the test is inconclusive and we can't conclude either way.

Test 9 (The Alternating Series Test). *Suppose $\{a_n\}$ is a sequence of positive terms ($a_n > 0$), with:*

- $0 < a_{n+1} < a_n$, *i.e. the sequence is decreasing.*
- $\lim_{n\to\infty} a_n = 0$.

Then the series $\sum_{n=1}^{\infty}(-1)^n a_n$ (or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$) converges.

(1) *When to use it.* Whenever you want to study a series of the form $\sum_{n=1}^{\infty}(-1)^n a_n$, i.e. the sign of the general term is alternating.
(2) *How to use it.* Check that $a_n > 0$, $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} < a_n$. If the three are true, then the series converges. If any of them fails, the test is inconclusive.

Example 0.0.10. Does the series following series converge?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Here $a_n = 1/n > 0$, thus $a_{n+1} = 1/(n+1)$. First:

$$\frac{1}{n+1} < \frac{1}{n}$$

because $n+1 > n$ for all n. We calculate the limit $\lim_{n\to\infty} 1/n = 0$ and we can conclude that the series converges.

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