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STATE AGREEMENT FOR CONTINUOUS-TIME COUPLED NONLINEAR SYSTEMS*

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Abstract. Two related problems are treated in continuous time. First, the state agreement problem is studied for coupled nonlinear differential equations. The vector fields can switch within a finite family. Associated to each vector field is a directed graph based in a natural way on the interaction structure of the subsystems. Generalizing the work of Moreau, under the assumption that the vector fields satisfy a certain subtangentiality condition, it is proved that asymptotic state agreement is achieved if and only if the dynamic interaction digraph has the property of being sufficiently connected over time. The proof uses nonsmooth analysis. Second, the rendezvous problem for kinematic point-mass mobile robots is studied when the robots' fields of view have a fixed radius. The circumcenter control law of Ando et al. [*IEEE Trans. Robotics Automation*, 15 (1999), pp. 818–828] is shown to solve the problem. The rendezvous problem is a kind of state agreement problem, but the interaction structure is state dependent.

Key words. state agreement, rendezvous, interacting nonlinear systems, time-varying interaction, asymptotical stability

AMS subject classifications. 93C10, 93D20, 37N35, 05C20

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1. Introduction.

This paper studies a dynamical system that is the interconnection of subsystems. Examples are abundant in biology, physics, engineering, ecology, and social science: e.g., a biochemical reaction network [14], coupled Kuramoto oscillators [17, 39], arrays of chaotic systems [44, 45], a swarm of organisms [12, 13], and a group of autonomous agents [16, 22, 23]. We model such systems by coupled nonlinear differential equations in state form. Pioneering work on such coupled dynamical systems from a structural point of view is that of Siljak, e.g., [35, 36].

State agreement means that the states of the subsystems are all equal. For example, [11] studies a group of individuals who must act together as a team; each individual has its own subjective probability distribution for the unknown value of some parameter. How the group might reach a consensus and form a common subjective probability distribution for the parameter is a state agreement problem. In other contexts, state agreement arises as *synchronization* in theoretical physics, e.g., [5, 30, 39, 42, 43], and *consensus* in computer science, particularly in distributed computing, e.g., [25].

Central to the state agreement problem is the graph describing the interaction structure in the interconnected system—that is, who is coupled to whom. And a central question is, What properties of the interaction graph lead to state agreement? Most existing work has dealt with static graphs with a particular topology, such as rings [6, 30], cyclic digraphs [32], and fully connected graphs [12, 13, 34], or with static graphs having an unspecified topology but a certain connectedness. Example frameworks are coupled cell systems [38], coupled oscillators [17, 45], multiagent systems [4, 31], and formations of unicycles [23]. Of course, a static graph simplifies the

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state agreement problem and allows one to focus on the difficulties caused by the nonlinear dynamics of the nodes.

The more interesting situation is when the interaction graph is time varying. From the point of view of control theory, the most suitable mathematical model for these setups is a switched interconnected system. However, attempts to understand how the switching affects the collective system behavior had been hampered by the lack of suitable analysis tools. Recently, however, great strides have been made [16] by characterizing the convergence of infinite products of certain types of nonnegative matrices in a linear discrete-time setup with an undirected interaction graph. For the switched linear continuous-time system model and a directed graph, [22] uses the graph Laplacian and the properties of some special matrices to prove asymptotic state agreement under certain graphical conditions. In addition, [29] uses the common Lyapunov function technique for the switched linear continuous-time system and shows that balanced digraphs play a key role in addressing the average-consensus problem. Two other works on state agreement for linear continuous-time systems are [15], which deals with random networks, and [26], which addresses the deterministic time-varying case.

However, many real systems are nonlinear in addition to having time-varying interaction among subsystems. Examples are systems of coupled oscillators. For nonlinear interconnected systems with time-varying interaction, new tools are required. A novel approach is taken by Moreau in [27]: The framework is nonlinear and discrete-time, and the stability analysis is based upon a blend of graph-theoretic and system-theoretic tools, with the notion of convexity being key. The idea is, roughly speaking, that if every agent always moves toward the relative interior of the convex hull of the set of neighbor agents at each step, state agreement will be achieved. The result in [27] was recently generalized in [2] as follows: The setup is still a discrete-time system, but each agent moves towards the relative interior of a set which is a function, not necessarily the convex hull, of the present and past states of neighbor agents. In this way communication delays can be accommodated.

One concrete instance of the state agreement problem is the rendezvous problem for autonomous mobile robots. Suppose the robots' fields of view have a fixed radius. Then the robots may come into and go out of sensor range of each other, and the interaction graph is therefore state dependent instead of time dependent. For this problem, some distributed algorithms were proposed in [1, 41], with the objective of getting the robots to congregate at a common location (achieving *rendezvous*). These algorithms were extended to various synchronous and asynchronous stop-and-go strategies in [9, 19, 20].

This paper makes two main contributions. The first is the continuous-time counterpart to the result of Moreau [27]. We borrow heavily from Moreau's geometric concepts and proof structure; we suggest, however, that the continuous-time case presents some considerable challenges, as one will see from the details of our proof. Thus our contribution to this problem is primarily technical in nature. As an example application, we apply our result to make new conclusions about synchronization of coupled Kuramoto oscillators. The second contribution of this paper is a solution of the continuous-time rendezvous problem for kinematic point-mass robots; we use the circumcenter control law of Ando et al. [1] and give the first proof of convergence in continuous time.

2. Preliminaries. Here we assemble some known and some novel concepts related to convex sets and tangent cones, directed graphs, and Dini derivatives. In addition, we provide some fundamental properties associated with them.

2.1. Convex sets and tangent cones. References for this subsection are [3,37].

The convex hull of $\mathcal{S} \subset \mathbb{R}^m$ is denoted $\mathrm{co}(\mathcal{S})$. The convex hull of a finite set of points $x_1, \dots, x_n \in \mathbb{R}^m$ is a *polytope*, denoted $\mathrm{co}\{x_1, \dots, x_n\}$.

Let $\mathcal{S} \subset \mathbb{R}^m$ be convex. If $\mathcal{S}$ contains the origin, the smallest subspace containing $\mathcal{S}$ is the *carrier subspace*, denoted $\mathrm{lin}(\mathcal{S})$. The *relative interior* of $\mathcal{S}$, denoted $\mathrm{ri}(\mathcal{S})$, is the interior of $\mathcal{S}$ when it is regarded as a subset of $\mathrm{lin}(\mathcal{S})$ and the relative topology is used, and likewise for the *relative boundary*, denoted $\mathrm{rb}(\mathcal{S})$. If $\mathcal{S}$ does not contain the origin, it must be translated by an arbitrary vector: Let v be any point in $\mathcal{S}$ and let $\mathrm{lin}(\mathcal{S})$ denote the smallest subspace containing $\mathcal{S} - v$. Then $\mathrm{ri}(\mathcal{S})$ is the interior of $\mathcal{S}$ when it is regarded as a subset of the affine subspace $v + \mathrm{lin}(\mathcal{S})$, and similarly for $\mathrm{rb}(\mathcal{S})$.

A nonempty set $\mathcal{K} \subset \mathbb{R}^m$ is a *cone* if $\lambda y \in \mathcal{K}$ when $y \in \mathcal{K}$ and $\lambda > 0$. Let $\mathcal{S} \subset \mathbb{R}^m$ be a closed convex set and $y \in \mathcal{S}$. The *tangent cone* (often referred to as *contingent cone*) to $\mathcal{S}$ at y is the set

$$\mathcal{T}(y, \mathcal{S}) = \left\{ z \in \mathbb{R}^m : \liminf_{\lambda \to 0} \frac{\|y + \lambda z\|_{\mathcal{S}}}{\lambda} = 0 \right\},$$

where $\|y + \lambda z\|_{\mathcal{S}}$ denotes the distance from $y + \lambda z$ to $\mathcal{S}$. The *normal cone* to $\mathcal{S}$ at y is

$$\mathcal{N}(y, \mathcal{S}) = \{z^* : \langle z, z^* \rangle \leq 0 \ \forall z \in \mathcal{T}(y, \mathcal{S})\}.$$

Note that if y is in the interior of $\mathcal{S}$, then $\mathcal{T}(y, \mathcal{S}) = \mathbb{R}^m$. Thus the set $\mathcal{T}(y, \mathcal{S})$ is nontrivial only on $\partial \mathcal{S}$, the boundary of $\mathcal{S}$. In particular, if $\mathcal{S}$ contains only one point, y, then $\mathcal{T}(y, \mathcal{S}) = \{0\}$. In geometric terms the tangent cone for $y \in \partial \mathcal{S}$ is a cone centered at the origin which contains all vectors whose directions point from y "inside" (or they are "tangent to") the set $\mathcal{S}$.

LEMMA 2.1 (see [3]). *Let $\mathcal{S}_i, i = 1, \dots, n$ be convex sets in $\mathbb{R}^m$.*

(i) *If $y \in \mathcal{S}_1 \subset \mathcal{S}_2$, then*

$$\mathcal{T}(y, \mathcal{S}_1) \subset \mathcal{T}(y, \mathcal{S}_2) \quad \text{and} \quad \mathcal{N}(y, \mathcal{S}_2) \subset \mathcal{N}(y, \mathcal{S}_1).$$

(ii) *If $x_i \in \mathcal{S}_i$ $(i = 1, \dots, n)$, then*

$$\begin{aligned} \mathcal{T}\left((x_1, \dots, x_n), \mathcal{S}_1 \times \cdots \times \mathcal{S}_n\right) &= \mathcal{T}(x_1, \mathcal{S}_1) \times \cdots \times \mathcal{T}(x_n, \mathcal{S}_n), \\ \mathcal{N}\left((x_1, \dots, x_n), \mathcal{S}_1 \times \cdots \times \mathcal{S}_n\right) &= \mathcal{N}(x_1, \mathcal{S}_1) \times \cdots \times \mathcal{N}(x_n, \mathcal{S}_n). \end{aligned}$$

2.2. Directed graphs. For a directed graph (digraph for short) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ is the set of nodes and $\mathcal{E}$ is the set of arcs, we write $i \to j$ if there is a path from node i to node j. By definition, $i \to i$ for every node i. A *center* is a node i such that $i \to j$ for every node j, and $\mathcal{G}$ is *quasi-strongly connected* (QSC) if it has a center [7]. Finally, $\mathcal{G}$ is *fully connected* if for every two nodes i and j there is an arc from i to j.

2.3. Dini derivatives. Consider the nonautonomous system

$$\dot{y} = f(t, y), \tag{2.1}$$

where $\mathcal{D} \subset \mathbb{R}^m$ is a domain and $f : \mathbb{R} \times \mathcal{D} \to \mathbb{R}^m$. Let $V(t, y) : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ be a continuous function satisfying a local Lipschitz condition for y, uniformly with respect to t. Then we define

$$D_f^+ V(t, y) = \limsup_{\tau \to 0^+} \frac{V(t + \tau, y + \tau f(t, y)) - V(t, y)}{\tau}.$$

The function $D_f^+ V$ is called the *upper Dini derivative* of V along the trajectory of (2.1). Suppose that for an initial condition $y(0) = y^0$, (2.1) has a solution $y(t)$ defined on an interval $[0, \epsilon)$ and let $D^+V(t, y(t))$ be the upper Dini derivative of $V(t, y(t))$ with respect to t, i.e.,

$$D^+V(t, y(t)) = \limsup_{\tau \to 0^+} \frac{V(t+\tau, y(t+\tau)) - V(t, y(t))}{\tau}.$$

Let $t^* \in [0, \epsilon)$ and put $y(t^*) = y^*$. Then one has that (see [33])

$$D^+V(t^*, y(t^*)) = D_f^+ V(t^*, y^*).$$

LEMMA 2.2. *Let $\mathcal{I}_0 = \{1, 2, \dots, n\}$ and suppose for each $i \in \mathcal{I}_0$, $V_i : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ is of class C^1; let $V(t, y) = \max_{i \in \mathcal{I}_0} V_i(t, y)$; and let $\mathcal{I}(t) = \{i \in \mathcal{I}_0 : V_i(t, y(t)) = V(t, y(t))\}$ be the set of indices where the maximum is reached at time t. Then $D^+V(t, y(t))$ satisfies*

$$D^+V(t, y(t)) = \max_{i \in \mathcal{I}(t)} \dot{V}_i(t, y(t)).$$

The proof can be obtained from Danskin's theorem [8, 10].

3. The state agreement problem: Main results. Our setup is a general interconnection of nonlinear subsystems, where the vector fields can switch within a finite family. We associate to each vector field a directed graph based in a natural way on the interaction structure of the subsystems; this is called an *interaction digraph* in the present paper. Assuming that the vector fields satisfy a certain subtangentiality condition, we show that asymptotic state agreement is achieved if and only if the dynamic interaction digraph has the property of being sufficiently connected over time, in a certain technical sense. Most of the proofs are deferred to section 5.

To formalize the notion of a switched interconnected system, first consider a family of systems

$$\begin{aligned} \dot{x}_1 &= f_p^1(x_1, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= f_p^n(x_1, \dots, x_n), \end{aligned}$$

where $x_i \in \mathbb{R}^m$ is the state of subsystem i and where the index p belongs to a finite set $\mathcal{P}$. Notice that the subsystems share a common state space, $\mathbb{R}^m$. Introducing the *aggregate state* $x \in \mathbb{R}^{mn}$, we have the concise form

$$\dot{x} = f_p(x), \quad p \in \mathcal{P}, \tag{3.1}$$

where for each $p \in \mathcal{P}$, $f_p : \mathbb{R}^{mn} \to \mathbb{R}^{mn}$.

We now associate to each vector field f_p an interaction digraph $\mathcal{G}_p$ capturing the interaction structure of the n subsystems (agents).

DEFINITION 3.1 (interaction digraph). *The interaction digraph $\mathcal{G}_p = (\mathcal{V}, \mathcal{E}_p)$ consists of*

- *a finite set $\mathcal{V}$ of n nodes, each node i modeling agent i;*
- *an arc set $\mathcal{E}_p$ representing the links between agents. An arc from node j to node i indicates that agent j is a neighbor of agent i in the sense that f_p^i depends on x_j; i.e., there exist $x_j^1, x_j^2 \in \mathbb{R}^m$ such that*

$$f_p^i(x_1, \dots, x_j^1, \dots, x_n) \neq f_p^i(x_1, \dots, x_j^2, \dots, x_n).$$

The set of neighbors of agent i is denoted $\mathcal{N}_i(p)$.

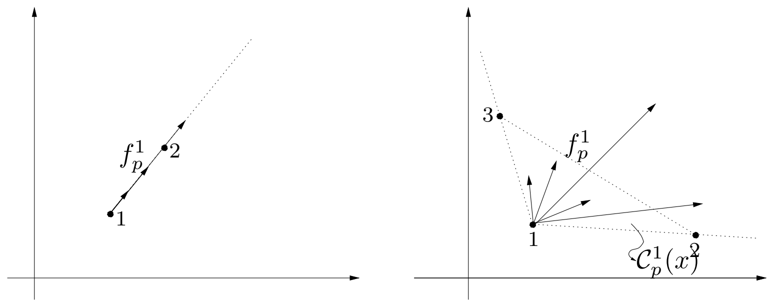

FIG. 3.1. *Some examples of vector fields f_p^i satisfying assumption* A2.

Let $\mathcal{C}_p^i(x) = \mathrm{co}\{x_i, x_j : j \in \mathcal{N}_i(p)\}$ denote the polytope in $\mathbb{R}^m$ formed by the states of agent i and its neighbors. Also, it is convenient to introduce a subset $\mathcal{S} \subset \mathbb{R}^m$ of the common state space that plays the role of a region of focus. In our state agreement problem, initial states of the agents will be in $\mathcal{S}$ and agreement will occur in $\mathcal{S}$. Let $\mathcal{I}_0$ denote the index set $\{1, \ldots, n\}$ and assume that, for each $i \in \mathcal{I}_0$ and each $p \in \mathcal{P}$, the vector fields $f_p^i : \mathbb{R}^{mn} \to \mathbb{R}^m$ satisfy the following two assumptions:

A1. f_p^i is locally Lipschitz on $\mathcal{S}^n$.

A2. For all $x \in \mathcal{S}^n$, $f_p^i(x) \in \mathrm{ri}\left(\mathcal{T}(x_i, \mathcal{C}_p^i(x))\right)$.

Assumption A2 is sometimes referred to as a *strict subtangentiality condition*. Figure 3.1 illustrates two example situations of A2. In the left-hand example, agent 1 has only one neighbor, agent 2; the convex hull $\mathcal{C}_p^1(x)$ is the line segment joining x_1 and x_2; the tangent cone $\mathcal{T}(x_1, \mathcal{C}_p^1(x))$ is the closed ray $\{\lambda(x_2 - x_1) : \lambda \geq 0\}$ (in the figure it is shown translated to x_1); the relative interior $\mathrm{ri}\left(\mathcal{T}(x_1, \mathcal{C}_p^1(x))\right)$ is the open ray $\{\lambda(x_2 - x_1) : \lambda > 0\}$; and A2 means that f_p^1 is nonzero and points in the direction of $x_2 - x_1$. In the right-hand example, agent 1 has two neighbors, agents 2 and 3; the convex hull $\mathcal{C}_p^1(x)$ is the triangle with vertices x_1, x_2, x_3; the tangent cone $\mathcal{T}(x_1, \mathcal{C}_p^1(x))$ is

$$\{\lambda_1(x_2 - x_1) + \lambda_2(x_3 - x_1) : \lambda_1, \lambda_2 \geq 0\}$$

(again, it is shown translated to x_1); the relative interior $\mathrm{ri}\left(\mathcal{T}(x_1, \mathcal{C}_p^1(x))\right)$ is

$$\{\lambda_1(x_2 - x_1) + \lambda_2(x_3 - x_1) : \lambda_1, \lambda_2 > 0\};$$

and A2 means that f_p^1 points into this open cone. In general, A2 requires that $f_p^i(x)$ have the form

$$\sum_{j \in \mathcal{N}_i(p)} \alpha_j(x)(x_j - x_i),$$

where $\alpha_j(x)$ are nonnegative scalar functions, and that $f_p^i(x)$, now viewed as a vector applied at the vertex x_i, not be tangent to the relative boundary of the convex set $\mathcal{C}_p^i(x)$.

When the index p in (3.1) is replaced by a piecewise constant function $\sigma : [0, \infty) \to \mathcal{P}$, we obtain a *switched interconnected system*

$$\dot{x}(t) = f_{\sigma(t)}\left(x(t)\right). \tag{3.2}$$

The function σ is called a *switching signal*. The case of infinitely fast switching (chattering), which would call for a concept of generalized solution, is not considered here. As a matter of fact, it can be shown that even piecewise constant switching signals $\sigma(t)$ do not have sufficient regularity for asymptotic agreement of the switched interconnected system (3.2) [21]. Let $\mathcal{S}_{dwell}$ denote the class of piecewise constant switching signals such that any consecutive discontinuities are separated by no less than some fixed positive constant τ_D, the *dwell time*. We make the following assumption:

A3. $\sigma(t) \in \mathcal{S}_{dwell}$.

Having replaced p by a switching signal $\sigma(t)$, we similarly replace the interaction digraph $\mathcal{G}_p$ by a dynamic interaction digraph $\mathcal{G}_{\sigma(t)}$.

Definition 3.2 (dynamic interaction digraph and union digraph). *Given a switching signal $\sigma(t)$, the* dynamic interaction digraph $\mathcal{G}_{\sigma(t)}$ *is the pair* $\left(\mathcal{V}, \mathcal{E}_{\sigma(t)}\right)$. *Given two real numbers $t_1 \leq t_2$, the* union digraph $\mathcal{G}([t_1, t_2])$ *is the digraph whose arcs are obtained from the union of the arcs in $\mathcal{G}_{\sigma(t)}$ over the time interval $[t_1, t_2]$.*

Definition 3.3. *A dynamic interaction digraph $\mathcal{G}_{\sigma(t)}$ is* uniformly quasi-strongly connected (UQSC) *if there exists $T > 0$ such that for all $t \geq 0$, the union digraph $\mathcal{G}([t, t+T])$ is QSC.*

Our main result, Theorem 3.8, is that the switched interconnected system achieves asymptotic state agreement on $\mathcal{S}$ if and only if the dynamic interaction digraph $\mathcal{G}_{\sigma(t)}$ is UQSC.

But first, the precise meaning of state agreement is given in the following definition.

Definition 3.4. *The switched interconnected system* (3.2) *has the property of*

(i) state agreement on $\mathcal{S}$ *if* $\forall \zeta \in \mathcal{S},\ \forall \varepsilon > 0,\ \exists \delta > 0$ *such that* $\forall t_0 \geq 0$,

$$(\forall i)\ \ (\|x_i(t_0) - \zeta\| \leq \delta) \wedge (x_i(t_0) \in \mathcal{S}) \Longrightarrow (\forall t \geq t_0)(\forall i)\ \|x_i(t) - \zeta\| \leq \varepsilon;$$

(ii) asymptotic state agreement on $\mathcal{S}$ *if it has the property of state agreement on $\mathcal{S}$ and in addition* $\forall \varepsilon > 0,\ \forall c > 0,\ \exists T > 0$ *such that* $\forall t_0 \geq 0$,

$$(\forall i)\ (\|x_i(t_0)\| \leq c) \wedge (x_i(t_0) \in \mathcal{S}) \Longrightarrow (\exists \zeta \in \mathcal{S})(\forall t \geq t_0 + T)(\forall i)\ \|x_i(t) - \zeta\| \leq \varepsilon;$$

(iii) global asymptotic state agreement *if it has the property of asymptotic state agreement on $\mathbb{R}^m$.*

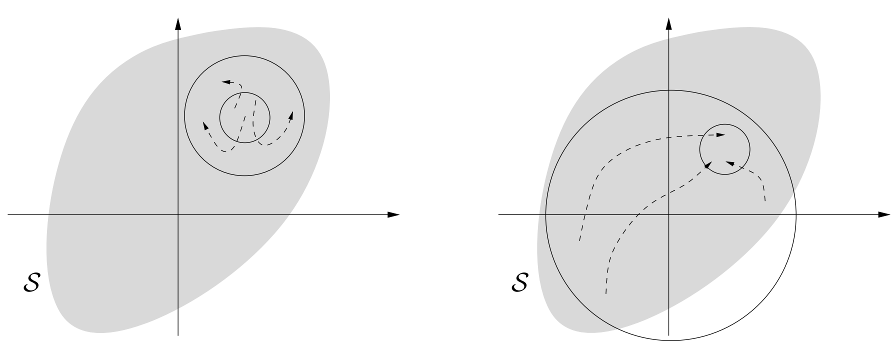

Fig. 3.2. *Asymptotic state agreement on $\mathcal{S}$.*

These definitions are illustrated in Figure 3.2 and can be stated, roughly speaking, as follows. State agreement (the left-hand figure) means that, for every point ζ in $\mathcal{S}$, the agents stay arbitrarily close to ζ if they start sufficiently close to ζ, uniformly with respect to the starting time. Asymptotic state agreement (the two figures together) means, in addition, that the agents converge to a common location in $\mathcal{S}$.

These state agreement definitions are related to stability with respect to a set. Let Ω denote the set of aggregate states such that the subsystem states are all equal and in $\mathcal{S}$, i.e.,

$$\Omega = \{x \in \mathbb{R}^{nm} : x_1 = \cdots = x_n \in \mathcal{S}\}.$$

Then state agreement is equivalent to uniform stability with respect to Ω.

Finally, we give the following new definition of positive invariance specially for interconnected systems.

DEFINITION 3.5. *A set $\mathcal{A} \subset \mathbb{R}^m$ is said to be* positively invariant *for the switched interconnected system* (3.2) *if*

$$(\forall t_0 \geq 0)(\forall i)\ x_i(t_0) \in \mathcal{A} \implies (\forall t \geq t_0)(\forall i)\ x_i(t) \in \mathcal{A}.$$

Our first result establishes the positive invariance property of any compact convex set in $\mathcal{S}$ without needing any property of the interaction digraph. This result can perhaps be understood intuitively as follows. For $m = 2$, all agents move in the plane. Let $\mathcal{A}$ be a compact convex set in $\mathcal{S}$ and assume all agents start in $\mathcal{A}$. Let $\mathcal{C}(t)$ denote the convex hull of the agents' locations at time t. Because $\mathcal{A}$ is convex, clearly $\mathcal{C}(0) \subset \mathcal{A}$. Now invoke assumption A2. An agent that is initially in the interior of $\mathcal{C}(0)$ can head off in any direction at $t = 0$, but an agent that is initially on the boundary of $\mathcal{C}(0)$ is constrained to head into its interior. In this way, $\mathcal{C}(t)$ is nonincreasing (if $t_2 > t_1$, then $\mathcal{C}(t_2) \subset \mathcal{C}(t_1)$), and $\mathcal{A}$ is therefore positively invariant for the switched interconnected system (3.2).

THEOREM 3.6. *Let $\mathcal{A} \subset \mathcal{S}$ be a compact convex set. Then $\mathcal{A}$ is positively invariant for the switched interconnected system* (3.2)*.*

The second result establishes state agreement of the system, again without needing any property of the interaction digraph.

THEOREM 3.7. *Suppose $\mathcal{S}$ is closed and convex. The switched interconnected system* (3.2) *has the property of state agreement on $\mathcal{S}$.*

Proof. Let $\zeta \in \mathcal{S}$ and $\varepsilon > 0$ be arbitrary, and let

$$\mathcal{A}_\varepsilon(\zeta) = \{y \in \mathcal{S} : \|y - \zeta\| \leq \varepsilon\}. \tag{3.3}$$

By Theorem 3.6, it follows that $\mathcal{A}_\varepsilon(\zeta)$ is positively invariant since it is a compact convex set in $\mathcal{S}$. We have thus proved that $\forall \zeta \in \mathcal{S},\ \forall \varepsilon > 0,\ \exists \delta = \varepsilon$ such that $\forall t_0 \geq 0$,

$$(\forall i)\ \ (\|x_i(t_0) - \zeta\| \leq \delta) \wedge (x_i(t_0) \in \mathcal{S}) \Longrightarrow (\forall t \geq t_0)(\forall i)\ \|x_i(t) - \zeta\| \leq \varepsilon.$$

The conclusion follows by Definition 3.4. ∎

Now comes our main result.

THEOREM 3.8. *Suppose $\mathcal{S}$ is closed and convex. The switched interconnected system* (3.2) *has the property of asymptotic state agreement on $\mathcal{S}$ if and only if the dynamic interaction digraph $\mathcal{G}_{\sigma(t)}$ is UQSC.*

This section concludes with a few remarks.

If $\mathcal{S} = \mathbb{R}^m$ in assumptions A1 and A2, then the switched interconnected system (3.2) has the global asymptotic state agreement property if and only if $\mathcal{G}_{\sigma(t)}$ is UQSC.

When the vector fields in the family (3.1) are nonautonomous, suppose assumptions A1 and A2 are replaced by the following (keeping assumption A3 the same):

A1$'$. $f_p^i(t,x)$ is locally Lipschitz with respect to x on $\mathcal{S}^n$ and piecewise continuous with respect to t.

A2$'$. For all $x \in \mathcal{S}^n$ and all $t \in \mathbb{R}$, $f_p^i(t,x) \in \operatorname{ri}\left(\mathcal{T}(x_i, \mathcal{C}_p^i(x))\right)$.

It can be shown [21] that Theorem 3.8 no longer holds in general.

In the special case that the interaction graph is fixed ($\sigma(t)$ is a constant signal), then the property of UQSC is equivalent to QSC. Thus, we arrive at the following special result.

Corollary 3.9. *Suppose $\sigma(t) = p$ and $\mathcal{S} = \mathbb{R}^m$. Then, the interconnected system* (3.2) *has the globally asymptotic state agreement property if and only if $\mathcal{G}_p$ is QSC.*

For this special case we can actually relax the assumptions on the vector fields $f_p^i : \mathbb{R}^{mn} \to \mathbb{R}^m$ as follows:

A1$''$. f_p^i is continuous on $\mathbb{R}^{mn}$.

A2$''$. For all $x \in \mathbb{R}^{mn}$, $f_p^i(x) \in \mathcal{T}\left(x_i, \mathcal{C}_p^i(x)\right)$. Moreover, $f_p^i(x) \neq 0$ if $\mathcal{C}_p^i(x)$ is not a singleton and x_i is its vertex.

A sketch of the proof can be found in [24]. Unlike the proof of Theorem 3.8 here (see section 5), the proof in [24] relies on LaSalle's invariance principle. Finally, it is worth pointing out that assumption A1$''$ is too weak for sufficiency in Theorem 3.8 when the interaction digraph is dynamic [21].

Application: Synchronization of coupled oscillators. The Kuramoto model [17, 39] describes the dynamics of a set of n oscillators with angles θ_i with natural frequencies ω_i. The time evolution of the ith oscillator is given by

$$\dot{\theta}_i = \omega_i + k_i \sum_{j \in \mathcal{N}_i(t)} \sin(\theta_j - \theta_i),$$

where $k_i > 0$ is the coupling strength and $\mathcal{N}_i(t)$ is the set of neighbors of oscillator i at time t. The interaction structure can be general up to this point in the paper; that is, $\mathcal{N}_i(t)$ can be an arbitrary set of other nodes and can be dynamic.

The neighbor sets $\mathcal{N}_i(t)$ define $\mathcal{G}_{\sigma(t)}$ and the switched interconnected system

$$\dot{\theta}(t) = f_{\sigma(t)}\left(\theta(t)\right),$$

where $\theta = (\theta_1, \ldots, \theta_n)$ and $\sigma(t)$ is a suitable switching signal. For identical frequencies (i.e., $\omega_i = \omega\ \forall i$), the transformation $x_i = \theta_i - \omega t$ yields

$$\dot{x}_i = k_i \sum_{j \in \mathcal{N}_i(t)} \sin(x_j - x_i), \quad i = 1, \ldots, n. \tag{3.4}$$

Let a, b be any real numbers such that $0 \leq b - a < \pi$, and define $\mathcal{S} = [a, b]$. It can be checked that A1 and A2 are satisfied. Suppose $\sigma(t)$ here is regular enough to satisfy A3. Then from Theorem 3.8 it follows that, if and only if $\mathcal{G}_{\sigma(t)}$ is UQSC, the switched interconnected system (3.4) has the property of asymptotic state agreement on $\mathcal{S}$. This implies that there exists $\bar{x} \in \mathbb{R}$ such that the oscillators asymptotically synchronize:

$$\theta_i(t) \to \bar{x} + \omega t, \quad \dot{\theta}_i(t) \to \omega.$$

This extends Theorem 1 in [17], which assumes the interaction graph is undirected and static and the initial state $\theta_i(0) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for all i.

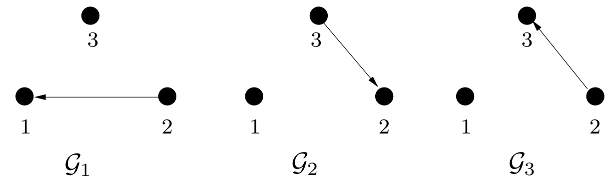

FIG. 3.3. *Three interaction digraphs* $\mathcal{G}_p$, $p = 1, 2, 3$.

As an example, three Kuramoto oscillators with time-varying interaction are simulated. The initial conditions are $\theta_1 = 0$, $\theta_2 = 1$, $\theta_3 = -1$. The natural frequency ω_i equals 1, and the coupling strength k_i is set to 1 for all i. The interaction structure switches among three possible interaction structures periodically, as shown in Figure 3.3. It can be checked that $\mathcal{G}_{\sigma(t)}$ is UQSC. Thus these three oscillators achieve asymptotical synchronization by the main theorem. Figure 3.4 shows the plots of $\sin(\theta_i)$, $i = 1, 2, 3$, and of the switching signal $\sigma(t)$. Synchronization is evident.

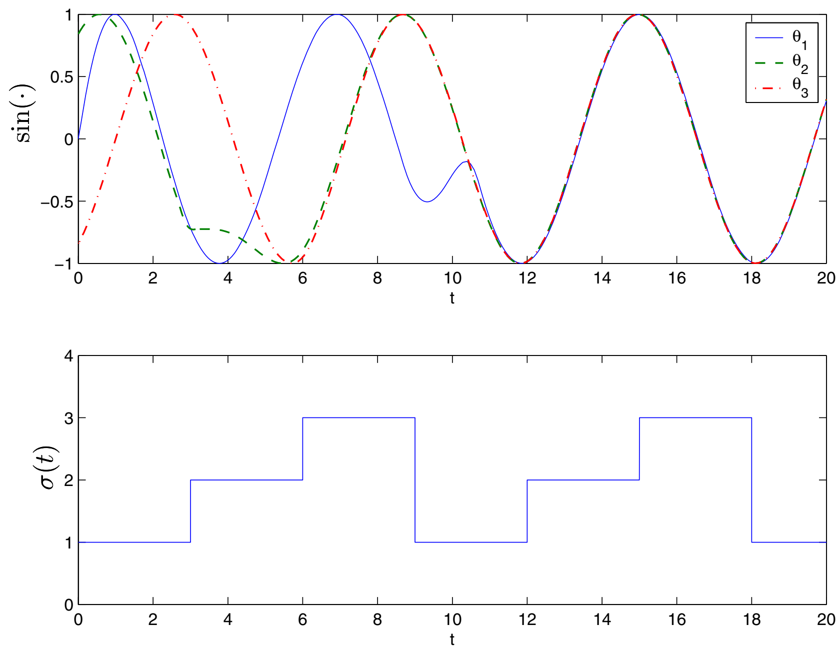

FIG. 3.4. *Synchronization of three oscillators with a dynamic interaction structure.*

4. The rendezvous problem. Now we turn to the second main topic: the rendezvous problem for autonomous mobile robots moving in continuous time. The problem here is different because connectivity is state dependent instead of time dependent a priori.

Suppose there are n robots, each having the simple kinematic model of velocity control: $\dot{x}_i = u_i$, where $x_i \in \mathbb{R}^m$ is the position of robot i. Assume that, due to the limited field of view of its sensor, each robot can sense only the relative positions of its neighbor robots within radius r. Letting $\mathcal{N}_i(x)$ denote the set of neighbors of robot i, where x is the aggregate state of n robots, we thus have that $\{y_{ij} = x_j - x_i : \ j \in$

$\mathcal{N}_i(x)\}$ is the information available to robot i. The *rendezvous problem* is to design local distributed control laws u_i, functions of $\{y_{ij} : j \in \mathcal{N}_i(x)\}$, such that all states $\{x_i : i = 1, \dots, n\}$ converge to a common value $\bar{x} \in \mathbb{R}^m$.

The interaction digraph is state dependent, $\mathcal{G}_{\sigma(x)}$, because of the proximity sensors, and the switched interconnected system takes the form

$$\dot{x} = f_{\sigma(x)}(x), \tag{4.1}$$

where $\sigma : \mathbb{R}^{mn} \to \mathcal{P}$. Let us *fix an initial state* $x^0 \in \mathbb{R}^{mn}$ and assume that (4.1) has a solution $x(t)$ defined for all $t \geq 0$. Then the state-dependent switching rule can be viewed as a time-dependent switching rule $\sigma(x(t))$, and the interaction graph becomes time dependent too, $\mathcal{G}_{\sigma(x(t))}$.

If some robots are initialized so far away from the rest that they never acquire information from them, then the rendezvous problem obviously cannot be solved. This corresponds to the situation where $\mathcal{G}_{\sigma(x(0))}$ is not QSC. Thus it is natural to assume that $\mathcal{G}_{\sigma(x(0))}$ *is* QSC. Moreover, we wish the control laws u_i to be devised such that $\mathcal{G}_{\sigma(x(t))}$ does not lose this property in the future, even though the controller may cause changes in $\mathcal{G}_{\sigma(x(t))}$. Intuitively, u_i should make the maximum distance between robot i and its neighbors nonincreasing.

Let $\mathcal{I}_i(x)$ denote the set of neighbor robots $j \in \mathcal{N}_i(x)$ that have maximum distance from robot i (generically $\mathcal{I}_i(x)$ is a singleton).

PROPOSITION 4.1. *Assume that for each i the control law u_i satisfies*

$$(\forall x) \quad \max_{j \in \mathcal{I}_i(x)} (x_i - x_j)^T u_i \leq 0. \tag{4.2}$$

If $\mathcal{G}_{\sigma(x^0)}$ is QSC and a solution $x(t)$ to (4.1) *exists for all $t \geq 0$, then $\mathcal{G}_{\sigma(x(t))}$ is QSC for all $t \geq 0$.*

Proof. Define

$$V(x) = \max_i \max_{j \in \mathcal{N}_i(x)} \|x_i - x_j\|^2.$$

Notice that $V(x) \leq r$, where r is the radius of the field of view of each robot. Also, define

$$\mathcal{I}(x) = \left\{(i,j) : \; V(x) = \|x_i - x_j\|^2, \; j \in \mathcal{N}_i(x)\right\},$$

the set of pairs of indices where the maximum is reached. By Lemma 2.2

$$\begin{aligned} D^+V(x(t)) &= 2 \max_{(i,j) \in \mathcal{I}(x)} \left[(x_i - x_j)^T u_i + (x_j - x_i)^T u_j\right] \\ &\leq 2 \max_{(i,j) \in \mathcal{I}(x)} (x_i - x_j)^T u_i + 2 \max_{(i,j) \in \mathcal{I}(x)} (x_j - x_i)^T u_j. \end{aligned}$$

It follows from (4.2) that

$$\max_{(i,j) \in \mathcal{I}(x)} (x_i - x_j)^T u_i \leq 0 \quad \text{and} \quad \max_{(i,j) \in \mathcal{I}(x)} (x_j - x_i)^T u_j \leq 0.$$

Hence $D^+V(x(t)) \leq 0$ for all $t \geq 0$, which means the already linked arcs will never be disconnected and therefore the conclusion follows. ∎

Next, we show that if the distributed control law u_i satisfies (4.2) as well as assumptions A1$''$ and A2$''$, then a solution $x(t)$ to (4.1) exists for all $t \geq 0$, and the robots rendezvous.

PROPOSITION 4.2. *Suppose* $\mathcal{G}_{\sigma(x^0)}$ *is QSC. If* u_i *satisfies* (4.2) *as well as* A1$''$ *and* A2$''$, *then the robots rendezvous.*

Proof. If $\mathcal{G}_{\sigma(x^0)}$ is fully connected, then $\mathcal{G}_{\sigma(x(t))}$ is fixed for all time $t \geq 0$ since no link will be dropped, by Proposition 4.1, and no link can be added. Then the conclusion follows from Corollary 3.9.

If instead $\mathcal{G}_{\sigma(x^0)}$ is not fully connected, then $\mathcal{G}_{\sigma(x(t))}$ is dynamic and switches for a finite number of times. To prove this, suppose by contradiction that for all $t \geq 0$, $\mathcal{G}_{\sigma(x(t))} = \mathcal{G}_{\sigma(x^0)}$. Then by Corollary 3.9, all the robots converge to a common location. So $\mathcal{G}_{\sigma(x(t))}$ will become fully connected at some time t, which contradicts the assumption that $\mathcal{G}_{\sigma(x(t))} = \mathcal{G}_{\sigma(x^0)}$ is not fully connected. Hence, there is a $t_1 \geq 0$ such that $\mathcal{G}_{\sigma(x(t_1))}$ has more links than $\mathcal{G}_{\sigma(x^0)}$ because no link will be dropped by Proposition 4.1. Repeating this argument a finite number of times eventually leads to the existence of t_i such that $\mathcal{G}_{\sigma(x(t_i))}$ is fully connected, and thus, it is fixed after t_i. Then the conclusion follows from Corollary 3.9 by treating $(t_i, x(t_i))$ as the initial condition. $\square$

The control law given next is based on the algorithm first proposed in [1].

PROPOSITION 4.3. *A possible choice of* u_i *satisfying condition* (4.2) *as well as assumptions* A1$''$ *and* A2$''$ *is* $u_i = e(0, y_{ij} : j \in \mathcal{N}_i(x))$, *the Euclidean center of the set* $\mathcal{Z} = \{0, y_{ij}, j \in \mathcal{N}_i(x)\}$.

Proof. The Euclidean center of the set $\mathcal{Z}$ is the unique point w that minimizes the function $g(w) := \max_{z\in\mathcal{Z}} \|w - z\|$. Interpreted geometrically, $e(\cdot)$ is the center of the smallest m-sphere that contains the set of points $\{0, y_{ij}, j \in \mathcal{N}_i(x)\}$. Furthermore, it can be easily shown that it lies in the polytope $\widetilde{\mathcal{C}}_p^i = \mathrm{co}\{0, y_{ij}, j \in \mathcal{N}_i(x)\}$ but not at its vertices if the polytope is not a singleton. Thus,

$$e\left(0, y_{ij} : j \in \mathcal{N}_i(x)\right) = \arg\min_{w\in\widetilde{\mathcal{C}}_p^i} \left(\max_{z\in\mathcal{Z}} \|w - z\| \right).$$

Then, by the maximum theorem [40], the function $e(\cdot)$ is continuous (but not locally Lipschitz by some other arguments), and hence u_i satisfies assumption A1$''$.

Next, $e(\cdot) \in \widetilde{\mathcal{C}}_p^i$ implies $e(\cdot) \in \mathcal{T}(0, \widetilde{\mathcal{C}}_p^i)$. Also, notice that $\mathcal{C}_p^i(x) = \mathrm{co}\{x_i, x_j : j \in \mathcal{N}_i(x)\}$ is the translation of $\widetilde{\mathcal{C}}_p^i$ to the point x_i. Hence, $e(\cdot) \in \mathcal{T}(x_i, \mathcal{C}_p^i(x))$. In addition, if $\mathcal{C}_p^i(x)$ is not a singleton and x_i is its vertex, this means that $\widetilde{\mathcal{C}}_p^i$ is not a singleton and 0 is its vertex. Then by the fact that $e(\cdot)$ lies in $\widetilde{\mathcal{C}}_p^i$ but not at its vertices, it follows that $u_i = e(\cdot) \neq 0$. Thus u_i satisfies assumption A2$''$.

Finally, u_i satisfies (4.2). This can be seen from geometry. We show the case $m = 2$ for illustration. If $u_i = 0$, then it trivially satisfies (4.2). If $u_i \neq 0$, then the picture is as in Figure 4.1. The solid circle C_1 is the smallest circle enclosing the points 0 and $y_{ij}, j \in \mathcal{N}_i(x)$. The dotted circle C_2 is centered at the origin and goes through the intersection points between C_1 and its diameter, which is perpendicular to u_i. We know that if there are some y_{ij} in the closed shaded area, then one of them achieves the maximal distance from the origin among all $y_{ij}, j \in \mathcal{N}_i(x)$. On the other hand, there is at least one $j \in \mathcal{N}_i(x)$ such that y_{ij} is in the closed semicircle of C_1, since otherwise it is not the smallest circle. Hence, y_{ij} lies in the closed shaded area if $j \in \mathcal{I}_i(x)$. Moreover, the angle between u_i and such y_{ij} is less than $\pi/2$. This implies that $\max_{j\in\mathcal{I}_i(x)}(x_i - x_j)^T u_i \leq 0$. $\square$

5. Proofs of the main results in section 3. Our proofs rely heavily on nonsmooth analysis involving the Dini derivative. They are partly inspired by a result of Narendra and Annaswamy [28], who show that with $\dot{V}(x, t) \leq 0$ uniform

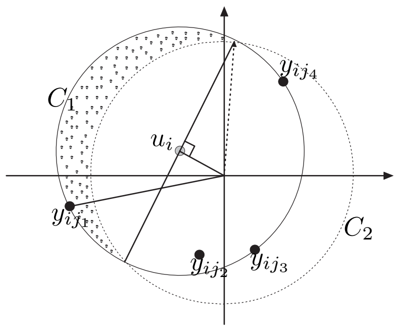

FIG. 4.1. *The smallest enclosing circle.*

asymptotic stability can be proved if there exists a positive T such that for all t, $V(x(t+T), t+T) - V(x(t), t) \leq -\gamma(\|x(t)\|) < 0$, where γ is a class $\mathcal{K}$ function. The difference here is that we deal with stability with respect to a *set*—the set of aggregate states where the subsystem states are all equal—rather than stability of an equilibrium point; an additional complication is that the natural V-functions are nondifferentiable.

Nagumo's theorem concerning set invariance is stated first, for later reference.

THEOREM 5.1 (see [3]). *Consider the system $\dot{y} = F(y)$, with $F : \mathbb{R}^l \to \mathbb{R}^l$, and let $\mathcal{Y} \subset \mathbb{R}^l$ be a closed convex set. Assume that, for each y^0 in $\mathcal{Y}$, there exists $\epsilon(y^0) > 0$ such that the system admits a unique solution $y(t, y^0)$ defined for all $t \in \big[0, \epsilon(y^0)\big)$. Then,*

$$y^0 \in \mathcal{Y} \Longrightarrow \big(\forall t \in \big[0, \epsilon(y^0)\big)\big) \;\; y(t, y^0) \in \mathcal{Y}$$

if and only if $F(y) \in \mathcal{T}(y, \mathcal{Y})$ for all $y \in \mathcal{Y}$.

Proof of Theorem 3.6. Let $\mathcal{A}$ be any compact convex set in $\mathcal{S}$ and consider any initial state $x^0 \in \mathcal{A}^n$ and any initial time t_0. For any piecewise constant switching signal $\sigma(t)$, let $x(t, t_0, x^0)$ be the solution of the switched interconnected system (3.2) with $x(t_0) = x^0$, and let $[t_0, t_0 + \epsilon(t_0, x^0))$ be its maximal interval of existence.

For any point $x \in \mathcal{A}^n$, it is obvious that $\mathcal{C}_p^i(x) \subset \mathcal{A}$ for all $i \in \mathcal{I}_0$ and $p \in \mathcal{P}$, by convexity of $\mathcal{A}$. Thus, by property (i) in Lemma 2.1,

$$f_p^i(x) \in \operatorname{ri}\big(\mathcal{T}(x_i, \mathcal{C}_p^i(x))\big) \subset \mathcal{T}(x_i, \mathcal{A}) \; \forall i \in \mathcal{I}_0, \; \forall p \in \mathcal{P},$$

and by property (ii) in the same lemma,

$$g(t, x) := f_{\sigma(t)}(x) \in \mathcal{T}(x, \mathcal{A}^n) \; \forall t \in \mathbb{R}, \; \forall x \in \mathcal{A}^n.$$

Set $y = (t, x)$ and construct the augmented system

$$\dot{y} = F(y) := \begin{bmatrix} 1 \\ g(y) \end{bmatrix}. \tag{5.1}$$

Since $g(t, x)$ admits a unique solution $x(t, t_0, x^0)$ defined for all $t \in [t_0, t_0 + \epsilon(t_0, x^0))$, it follows that for all $y^0 = (t_0, x^0) \in \mathbb{R} \times \mathcal{A}^n$, the augmented system (5.1) has a unique solution $y(t, y^0)$ defined on $[0, \epsilon(y^0))$. Moreover,

$$F(y) \in \mathcal{T}(t, \mathbb{R}) \times \mathcal{T}(x, \mathcal{A}^n) = \mathcal{T}(y, \mathbb{R} \times \mathcal{A}^n) \; \forall y \in \mathbb{R} \times \mathcal{A}^n.$$

Since $\mathbb{R} \times \mathcal{A}^n$ is closed and convex, by Theorem 5.1 it follows that

$$y^0 = (t_0, x^0) \in \mathbb{R} \times \mathcal{A}^n \Longrightarrow (\forall \tau \in [0, \epsilon(y^0)))\ y(\tau) \in \mathbb{R} \times \mathcal{A}^n. \tag{5.2}$$

The solution $y(\tau)$ to (5.1) with initial condition $y^0 = (t_0, x^0)$ is related to the solution $x(t)$ to $\dot{x} = g(t, x)$ with initial condition $x(t_0) = x^0$ as follows:

$$(\forall t \in [t_0, t_0 + \epsilon(t_0, x^0)))\ (t, x(t)) = y(t - t_0).$$

We thus rewrite condition (5.2) as

$$t_0 \in \mathbb{R} \text{ and } x^0 \in \mathcal{A}^n \Longrightarrow (\forall t \in [t_0, t_0 + \epsilon(t_0, x^0)))\ x(t) \in \mathcal{A}^n.$$

Since the set $\mathcal{A}^n$ is compact, it follows by Theorem 2.4 in [18] that, for all $x^0 \in \mathcal{A}^n$ and all t_0, $\epsilon(t_0, x^0) = \infty$ and the set $\mathcal{A}$ is positively invariant for the switched interconnected system (3.2) by definition 3.5. ∎

Now we need some additional notation. First, a hypercube in $\mathbb{R}^m$:

$$\mathcal{A}_r(z) = \{y \in \mathbb{R}^m : \|y - z\|_\infty \le r\}.$$

Let $c > 0$ be large enough that $\mathcal{S}_c := \mathcal{S} \cap \mathcal{A}_c(0)$ is not empty. Now consider any $x = (x_1, \dots, x_n)$, $x_i \in \mathcal{S}_c$. Each x_i lives in $\mathbb{R}^m$. Let $\mathcal{C}(x)$ denote the convex hull of the points $x_1, \dots, x_n$; $\mathcal{C}(x)$ is a polytope in $\mathbb{R}^m$.

To simplify notation, we focus on the first axis in $\mathbb{R}^m$. Along this axis, let $a_1(x)$ and $b_1(x)$ denote the upper and lower ordinates of $\mathcal{C}(x)$, as in Figure 5.1. The set $\{y \in \mathcal{C}(x) : y_1 = a_1(x)\}$ is the first upper boundary of $\mathcal{C}(x)$. Finally, for small enough $r > 0$, define

$$\mathcal{H}_r(x) = \{y \in \mathcal{C}(x) : y_1 \le a_1(x) - r\}.$$

The setup is summarized in Figure 5.1.

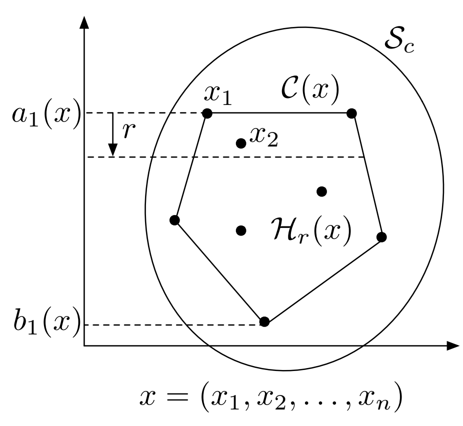

FIG. 5.1. *Illustration to define notation: $\mathcal{C}(x)$ is the convex hull of the points $x_1, \dots, x_n$; $a_1(x), b_1(x)$ are its upper and lower ordinates; $\mathcal{H}_r(x)$ is the part of the convex hull below the line with ordinate $a_1(x) - r$.*

Now we need two technical lemmas for which we assume that the hypotheses of Theorem 3.8 hold. Due to space limitation, we have to omit the proofs and refer the reader to [21].

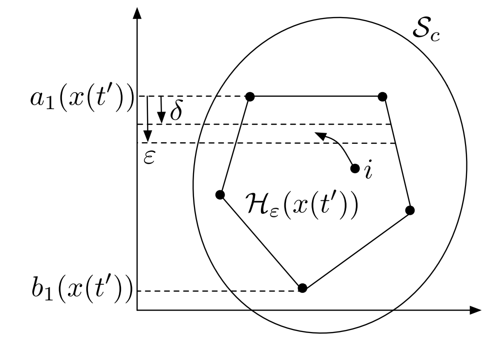

FIG. 5.2. *Illustration for Lemma 5.2: Agent i is in $\mathcal{H}_\varepsilon(x(t'))$ at time t_1, and it cannot get into the upper layer of width δ in the near future.*

The first lemma is illustrated in Figure 5.2.

LEMMA 5.2. *For every sufficiently large $c > 0$, there exists a class $\mathcal{KL}$ function $\gamma : [0, 2c] \times [0,\infty) \to [0,\infty)$ such that $\gamma(\triangle, 0) = \triangle$ and such that the following is true: For every $(t', x(t')) \in \mathbb{R} \times \mathcal{S}_c^n$, every $\varepsilon > 0$ sufficiently small, and every $T > 0$, if $x_i(t_1) \in \mathcal{H}_\varepsilon(x(t'))$ at $t_1 \geq t'$, then $x_i(t) \in \mathcal{H}_\delta(x(t'))$ for all $t \in [t_1, t_1 + T]$, where $\delta = \gamma(\varepsilon, T)$.*

The second lemma is illustrated in Figure 5.3.

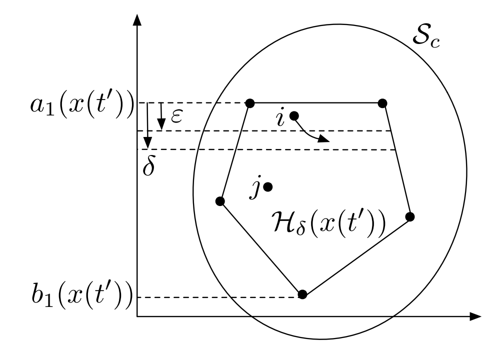

FIG. 5.3. *Illustration for Lemma 5.3: Agent i has the neighbor j in $\mathcal{H}_\delta(x(t'))$ and will consequently be pulled into $\mathcal{H}_\varepsilon(x(t'))$.*

LEMMA 5.3. *For every sufficiently large $c > 0$, there exists a class $\mathcal{K}$ function $\varphi : [0, 2c] \to [0,\infty)$ such that $\varphi(\triangle) < \triangle$ for $\triangle \neq 0$ and such that the following is true: For every $(t', x(t')) \in \mathbb{R} \times \mathcal{S}_c^n$ and every $\delta > 0$ sufficiently small, if there exist a pair (i, j) and a $t_1 \geq t'$ such that $j \in \mathcal{N}_i(t)$ and $x_j(t) \in \mathcal{H}_\delta(x(t'))$ for all $t \in [t_1, t_1 + \tau_D]$, then there exists a $t_2 \in [t', t_1 + \tau_D]$ such that $x_i(t_2) \in \mathcal{H}_\varepsilon(x(t'))$, where $\varepsilon = \varphi(\delta)$.*

Proof of Theorem 3.8. (Necessity.) To prove the contrapositive form, assume that $\mathcal{G}_{\sigma(t)}$ is not UQSC. That is, for every $T > 0$ there exists $t^* \geq 0$ such that $\mathcal{G}([t^*, t^*+T])$ is not QSC; i.e., it does not have a center. Then, in $\mathcal{G}([t^*, t^*+T])$ there are two nodes i^* and j^* such that for every node k either $k \not\rightarrow i^*$ or $k \not\rightarrow j^*$. Let $\mathcal{V}_1$ be the set of nodes l such that $l \to i^*$ and let $\mathcal{V}_2$ be the set of nodes l such that $l \to j^*$. Obviously, $\mathcal{V}_1$ and $\mathcal{V}_2$ are disjoint. Moreover, for each node $i \in \mathcal{V}_1$ (resp., $\mathcal{V}_2$), the set of neighbors of agent i in $\mathcal{G}([t^*, t^* + T])$ is a subset of $\mathcal{V}_1$ (resp., $\mathcal{V}_2$). This implies that, for all $t \in [t^*, t^* + T]$ and for all $(i, j) \in \mathcal{V}_1 \times \mathcal{V}_2$,

$$\mathcal{N}_i(\sigma(t)) \subseteq \mathcal{V}_1 \quad \text{and} \quad \mathcal{N}_j(\sigma(t)) \subseteq \mathcal{V}_2.$$

Choose any $z_1, z_2 \in \mathcal{S}$ such that $z_1 \neq z_2$. Let $t_0 = t^*$ and pick any initial condition $x(t_0)$ such that

$$x_i(t_0) = \begin{cases} z_1 & \text{if } i \in \mathcal{V}_1, \\ z_2 & \text{if } i \in \mathcal{V}_2. \end{cases}$$

Then, by assumption A2, for all $t \in [t_0, t_0 + T]$,

$$x_i(t) = \begin{cases} z_1 & \forall i \in \mathcal{V}_1, \\ z_2 & \forall i \in \mathcal{V}_2. \end{cases}$$

Let $c = \max_i \|x_i(t_0)\|$ and let ε be a positive scalar smaller than $\|z_1 - z_2\|/2$. We have thus found $\varepsilon > 0$ and $c > 0$ such that, for all $T > 0$, there exists $t_0 = t^*$ such that

$$(\forall i)\ (\|x_i(t_0)\| \leq c) \wedge (x_i(t_0) \in \mathcal{S}),\ \text{but}\ (\forall \zeta \in \mathcal{S})(\exists t = t_0 + T)(\exists i)\ \|x_i(t) - \zeta\| > \epsilon.$$

Thus system (3.2) does not have the property of asymptotic state agreement on $\mathcal{S}$.

(Sufficiency.) Assume $\mathcal{G}_{\sigma(t)}$ is UQSC. By Theorem 3.7 the switched interconnected system (3.2) has the property of state agreement on $\mathcal{S}$, so it remains to show that $\forall \varepsilon > 0,\ \forall c > 0,\ \exists T^* > 0$ such that $\forall t_0 \geq 0$

$$(\forall i)\ x_i(t_0) \in \mathcal{S}_c \Longrightarrow (\exists \zeta \in \mathcal{S})(\forall t \geq t_0 + T^*)(\forall i)\ x_i(t) \in \mathcal{A}_\varepsilon(\zeta). \tag{5.3}$$

Let $\varepsilon > 0,\ c > 0$ be arbitrary. There exist a class $\mathcal{KL}$ function γ and a class $\mathcal{K}$ function φ satisfying the properties in Lemmas 5.2 and 5.3, respectively. For any given $t_0 \geq 0$ and $x^0 \in \mathcal{S}_c^n$, consider the solution $x(t)$ of (3.2) with $x(t_0) = x^0$ and the nonnegative function $V_j(x) := a_j(x) - b_j(x)$, $j = 1, \ldots, m$. Thus $V_j(x(t))$ equals the width in the jth direction of the convex hull of the agents at time t. By Theorem 3.6, for every $t \geq t' \geq t_0$, $x_i(t) \in \mathcal{C}(x(t')) \subset \mathcal{S}_c$ for all i. It follows that $V_j(x(t))$ is nonincreasing along the trajectory $x(t)$.

Since $\mathcal{G}_{\sigma(t)}$ is UQSC, there is a $T' > 0$ such that for each t the union digraph $\mathcal{G}([t, t + T'])$ is QSC. Let $T = T' + 2\tau_D$, where τ_D is the dwell time.

Claim. There exists a class $\mathcal{K}$ function η such that for every $t' \geq t_0$

$$V_1(x(t' + \bar{T})) - V_1(x(t')) \leq -\eta\left(V_1(x(t'))\right), \tag{5.4}$$

where $\bar{T} = 2nT$.

Let us postpone the proof of this claim and see how the theorem follows from the claim. From (5.4) we have

$$V_1\left(x(t_0 + k\bar{T})\right) \leq V_1\left(x(t_0)\right) - \eta\left(V_1(x(t_0))\right) - \cdots - \eta\left(V_1(x(t_0 + (k-1)\bar{T}))\right).$$

Notice that $x^0 \in \mathcal{S}_c^n(0)$ implies $V_1(x^0) \leq 2c$. In addition, considering the facts that η is a class $\mathcal{K}$ function and that $V_1(x(t))$ is nonincreasing, one obtains

$$V_1(x(t_0 + k\bar{T})) \leq 2c - k\eta(V_1(x(t_0 + k\bar{T}))).$$

This means there is a $T_1^* = k\bar{T} > 0$ (k large enough) such that $V_1(x(t)) < 2\varepsilon$ for all $t \geq t_0 + T_1^*$. For each $j = 2, \ldots, m$, by the same argument, there is a $T_j^* > 0$ such that $V_j(x(t)) < 2\varepsilon$ for all $t \geq t_0 + T_j^*$. Let $T^* = \max_j T_j^*$. Thus $V_j(x(t)) < 2\varepsilon$ for all $t \geq t_0 + T^*$ and all $j = 1, \ldots, m$. This in turn implies that there exists a $\zeta \in \mathcal{S}$ such that $x_i(t) \in \mathcal{A}_\varepsilon(\zeta)$ for all i and all $t \geq t_0 + T^*$. This proves (5.3).

Now we prove the claim. Inequality (5.4) says that the width, along the first axis, of the convex hull of the agents reduces measurably from time t' to time $t' + \bar{T}$. The proof is intricate and involves applying Lemmas 5.2 and 5.3 alternately.

We begin by constructing a family of parameters, $\varepsilon_1, \delta_1, \varepsilon_2, \ldots, \varepsilon_{n-1}, \delta_{n-1}, \varepsilon_n$. First, ε_1 is taken to be half the width at time t': $\varepsilon_1 = V_1(x(t'))/2$. Then δ_1 is produced by applying Lemma 5.2: $\delta_1 = \gamma(\varepsilon_1, \bar{T})$. Then ε_2 comes from Lemma 5.3, $\varepsilon_2 = \varphi(\delta_1)$, and δ_2 comes from Lemma 5.2, $\delta_2 = \gamma(\varepsilon_2, \bar{T})$. Continuing, we set

$$\begin{aligned}
\varepsilon_3 &= \varphi(\delta_2), \\
\delta_3 &= \gamma(\varepsilon_3, \bar{T}) \\
&\vdots \\
\varepsilon_{n-1} &= \varphi(\delta_{n-2}), \\
\delta_{n-1} &= \gamma(\varepsilon_{n-1}, \bar{T}), \\
\varepsilon_n &= \varphi(\delta_{n-1}).
\end{aligned}$$

Define $\bar{\gamma}(\cdot) := \gamma(\cdot, \bar{T})$. Then ε_n can be written as

$$\varepsilon_n = \eta\left(V_1(x(t'))\right),$$

where $\eta(\cdot) := \varphi \circ \bar{\gamma} \circ \cdots \circ \varphi \circ \bar{\gamma}(\cdot/2)$. It is a class $\mathcal{K}$ function since $\bar{\gamma}$ and φ both are. Since γ is class $\mathcal{KL}$ with the property $\gamma(\triangle, 0) = \triangle$ and $\bar{T} > 0$, it follows that $\delta_k < \varepsilon_k$. In addition, $\varepsilon_{k+1} < \delta_k$ because $\varphi(\triangle) < \triangle$ for $\triangle \neq 0$. Thus,

$$0 < \varepsilon_n < \delta_{n-1} < \cdots < \delta_1 < \varepsilon_1.$$

These parameters are used as in Figure 5.4.

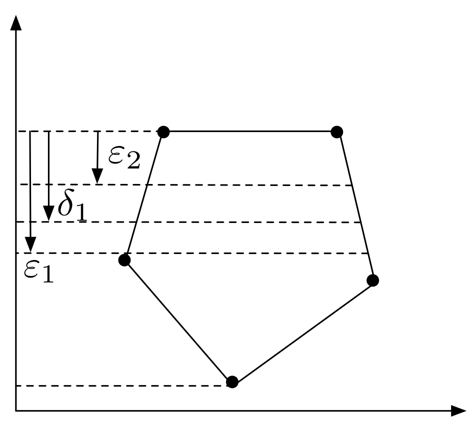

FIG. 5.4. *The parameters $\varepsilon_1, \delta_1, \varepsilon_2, \ldots, \varepsilon_{n-1}, \delta_{n-1}, \varepsilon_n$ with respect to the convex hull and the first axis.*

FIG. 5.5. *The time interval* $[t', t' + \bar{T}]$.

Let $\mathcal{V}_1$ and $\mathcal{V}_1^*$ be a partition of the node set $\mathcal{V}$ such that $i \in \mathcal{V}_1$ if $x_i(t') \in \mathcal{H}_{\varepsilon_1}$ and $i \in \mathcal{V}_1^*$ otherwise. Thus $\mathcal{V}_1$ is the set of agents located in the lower half of the convex hull in Figure 5.4 at time t'.

Next, we apply the two lemmas to construct a sequence of times at which certain events are known to occur. In what follows, hopefully without causing confusion, we use $\mathcal{H}_r$ to denote $\mathcal{H}_r(x(t'))$ for simplicity. As shown in Figure 5.5, let

$$\begin{aligned} \tau_1 &= t' + \tau_D, \\ \tau_2 &= t' + T + \tau_D \\ &\vdots \\ \tau_{2n} &= t' + (2n-1)T + \tau_D. \end{aligned}$$

For each $k = 1, \dots, 2n$, the digraph $\mathcal{G}([\tau_k, \tau_k + T'])$ is QSC, and therefore it has a center, say c_k. Now c_k is either in $\mathcal{V}_1$ or in $\mathcal{V}_1^*$; thus at least n elements in $\{c_1, \dots, c_{2n}\}$ lie in either $\mathcal{V}_1$ or $\mathcal{V}_1^*$. Assume without loss of generality that they lie in $\mathcal{V}_1$; thus there exist indices $1 \le k_1 < \cdots < k_n \le 2n$ such that $c_{k_i} \in \mathcal{V}_1$.

At time t', by definition, $\mathcal{H}_{\varepsilon_1}$ has at least one agent (see Figure 5.4). Moreover, by Lemma 5.2, for all i

$$x_i(t') \in \mathcal{H}_{\varepsilon_1} \Longrightarrow x_i(t) \in \mathcal{H}_{\delta_1} \ \forall t \in [t', t' + \bar{T}]. \tag{5.5}$$

Since $\mathcal{G}([\tau_{k_1}, \tau_{k_1} + T'])$ has a center c_{k_1} in $\mathcal{V}_1$, there exists a pair $(i, j) \in \mathcal{V}_1^* \times \mathcal{V}_1$ such that j is a neighbor of i in this digraph; otherwise there is no link from j to i for any $i \in \mathcal{V}_1^*$ and $j \in \mathcal{V}_1$, which contradicts the fact that the digraph has a center in $\mathcal{V}_1$. This further implies that there is a $\tau \in [\tau_{k_1}, \tau_{k_1} + T']$ such that $j \in \mathcal{N}_i(\tau)$. Since $\tau \in [\tau_{k_1}, \tau_{k_1} + T'] = [t' + (k_1 - 1)T + \tau_D, t' + k_1 T - \tau_D]$, it follows that $[\tau - \tau_D, \tau + \tau_D] \subset [t' + (k_1 - 1)T, t' + k_1 T]$. Since $\sigma(t) \in \mathcal{S}_{dwell}(\tau_D)$, there is an interval $[\bar{\tau}, \bar{\tau} + \tau_D]$, which contains τ and is a subinterval of $[t', t' + k_1 T]$, such that $j \in \mathcal{N}_i(t)$ for all $t \in [\bar{\tau}, \bar{\tau} + \tau_D]$. In addition, since $j \in \mathcal{V}_1$ or, what is the same, $x_j(t') \in \mathcal{H}_{\varepsilon_1}$, from (5.5) we know that $x_j(t) \in \mathcal{H}_{\delta_1}$ for all $t \in [t', t' + \bar{T}]$ (and of course for all $t \in [\bar{\tau}, \bar{\tau} + \tau_D]$). Thus, by Lemma 5.3, there exists $t_1 \in [t', \bar{\tau} + \tau_D] \subseteq [t', t' + k_1 T]$ such that $x_i(t_1) \in \mathcal{H}_{\varepsilon_2}$.

So we have shown on the one hand that the agents not in $\mathcal{H}_{\varepsilon_1}$ at t' are in $\mathcal{H}_{\varepsilon_2}$ at t_1. On the other hand, the agents in $\mathcal{H}_{\varepsilon_1}$ at t' remain in $\mathcal{H}_{\delta_1}$ at t_1 from (5.5), and therefore remain in $\mathcal{H}_{\varepsilon_2}$ at t_1 because $\mathcal{H}_{\delta_1} \subset \mathcal{H}_{\varepsilon_2}$. Hence, at time t_1, $\mathcal{H}_{\varepsilon_2}(x(t'))$ has at least two agents.

Let $\mathcal{V}_2$ and $\mathcal{V}_2^*$ be a partition of the node set $\mathcal{V}$ such that $i \in \mathcal{V}_2$ if $x_i(t_1) \in \mathcal{H}_{\varepsilon_2}$ and $i \in \mathcal{V}_2^*$ otherwise. Note that by (5.5)

$$k \in \mathcal{V}_1 \Longrightarrow x_k(t') \in \mathcal{H}_{\varepsilon_1} \underset{(5.5)}{\Longrightarrow} x_k(t_1) \in \mathcal{H}_{\delta_1} \subset \mathcal{H}_{\varepsilon_2} \Longrightarrow k \in \mathcal{V}_2,$$

so $\mathcal{V}_1 \subset \mathcal{V}_2$. In particular c_{k_2}, the center node of $\mathcal{G}([\tau_{k_2}, \tau_{k_2} + T'])$, is in $\mathcal{V}_2$ because it is in $\mathcal{V}_1$. Then we can apply the same argument to conclude that there are a

$t_2 \in [t_1, t' + k_2 T]$ and an i in $\mathcal{V}_2^*$ such that $x_i(t_2) \in \mathcal{H}_{\varepsilon_3}$ and therefore, $\mathcal{H}_{\varepsilon_3}$ has at least three agents at t_2.

Repeating this argument $n-1$ times leads to the result that there is a $t_{n-1} \in [t', t' + k_{n-1}T] \subset [t', t' + \bar{T}]$ such that $\mathcal{H}_{\varepsilon_n}$ has n agents at t_{n-1}. Hence,

$$V_1(x(t_{n-1})) \leq V_1(x(t')) - \varepsilon_n = V_1(x(t')) - \eta(V_1(x(t'))),$$

and (5.4) follows. $\square$

6. Conclusions. In this paper we first studied the state agreement problem for a class of switched interconnected large-scale systems with a family of admissible vector fields. The interconnection structure is time varying and independent of the state. The key assumption about the vector fields, A2, generalizes Moreau's assumption in discrete time. Necessary and sufficient conditions, in terms of the interaction graph, are obtained to assure that the system achieves asymptotic state agreement. These results can be understood as connective stability, as in the framework of [36]. Achieving asymptotic state agreement of a large-scale interconnected system is robust with respect to either the coupling structure or parameter values. In addition, our results and analysis may be of independent interest in the field of switched systems.

Second, we studied the rendezvous problem in continuous time. The interconnection structure is defined in terms of the distances between agents and hence is state independent. We proved that the circumcenter control law is a solution to the problem.

The notion of state agreement in this paper is that the states of the subsystems are all equal and constant. This notion can potentially be generalized in the following two directions. First, state agreement could mean equality of all the trajectories of the subsystems. In other words, the trajectories of a collection of subsystems will follow, after some transient, the same path in time. This would be of interest in formation control of multiagent systems. Second, state agreement could mean equality of all the states after suitable state transformations. An example is a biochemical reaction network studied in [21].

In many state-agreement problems, the interaction graphs are bidirectional. For such cases, it is reasonable to conjecture that interconnected systems enjoy several special properties. For instance, results similar to those in Theorem 3.8 may be obtained with weaker assumptions on the smoothness of the vector fields.

Finally, we conjecture in the spirit of [2] that our result could be generalized by replacing $\mathcal{C}_p^i(x)$ in assumption A2 by a set-valued map satisfying suitable properties.

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