Costly Interpretation of Asset Prices

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Abstract

We propose a model in which investors cannot costlessly process information from asset prices. At the trading stage, investors are boundedly rational and their interpretation of prices injects noise into the price, generating a source of endogenous noise trading. Compared to the standard rational expectations equilibrium, our setup features price momentum and yields higher return volatility and excessive trading volume. In an overall equilibrium, investors optimally choose sophistication levels by balancing the benefit of beating the market against the cost of acquiring sophistication. Investors tend to over-acquire sophistication. There can exist strategic complementarity in sophistication acquisition, leading to multiple equilibria.

Key words: Investor sophistication, asset prices, disagreement, trading volume, noise trading, welfare, multiplicity

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1 Introduction

Data can be viewed as information only after it has been analyzed. Interpreting data is often costly in terms of time, effort, and other resources. This is particular true for market data given the complexity of modern financial markets. In the existing frameworks—such as the traditional (noisy) rational expectations equilibrium (REE) model (e.g., Radner, 1979; Grossman and Stiglitz 1980; Hellwig, 1980), and the more recent REE-disagreement hybrid models (e.g., Banerjee, 2011)—investors perfectly comprehend the price function and thus can costlessly read into the price to uncover value-relevant information. Apparently, such an argument requires a high degree of sophistication on the part of market participants.[1] What if interpreting price information is costly and investors commit errors in the inference process? How to determine the sophistication levels of investors in interpreting prices? How does investor sophistication affect market prices, trading volume, and investor welfare? In this paper, we propose a structural model to address these questions. We show that the errors committed by traders in making inferences from prices inject endogenous noise into the price system and provide a behavioral foundation for noise trading.

In our model, a continuum of investors interact with each other in two periods. In the second period, investors trade on private information in a financial market. As in the standard REE, the asset price aggregates information and investors make inference from the price. However, at the trading stage, investors are boundedly rational and do not fully understand the price function. We discipline their beliefs using a "receiver noise" approach as in Myatt and Wallace

[1] As discussed by Guesnerie (1992), this comprehension is broadly justified in two ways: the "eductive" justification that relies on the understanding of the logic of the situation faced by economic agents and that is associated with mental activity of agents aiming at "forecasting the forecasts of others;" and the "evolutive" justification that emphasizes the learning possibilities offered by the repetition of the situation and that is associated with the convergence of several versions of learning processes. See Section 7.1 in Vives (2008).

(2012). A fully sophisticated investor would extract the best signal possible from the price (which is endogenously determined in equilibrium), while a less sophisticated investor introduces noise in interpreting the price. After investors form their beliefs based on the personalized price signals, they behave as rational Bayesian and make optimal investments in response to their own beliefs. Through market clearing, investors' optimal asset investments in turn endogenously determine the equilibrium price function and hence the best price signal (i.e., the "truth" in investors' personalized price signals).

In the first period, investors optimally choose their sophistication levels to maximize ex ante expected utilities. On the one hand, increasing sophistication reduces the bounded rationality at the later trading stage, which therefore benefits investors ex ante. On the other hand, acquiring sophistication is costly. For instance, if we think of investors as individual investors, then in order to become more sophisticated, investors may need better education/training (which will cost wealth) or simply need to think harder (which will be involved with mental costs). The optimal sophistication level is determined by balancing the benefit from reduced bounded rationality against the cost of sophistication acquisition.

Investors in our setting can also be interpreted as financial institutions such as hedge funds or mutual funds. Each institution has both a trading desk and a research department. The trading desk is responsible for trading assets but it relies on the institution's research department to generate information from prices. Even if the research department is able to extract the correct signal from prices in the form of research reports, when it passes the signal to the trading desk, the trading desk may add noise in comprehending the reports (which leads to the receiver-noise approach in forming traders' beliefs). The research department can train the trading desk to improve the understanding

of the research reports, which corresponds to a higher sophistication level of investors at the trading stage in our model.

We first analyze the equilibrium in the financial market, which can be viewed as an REE extended with bounded rationality. We find that costly price interpretation can inject noise into the price system. This result relates to De Long, Shleifer, Summers, and Waldmann (1990, DSSW) who show that the misperception of irrational traders about asset fundamentals can pose "noise trader risk" to rational arbitrageurs. We extend the idea to an asymmetric information setting through imperfect price interpretation. Specifically, in our setting, the equilibrium price is a linear function of the asset fundamental and a noise term. The fundamental element comes from aggregating investors' private value-relevant information, which is the root reason why investors care to learn from the price. The noise term in the price arises from a common error in investors' personalized price signals, which is meant to capture the idea that in processing price data, investors may suffer a common cognitive error (such as "sentiment"/"misperception") or technical error (such as a pricing error in commonly used factor models). When investors become more sophisticated, they understand better the true price signal and their trading brings less noise into the price. As investor sophistication approaches to infinity, the asset price approaches the standard REE.

Compared to the standard REE (in which investors are infinitely sophisticated), costly interpretation of prices leads to price momentum (future returns depend positively on the current price), excessive return volatility, and excessive trading volume. This result is consistent with the existing empirical evidence (e.g., Jegadeesh and Titman (1993) and Moskowitz, Ooi, and Pedersen (2012) on momentum; Shiller (1981) and LeRoy and Porter (1981) on excess volatility; and Odean (1999) and Barber and Odean (2000) on excessive trading). In addi-

tion, this result also demonstrates that our setup qualitatively differs from the traditional models with exogenous noise trading such as Grossman and Stiglitz (1980) and Hellwig (1980). For instance, in Hellwig (1980), asset returns exhibit reversals—a high price predicts a future price decline—which is opposite of our prediction (see also Section 4.2.1 in Vives (2008)).

As investors become gradually more sophisticated, return volatility generally decreases, while both disagreement and trading volume can exhibit a hump shape. This finding echoes Garfinkel (2009) who finds that volume is a better proxy for disagreement than return volatility. It also helps to reconcile the contradictory evidence on the relation between disagreement and return volatility. For instance, Frankel and Foot (1990) and Anderson, Ghysels, and Juergens (2005) document a positive disagreement-volatility relation, while Garfinkel (2009) documents a negative relation.

After analyzing the financial market equilibrium, we turn to examine how sophistication levels are determined in an overall equilibrium. From an individual's perspective, the incentive to acquire sophistication comes primarily from beating the average sophistication level across the market, which allows the investor to interpret the price better and trade better (i.e., more likely to buy low and sell high). However, this race in sophistication forms a fallacy of composition, because all investors end up with the same equilibrium sophistication level and no one can gain from beating the market in equilibrium. This misalignment between private and social incentives implies that investors tend to over invest in sophistication acquisition in our setting.

Acquiring sophistication by all investors affects equilibrium welfare both directly and indirectly. The direct effect works through incurring sophistication-acquisition cost. The indirect effect works through affecting a welfare loss driven by speculative trading. Specifically, in our setting, investors do not trade to

share risks and thus, their equilibrium positions only reflect the noise terms in their private information, which is a form of "winner's curse" as pointed out by Biais, Bossaerts, and Spatt (2010). This winner's curse harms investor welfare; it manifests itself as a product of trading size and return volatility, both of which can be affected by sophistication acquisition. In particular, since more sophisticated investors understand the price better, their trading brings more information than noise into the price. As a result, the price is closer to the asset fundamental, which helps to protect investors. Nonetheless, due to the interactions among various forces, the overall welfare effect of sophistication acquisition is generally ambiguous.

We also find strategic complementarity in sophistication acquisition, which leads to the possibility of multiple equilibria. Specifically, when a representative investor decides to become more sophisticated in reading the price, price informativeness increases and the price conveys more information, which increases the marginal value of attending to price data at the trading stage. This in turn further strengthens investors' ex-ante incentives to acquire sophistication. This strategic complementarity implies that multiple sophistication levels can be sustained in equilibrium. Thus, when an exogenous parameter, for instance, the cost of achieving sophistication, changes, there can be jumps in equilibrium sophistication levels. This can correspond to waves of development of algorithmic trading in reality in response to exogenous shocks to the economy, say, some regulation changes.

The plan of the paper is as follows. Section 2 reviews related literature. Section 3 presents the model and the equilibrium concept. Section 4 studies the equilibrium in the financial market for given sophistication levels of investors. Section 5 determines the overall equilibrium including the investor's sophistication level and examines investor welfare and potential multiplicity. Section 6

concludes the paper. Proofs and additional materials are gathered in an online appendix.

2 Related Literature

2.1 Literature on Ignoring Market Information

There is a recent literature on environment complexity that makes agents fail to account for the informational content of other players' actions in game settings. Eyster and Rabin (2005) develop the concept of "cursed equilibrium," which assumes that each player correctly predicts the distribution of other players' actions, but underestimates the degree to which these actions are correlated with other players' information. Esponda (2008) extends Eyster and Rabin's (2005) concept to "behavioral equilibrium" by endogenizing the beliefs of cursed players. Esponda and Pouzo (2016) propose the concept of "Berk-Nash equilibrium" to capture that people can have a possibly misspecified view of their environment. Although these models are cast in a game theoretical framework, the spirit of our financial market model is similar. In our model, investors' interactions are mediated by an asset price, which can be viewed a summary statistic for all the other players' actions.

The most closely related study is perhaps Eyster, Rabin, and Vayanos (2017) who have applied the cursed equilibrium concept to a financial market setting and labeled the resulting equilibrium as the cursed expectations equilibrium (CEE). In a CEE, an investor is a combination of a fully rational REE investor (who correctly reads information from the price) and a naive Walrasian investor (who totally neglects the information in the asset price). Thus, the notion of "cursedness" in Eyster et al. (2017) is conceptually related to the notion of

"sophistication/attention" in our setting, since both notions aim to capture the fact that investors sometimes partially ignore information contained in asset prices.[2] Our paper complements Eyster et al. (2017) in three important ways. First, their central results refer to explaining trading volume. For example, showing that as the number of traders diverges to infinity, the total trading volume goes to infinity in their framework. Instead, at the trading stage, we conduct a comparative static analysis with respect to investors' sophistication level, rather than with respect to the number of traders (which in our case is a continuum and therefore we face no tension between assuming a finite number of trders and assuming that they are price takers). This exercise allows us to compare our setting to a fully REE benchmark. Moreover, this exercise helps to differentiate our framework from Eyster et al. (2017) in terms of testable volume predictions. Specifically, Eyster et al. (2017) predict that trading volume always increases with the degree of cursedness (i.e., decreases with the degree of sophistication). By contrast, in our model, the volume-sophistication pattern can be either hump-shaped or downward sloping, depending on the precision of investors' private information.

Second and more importantly, our setting has an extra stage to determine the equilibrium level of investor sophistication, and we find strategic complementarity in sophistication acquisition, while Eyster et al. (2017) do not explore these issues. Third, our analysis incorporates a common error in interpreting prices, which generates a form of endogenous noise trading. This result leads to the complementarity result in the sophistication acquisition stage, and again, these features are absent in Eyster et al.

[2] See Section 5 of Eyster et al. (2017) for extensive evidence that people do not sufficiently heed the information content of others' behavior and of financial markets. Addoum and Murfin (2017) recently document that equity market participants fail to account for information reflected in publicly posted loan prices.

Peng and Xiong (2006) have considered a representative agent framework in which the attention-constrained agent is not allowed to learn information from prices. In their supplementary material, both Mondria (2010) and Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016) have analyzed REE settings in which investors ignore the information from prices. In particular, Kacperczyk et al. (2016) show that if it requires capacity for investors to process information from prices, then investors would choose not to process that information and to obtain independent signals instead. In our setting, investors optimally determine the sophistication level which in turn determines how much information they will extract from the price.

Banerjee, Kaniel, and Kremer (2009) and Banerjee (2011) have combined REE and disagreement frameworks to allow investors to underestimate the precision of other investors' private information (and hence also labeled as "dismissiveness" models). A dismissive investor can be roughly viewed as a combination of a fully sophisticated and a naive agent, and thus conceptually related to our investors at the trading stage. However, in the dismissiveness model, investors can still read perfectly from the price function and they only disagree about the distribution of other investors' signals.

2.2 Literature on Correlated Errors in Beliefs

As stated, we model investor sophistication by the degree of individual noise added to the best signal possible extracted from the price following a similar approach to Myatt and Wallace (2002). We extend Myatt and Wallace (2002) by introducing a common term into receiver's noise, which in turn endogenously determines the accuracy of the best price signal (see Section 3.1 for a fuller discussion).

The common term in receiver's noise can also be understood as a form

of "sentiment" or "misperception," which therefore connects our paper to the behavioral economics literature (see Shleifer (2000) and Barberis and Thaler (2003) for excellent surveys). In particular, the way we model investors' beliefs shares similarity with DSSW (1990). In DSSW(1990), irrational noise traders misperceive future asset payoffs, and because this misperception is identical across all noise traders, it generates noise trader risk to rational arbitrageurs in financial markets. In our setting, investors suffer misperception when they try to read information from the price and the misperception generates endogenous noise trading that in turn determines the accuracy of price information. In a way, our analysis can be viewed as DSSW cast in an asymmetric information model with endogenous sophistication. Recently, Gârleanu and Pedersen (2016) propose a model to show market efficiency is closely connected to the efficiency of asset management. In our model, market efficiency is determined by how investors (institutions or retail investors) interpret the asset price.

Hassan and Mertens (2011, 2017) have proposed a "near-rational" approach to endogenize noise trading in REE settings. A near-rational agent has wrong perceptions of the first-order moment but has the correct perception of all higher moments. In particular, a near-rational agent's expectation about a random variable is the rational expectation swayed by a common error ("sentiment") and an agent-specific error. In our setting, we do not model perceived moments directly, but instead, we model investors' beliefs based on signals. As a result, when our investors predict fundamentals, both their perceived expectations and variances will differ from those of a fully rational investor. In addition, unlike Hassan and Mertens (2011, 2017) who specify misperception about the exogenous asset fundamental or productivity, we instead specify misperception about the asset price, which itself is an endogenous variable whose statistical properties are in turn affected by investors' misperception. This difference generates

some novel theory insight such as strategic complementarity in sophistication acquisition.

3 A Model of Costly Interpretation of Asset Prices

3.1 Setup

Environment We consider an economy with three dates, $t = 0, 1$, and 2. At $t = 1$, two assets are traded in a competitive market: a risk-free asset and a risky asset. The risk-free asset has a constant value of 1 and is in unlimited supply. The risky asset is traded at an endogenous price $\tilde{p}$ and is in zero supply. It pays an uncertain cash flow at date 2, denoted $\tilde{V}$. We assume that $\tilde{V}$ has two elements, a learnable element $\tilde{v}$ and an unlearnable element $\tilde{\xi}$, which are mutually independent and normally distributed. That is, $\tilde{V} = \tilde{v} + \tilde{\xi}$, where $\tilde{v} \sim N\left(0, \tau_v^{-1}\right)$ and $\tilde{\xi} \sim N\left(0, \tau_\xi^{-1}\right)$, with $\tau_v > 0$ and $\tau_\xi > 0$.

There is a continuum $[0, 1]$ of investors who derive expected utility from their date-2 wealth. They have constant-absolute-risk-aversion (CARA) utility with a risk aversion coefficient of $\gamma > 0$. As we mentioned in the introduction, investors can represent either retail investors or financial institutions. Investors have fundamental information and trade on it. Specifically, at the beginning of date 1, prior to trading, investor i is endowed with the following private signal about the learnable element $\tilde{v}$ in the asset payoff:

$$\tilde{s}_i = \tilde{v} + \tilde{\varepsilon}_i, \text{ with } \tilde{\varepsilon}_i \sim N\left(0, \tau_\varepsilon^{-1}\right),$$

where $\tau_\varepsilon > 0$, and $(\tilde{v}, \tilde{\xi}, \{\tilde{\varepsilon}_i\}_i)$ are mutually independent. We will refer to both the learnable element $\tilde{v}$ and the total asset payoff $\tilde{V}$ as "fundamentals." The unlearnable element $\tilde{\xi}$ reflects the notion that investors collectively do not know

the true payoff from the risky asset.

Each investor has two selves, self 0 and self 1, who make decisions at dates 0 and 1, respectively. The two selves behave in the sense of Kahneman's (2011) two thinking systems. Self 1 engages in fast but noisy thinking; she makes *trading* decisions in the date-1 financial market and is *boundedly rational* in interpreting the information content of prices, adding noise in the process. Self 0 engages in slow and deliberative thinking; she is *fully rational,* extracts the best signal about fundamentals from prices, and determines the future self's sophistication level.[3] Alternatively, we can interpret the two selves of our investors as the research department (self 0) and trading desk (self 1) of an investment institution. The trading desk is responsible for trading assets and it relies on the institution's research department to generate information from the prices. Research departments are able to extract the best signal from the price in the form of research reports, but when they pass the signal to trading desks, trading desks add noise in comprehending the reports.

Self 1's belief specification One key feature of REE is that investors look into the asset price to make inference about fundamentals, which is usually modeled as a statistical signal, $\tilde{s}_p$, about the asset fundamental $\tilde{V}$. In standard REE models, investors are sophisticated enough to understand the statistical properties of the price function that links the price $\tilde{p}$ to the fundamental $\tilde{V}$ and thus, they can convert the price $\tilde{p}$ into a best signal $\tilde{s}_p$ to extract information about $\tilde{V}$. In practice, it is questionable that the information in asset prices in modern financial markets can be fully understood by market participants. A better understanding of the market structure needs more effort. Even worse, the very act of extracting information from the price can bring noise into the

[3] In Eyster et al. (2017) partial cursedness is represented as a geometric average of rational self and a fully cursed self.

price, as interpreting prices can involve errors.

In out setting, we maintain REE's key element that investors make inference from prices but relax the restriction that investors can do so costlessly. To capture this idea, we endow self 1 of each investor (trading desk or the self who makes trading decisions in the financial market) with a reduced-form belief specification which adds noise to the best signal that can be derived from prices $\tilde{s}_p$ that is understood by self 0 (research department). Specifically, self 1 of investor i interprets the information in the price $\tilde{p}$ with additional noise:

$$\tilde{s}_{p,i} = \underbrace{\tilde{s}_p}_{\text{best signal}} + \underbrace{\tilde{x}_i}_{\text{noise}}, \text{ with } \tilde{x}_i \sim N\left(0, \tau_x^{-1}\right). \tag{1}$$

Here, $\tilde{s}_p$ is the best signal implied by the price, which is also the best signal that a fully sophisticated investor (self 0 or research department) can obtain in a standard REE setting. Variable $\tilde{x}_i$ is the noise in processing the price data, which can come from fast mental reasoning or from technology capacity. We do not model where specification (1) comes from and thus it is a reduced-form belief formation. The standard REE concept corresponds to a situation in which investors can costlessly process the price inference problem, so that the noise $\tilde{x}_i$ degenerates to 0.

We further specify that noise term $\tilde{x}_i$ in (1) admits a factor structure:

$$\tilde{x}_i = \tilde{u} + \tilde{e}_i, \text{ with } \tilde{u} \sim N\left(0, \tau_u^{-1}\right) \text{ and } \tilde{e}_i \sim N\left(0, \tau_e^{-1}\right), \tag{2}$$

where $(\tilde{u}, \{\tilde{e}_i\}_i)$ is mutually independent and independent of all other random variables.[4] Note that, by equations (1) and (2), we have $\tau_x^{-1} = \tau_u^{-1} + \tau_e^{-1}$. In (2), the idiosyncratic noise $\tilde{e}_i$ is specific to investor i. The common noise $\tilde{u}$ in investors' price signals may represent waves of optimism and pessimism, which corresponds to the notion of "sentiment" in the behavioral economics

[4]Han and Sangiorgi (2015) have recently provided a search-based microfoundation for the information structure (2) using an urn model with an asymptotic approach.

literature (e.g., DSSW, 1990; Baker and Wurgler, 2007; Angeletos and La'o, 2013; Benhabib, Wang, and Wen, 2015). For instance, DSSW (1990) assume that all noise traders misperceive future asset payoff with a common error that generates noise trader risk to rational arbitrageurs. The term $\tilde{u}$ in our setting can also arise from a common error in data-processing algorithms.[5] As we will show shortly, the random variable $\tilde{u}$ will enter the price endogenously as noise trading in the noisy REE literature (e.g., Grossman and Stiglitz, 1980; Hellwig, 1980). In addition, we will show that even very small noises $\tilde{u}$ and $\tilde{e}_i$ in investors' personalized beliefs can have significant effect on market outcomes.

Sophistication (attention) Investors can study market data more intensively to reduce their noise $\tilde{x}_i$ in (1), thereby bringing the price signal $\tilde{s}_{p,i}$ closer to the best signal $\tilde{s}_p$. The reduction in noise depends on their sophistication levels. Thus, the variance of noise $\tilde{x}_i$ perceived by an investor represents the degree of her bounded rationality, which is negatively related to the investor's sophistication level. We follow Kim and Verrecchia (1994) and model this noise-reduction process due to sophistication as investors gleaning private information about $\tilde{x}_i$. Specifically, self 1 of investor i can study the market and obtain the following signal about $\tilde{x}_i$:

$$\tilde{z}_i = \tilde{x}_i + \tilde{\eta}_i \text{ with } \tilde{z}_i \sim N(0, \tau_{\eta_i}^{-1}), \tag{3}$$

where $\tilde{\eta}_i$ is independent of all other random variables and independent of each other. Conditional on $\tilde{z}_i$, the noise in investor i's price signal $\tilde{s}_{p,i}$ has a posterior

[5] Recent empirical literature documents that, by revealed presence, both individual investors and institutional investors are using factor models, such as the Capital Asset Pricing Model (CAPM), in their investment decisions (e.g., Berk and van Binsbergen, 2016; Blocher and Molyboga, 2017). Variable $\tilde{u}$ in our setting corresponds to the deviation of these commonly used factor models from the true underlying model.

distribution

$$\tilde{x}_i|\tilde{z}_i \sim N\left(\tau_{\eta_i}\left(\tau_x+\tau_{\eta_i}\right)^{-1}\tilde{z}_i,\left(\tau_x+\tau_{\eta_i}\right)^{-1}\right), \tag{4}$$

which indeed has a variance $\left(\tau_x+\tau_{\eta_i}\right)^{-1}$ smaller than the prior variance τ_x^{-1} (i.e., self 1's bounded rationality is reduced from τ_x^{-1} to $\left(\tau_x+\tau_{\eta_i}\right)^{-1}$).

Precision τ_{η_i} controls investor i's ability or "sophistication" level in interpreting asset prices. When $\tau_{\eta_i}=\infty$, self 1 of investor i is fully rational and she can interpret the asset price costlessly, which reduces our economy to the traditional REE setting. We assume that improving sophistication τ_{η_i} is costly, which is reflected by a smooth, increasing, and weakly convex cost function of precision, $C\left(\tau_{\eta_i}\right)$ (similar to the literature, e.g., Verrecchia (1982), Vives (2008), and Myatt and Wallace (2012)). Although the cost is denoted in wealth terms, it can measure both monetary cost (such as expenses of attending educational programs) and mental cost (such as thinking harder to be more attentive). In Appendix C, we have considered a variation in which investors can spend a fixed cost to completely remove the receiver noise $\tilde{x}_i$, and there we show that our main results are robust in that alternative setting.

Alternatively, we can interpret sophistication parameter τ_{η_i} as attention: if investors do not pay attention then there is limited learning from the price, but to pay attention is costly. For instance, in the language of Pavan (2014), parameter τ_{η_i} can be thought of as the time investor i devotes to the information source (which is the price in our context) and $C\left(\tau_{\eta_i}\right)$ denotes the attention cost incurred by the investor.[6] For our analysis, it does not matter which interpre-

[6]Some studies in the rational inattention literature further adopt an entropy-based cost function (e.g., Hellwig, Kohls, and Veldkamp, 2012; Myatt and Wallace, 2012; Matějka and McKay, 2015): the amount of information transmitted is captured by the concept of mutual information; the mutual information uses an agent's attention capacity and an agent can incur a cost to increase the attention capacity. In our context, the mutual information is given by $K\equiv\frac{1}{2}\log\left[\frac{Var(\tilde{s}_p|\tilde{s}_{p,i})}{Var(\tilde{s}_p|\tilde{s}_{p,i},\tilde{z}_i)}\right]$, which captures how much information is transmitted after the investor processes price data. The investor incurs a cost $C(K)$ to process price information

tation (sophistication or attention) makes more sense. We use the two words "sophistication" and "attention" interchangeably, although the language we use in the rest of the paper follows mostly the first interpretation of sophistication.

The optimal sophistication decision on τ_{η_i} is made by self 0 at date 0. Self 0 of each investor is fully rational in choosing the sophistication level of her future self. This treatment is in the same spirit of "optimal expectations" studied by Brunnermeier and Parker (2005) and Brunnermeier, Papakonstantinou, and Parker (2016) and of "wishful thinking" studied by Leahy (2016). In this literature, forward-looking agents derive positive anticipatory utility from optimistic beliefs but suffer disutility from distorted decision making. The optimal beliefs are chosen to balance this benefit-cost trade-off to maximize average felicity, which is evaluated under the objective probability. In our setting, sophistication τ_{η_i} governs self 1's beliefs via equations (3) and (4) and thus, by choosing τ_{η_i}, self 0 is effectively disciplining her future self's biased belief. Self 0 optimally balances this discipline benefit against the mental costs of acquiring sophistication to determine the sophistication level τ_{η_i}.

As pointed out by Brunnermeier and Parker (2005) and Kahneman (2011), the division of rational self 0 and bounded rational self 1 is consistent with the view that some human behaviors are determined primarily by the slower, conscious processing of the prefrontal cortex (self 0), while others are determined by rapid and unconscious processing of the limbic system (self 1). It is possible that agents are unaware of this division and the decision of self 0 is made subconsciously. From a modeler's perspective, the determination of sophistication τ_{η_i} can be viewed as if self 0 trades off a better reading of prices in the later market against a mental cost $C\left(\tau_{\eta_i}\right)$ of acquiring sophistication. In this "as if"

more accurately. The recent experimental study by Dean and Neligh (2017) finds supporting evidence for rational inattention but not for the cost function based on mutual information.

argument, the fully rational self 0 anticipates that her future self will become boundedly rational in reading information from asset prices and thus today, she has an incentive to reduce the later bounded rationality at a commitment cost $C\left(\tau_{\eta_i}\right)$.

Our specification of belief and sophistication matches closely the attention structure in Myatt and Wallace (2012). In our institutional interpretation it is as if the research department sends the signal $\tilde{s}_p$ to the trading desk which adds receiver noise. Indeed, the term $\tilde{x}_i$ in (1) corresponds to the notion of "receiver noise" in Myatt and Wallace (2012) and extends it in three important ways. First, in equation (2), we allow both a common noise $\tilde{u}$ and an investor-specific noise $\tilde{e}_i$ in investor i's receiver noise, where Myatt and Wallace (2012) only deal with agent-specific receiver noise. Second, the quality or accuracy of the correct underlying signal $\tilde{s}_p$ is endogenous in our setting, while it is exogenous in Myatt and Wallace (2012). Third, Myatt and Wallace (2012) assume that paying attention τ_{η_i} can linearly increase the precision of receiver noise. Here, we employ a learning structure to endogenously generate a posterior receiver-noise precision that is linear in τ_{η_i}, as shown by equation (4).

3.2 Timeline and Equilibrium Concept

The timeline of our economy is as follows:

$t = 0$: Self 0 of each investor (research department) chooses τ_{η_i} to maximize ex ante utility. Self 0 is fully rational and so she computes the expected utility under the correct belief.

$t = 1$: Self 1 of each investor (trading desk) receives the private fundamental signal $\tilde{s}_i$, acquires the signal $\tilde{z}_i$ according to τ_{η_i} specified by self 0, and submits demand schedules. Self 1 is boundedly rational in reading information from the price, and thus she interprets the price as a signal $\tilde{s}_{p,i}$ in making inferences. This

implies that the demand schedule is $D\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)$. Market clears at price $\tilde{p}$.

$t=2$: Asset payoff $\tilde{V}$ is realized, and investors get paid and consume.

The overall equilibrium in our model is composed of a date-1 trading equilibrium in the financial market and a date-0 sophistication determination equilibrium. In the date-1 financial market equilibrium, self 1 of each investor maximizes her conditional subjective expected utility and the asset market clears for given sophistication levels τ_{η_i}. This equilibrium determines the price function and hence the best price signal $\tilde{s}_p$. In the sophistication determination stage, self 0 of each investor optimally chooses the sophistication level τ_{η_i} to maximize her ex-ante expected utility taking into account future equilibrium demands. Alternatively, in the institutional interpretation, the research department of the institution chooses a level of training of the trading desk. In Section 4, we will consider first a financial market equilibrium taking investors' sophistication level τ_{η_i} as given. In Section 5, we will deal with the overall equilibrium and the determination of sophistication levels.

4 Financial Market Equilibrium

At date 1, self 1 of each investor chooses investments in assets to maximize her subjective expected utility. Investors are price takers but still actively infer information from the price $\tilde{p}$, although adding individual noise in their inference process. Formally, investor i chooses investment D_i in the risky asset to maximize

$$E_i\left[-\exp(-\gamma\tilde{W}_i)\middle|\,\tilde{p},\tilde{s}_i,\tilde{z}_i\right], \tag{5}$$

with her final wealth $\tilde{W}_i$ given by

$$\tilde{W}_i=(\tilde{V}-\tilde{p})D_i-C\left(\tau_{\eta_i}\right), \tag{6}$$

where we have normalized her initial wealth level at 0 and take τ_{η_i} as given.

The operator $E_i\left[\cdot|\,\tilde{p},\tilde{s}_i,\tilde{z}_i\right]$ in (5) indicates that self 1 of investor i takes expectation with respect to her own (subjective) belief. Specifically, investor i observes $\{\tilde{p},\tilde{s}_i,\tilde{z}_i\}$ and needs to forecast her future wealth $\tilde{W}_i$. Since $\tilde{p}$ is in her information set, she takes $\tilde{p}$ as a known constant. Thus, in equation (6), the only random variable she faces is the fundamental $\tilde{V}$. When she predicts $\tilde{V}$, she extracts information from the price by interpreting $\tilde{p}$ as a signal $\tilde{s}_{p,i}$ according to (1). Endowed with signals $\{\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\}$, self 1 of investor i is a subjective expected utility investor, and in particular, she is Bayesian. As a consequence, in investor i's mind at date 1, the fundamental $\tilde{V}$ follows a normal distribution conditional on $\{\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\}$.

The CARA-normal setting implies that investor i's demand for the risky asset is

$$D\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)=\frac{E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-\tilde{p}}{\gamma Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)}, \tag{7}$$

where $E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)$ and $Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)$ are the conditional expectation and variance of $\tilde{V}$ given information $\{\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\}$. In (7), we have explicitly incorporated $\tilde{s}_{p,i}$ in the demand function to reflect the informational role of the price (i.e., the price helps to predict $\tilde{V}$) and used $\tilde{p}$ per se to capture the substitution role of the price (i.e., a higher price directly leads to a lower demand). Here, the dependence of $D\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)$ on the price $\tilde{p}$ in (7) reflects the fact that the investor knows each unit of the asset costs $\tilde{p}$, while her learning on fundamentals operates through the private signal $\tilde{s}_{p,i}$ or "price interpretation."

The financial market clears, i.e.,

$$\int_0^1 D\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)di=0 \text{ almost surely.} \tag{8}$$

This market-clearing condition, together with demand function (7), will deter-

mine an equilibrium price function,

$$\tilde{p} = p(\tilde{v}, \tilde{u}). \tag{9}$$

where $\tilde{v}$ and $\tilde{u}$ come from the aggregation of signals $\tilde{s}_i$, $\tilde{s}_{p,i}$, and $\tilde{z}_i$. In equilibrium, price function (9) will endogenously determine the informational content in the best signal $\tilde{s}_p$.

A financial market equilibrium for given sophistication levels $(\tau_{\eta_i})_{i\in[0,1]}$ is characterized by a price function $p(\tilde{v}, \tilde{u})$ and demand functions $D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)$, such that:

1. $D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)$ is given by (7), which maximizes investors' conditional subjective expected utilities given their beliefs at date 1;

2. The market clears almost surely, i.e., equation (8) holds;

3. Investors' date-1 beliefs are given by (1), (2), and (3), where $\tilde{s}_p$ in (1) is implied by the equilibrium price function $p(\tilde{v}, \tilde{u})$.

4.1 Equilibrium Construction

We consider a linear financial market equilibrium in which the price function takes the following form:

$$\tilde{p} = a_v \tilde{v} + a_u \tilde{u}, \tag{10}$$

where the coefficients a's are endogenous.

By equation (10), provided that $a_v \neq 0$ (which is true in equilibrium), the price $\tilde{p}$ is equivalent to the following signal in predicting the fundamental $\tilde{v}$:

$$\tilde{s}_p = \tilde{v} + \alpha\tilde{u} \text{ with } \alpha \equiv \frac{a_u}{a_v}, \tag{11}$$

which would be the best signal that a fully sophisticated investor can achieve. However, as we mentioned in Section 3.1, at date 1, investor i can not costlessly

process the price information, and she can only read a coarser signal as follows:

$$\tilde{s}_{p,i} = \tilde{s}_p + \tilde{x}_i = (\tilde{v} + \alpha\tilde{u}) + (\tilde{u} + \tilde{e}_i) = \tilde{v} + (\alpha + 1)\,\tilde{u} + \tilde{e}_i, \tag{12}$$

where the second equality follows from equations (1) and (2). In other words, in the inference process, our investors add noise to the best signal that a fully sophisticated investor could obtain.

Recall that after acquiring sophistication τ_{η_i} at date 0, investor i at date 1 can study market data to further purge the receiver noise $\tilde{x}_i$ in her personalized price signal $\tilde{s}_{p,i}$. This is represented by an access to the private signal $\tilde{z}_i$ in (3). By Bayes' rule, the two signals $\{\tilde{s}_{p,i}, \tilde{z}_i\}$ combine to generate the following signal $\tilde{s}_{pz,i}$ in predicting the fundamental $\tilde{v}$:

$$\begin{aligned}
\tilde{s}_{pz,i} &\equiv \tilde{s}_{p,i} - \frac{\tau_\eta\left(\tau_e + \tau_u + \alpha\tau_e\right)}{\tau_e\tau_u + \tau_e\tau_{\eta_i} + \tau_u\tau_{\eta_i}}z_i \\
&= \tilde{v} + \left(\alpha + \frac{\tau_e\left(\tau_u - \alpha\tau_{\eta_i}\right)}{\tau_e\tau_u + \tau_e\tau_{\eta_i} + \tau_u\tau_{\eta_i}}\right)\tilde{u} \\
&\quad + \frac{\tau_e\left(\tau_u - \alpha\tau_{\eta_i}\right)}{\tau_e\tau_u + \tau_e\tau_{\eta_i} + \tau_u\tau_{\eta_i}}\tilde{e}_i - \frac{\tau_\eta\left(\tau_e + \tau_u + \alpha\tau_e\right)}{\tau_e\tau_u + \tau_e\tau_{\eta_i} + \tau_u\tau_{\eta_i}}\tilde{\eta}_i.
\end{aligned}$$

The signal $\tilde{s}_{pz,i}$ summarizes the overall information that investor i can extract from the price after seeing $\tilde{z}_i$. It predicts $\tilde{v}$ with a precision given by

$$\tau_{p,i} = \frac{\tau_e\tau_u + \left(\tau_e + \tau_u\right)\tau_{\eta_i}}{\tau_u + \tau_e\left(\alpha + 1\right)^2 + \alpha^2\tau_{\eta_i}}. \tag{13}$$

Using Bayes' rule, we can compute

$$E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i) = E\left(\tilde{v}|\tilde{s}_i, \tilde{s}_{pz,i}\right) = \beta_{s,i}\tilde{s}_i + \beta_{p,i}\tilde{s}_{p,i} + \beta_{z,i}\tilde{z}_i, \tag{14}$$

$$Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i) = \left(\tau_v + \tau_\varepsilon + \tau_{p,i}\right)^{-1} + \tau_\xi^{-1}, \tag{15}$$

where the coefficients β's are given in the online appendix. Inserting these two expressions into (7), we can compute the expression of $D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)$, which is in turn inserted into (8) to compute the equilibrium price as a function of $\tilde{v}$ and $\tilde{u}$. Comparing coefficients with the conjectured price function (10), we can form a system of equations to determine the two unknown price coefficients a_v

and a_u.

Proposition 1 (Financial market equilibrium) *Suppose that investors have the same sophistication level (i.e., $\tau_{\eta_i} = \tau_\eta$, $i \in [0,1]$). There exists a unique linear equilibrium price function,*

$$\tilde{p} = a_v \tilde{v} + a_u \tilde{u},$$

where

$$a_v = \frac{\tau_\varepsilon + \tau_p}{\tau_v + \tau_\varepsilon + \tau_p} \text{ and } a_u = \frac{\tau_p}{\tau_v + \tau_\varepsilon + \tau_p} \frac{\tau_u \left(\tau_e + \alpha\tau_e + \alpha\tau_\eta\right)}{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta},$$

where $\tau_p = \frac{\tau_e\tau_u + (\tau_e+\tau_u)\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}$ and where $\alpha \equiv \frac{a_u}{a_v} \in \left(0, \frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon}\right)$ is uniquely determined by the positive real root of the following cubic equation:

$$\left(\tau_e\tau_\varepsilon + \tau_\varepsilon\tau_\eta\right)\alpha^3 + 2\tau_e\tau_\varepsilon\alpha^2 + \left(\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon\right)\alpha - \tau_e\tau_u = 0. \tag{16}$$

Note that in Proposition 1, we have $a_u > 0$ for any $\tau_\eta \in (0, \infty)$. That is, costly interpretation of asset prices brings an endogenous noise $\tilde{u}$ into the price system. As $\tau_\eta \to \infty$, investors become fully sophisticated and thus they can extract the best signal from the price. In this limiting case, the noise $\tilde{u}$ will vanish in the price function, which degenerates our economy to the full REE setup. It is worth noting that the full REE with $\tau_\eta = \infty$ is not implementable in demand functions.

Corollary 1 (Full REE) *Given $(\tau_e, \tau_u, \tau_v, \tau_\xi, \tau_\varepsilon) \in \mathbb{R}^5_{++}$, as $\tau_\eta \to \infty$, the price function converges almost surely to*

$$\tilde{p}^{REE} = \tilde{v}.$$

4.2 Implications of Investor Sophistication

In this subsection, we examine how investor sophistication affects asset prices, investor beliefs, and trading volume. We assume that all investors have a common sophistication level τ_η and conduct comparative static analysis with respect

to τ_η. In a full equilibrium setting, an increase in τ_η corresponds to a decrease in some parameter that governs the cost function $C(\tau_\eta)$, which will be explored later in Section 5.

4.2.1 Price Informativeness and Asset Returns

Price informativeness As standard in the literature (e.g., Vives, 2008; Peress, 2010), we can use the precision $\frac{1}{Var(\tilde{V}|\tilde{p})}$ of stock payoff conditional on its price to measure "price informativeness" (or "market efficiency," "informational efficiency," and "price efficiency").[7] By equation (10), applying Bayes' rule delivers $\frac{1}{Var(\tilde{V}|\tilde{p})} = \left[\left(\tau_v + \alpha^{-2}\tau_u\right)^{-1} + \tau_\xi^{-1}\right]^{-1}$. Since τ_v, τ_u, and τ_ξ are exogenous constants, we can measure price informativeness inversely by α: a high value of α corresponds to a low value of price informativeness.

We can show that price informativeness increases with investor sophistication (i.e., α decreases with τ_η). Intuitively, when investors pay a lot of attention to study price data, they know well the true price signal $\tilde{s}_p$, and thus their trading brings less noise $\tilde{u}$ into the price. This complementarity result has important implications for determining the sophistication level in Section 5.

In the left two panels of Figure 1, we use solid curves to plot price informativeness α against investor sophistication τ_η. As a comparison, the dashed lines plot the α-value in the standard REE economy (i.e., $\alpha = 0$ for $\tau_\eta = \infty$). In both panels, we observe (a) that costly interpretation of prices injects noise into the price as long as investors are not fully sophisticated (i.e., the solid curves lie above the dashed lines); and (b) that price informativeness increases with

[7] In our setting, we can also measure price informativeness from an investor's date-1 perspective, which is the precision $\frac{1}{Var(\tilde{V}|\tilde{s}_{pz,i})}$ of stock payoff conditional on personalized price signals $\tilde{s}_{pz,i}$. Nonetheless, the two price-informativeness measures $\frac{1}{Var(\tilde{V}|\tilde{p})}$ and $\frac{1}{Var(\tilde{V}|\tilde{s}_{pz,i})}$ do not differ much, when the precision τ_u and τ_e of the errors in investors' personalized signals are small (which is the focus of our analysis).

sophistication (i.e., the solid curves are downward sloping).

The parameter values in Figure 1 follow from the calibration exercise conducted by Kovalenkov and Vives (2014). Specifically, we interpret one period as one year and let the total asset payoff volatility match its historical value 20% of the aggregate stock market (i.e., $\sqrt{Var(\tilde{V})} = 20\%$). We assume that investors collectively can learn half of the asset uncertainty, so that $Var(\tilde{v}) = \frac{1}{2}Var(\tilde{V})$, which implies that $\tau_v = \tau_\xi = 50$. Regarding private information quality, Kovalenkov and Vives (2014) consider a range of signal-to-noise ratio $\frac{\tau_\varepsilon}{\tau_v}$, from as low as one basis point to as high as 16. We consider two possible values, $\frac{\tau_\varepsilon}{\tau_v} \in \{0.001, 1\}$, to respectively represent low quality and high quality private information. Given the choice of $\tau_v = 50$, this implies two possible values of $\tau_\varepsilon : \tau_\varepsilon \in \{0.05, 500\}$. We also follow Kovalenkov and Vives (2014) and set the risk aversion coefficient γ at 2. Finally, we choose $\tau_e = \tau_u = 50000$, which implies that investors make very small errors in forming their date-1 beliefs (note that $\frac{Var(\tilde{u})}{Var(\tilde{v})} = \frac{Var(\tilde{e})}{Var(\tilde{v})} = 0.001$). We do so deliberately to illustrate that even small errors in interpreting price information can aggregate into a significant effect on equilibrium outcomes.

Return volatility Buying the asset at the date-1 market costs $\tilde{p}$ per share. Holding it till date 2 generates a payoff $\tilde{V}$. Hence, the asset return per share is $\tilde{V} - \tilde{p}$. Return volatility is measured by the standard deviation of asset returns, $\sigma(\tilde{V} - \tilde{p})$.

In the middle two panels of Figure 1, we plot return volatility $\sigma(\tilde{V} - \tilde{p})$ against investor sophistication τ_η with solid curves. Again, the dashed lines correspond to the value in the standard REE economy with $\tau_\eta = \infty$. We make the following two observations. First, costly interpretation of prices generates higher return volatility than the full REE benchmark (i.e., the solid curves lie

above the dashed lines in both panels). This may help to address the volatility puzzle (Shiller, 1981; LeRoy and Porter, 1981), which states that it is difficult to explain the historical volatility of stock returns with any model in which investors are rational and discount rates are constant. Also note that the excess return volatility is non-negligible even though investors only make very small mistakes with $\frac{Var(\tilde{u})}{Var(\tilde{v})} = \frac{Var(\tilde{e})}{Var(\tilde{v})} = 0.001$. For instance, in the top panel with $\tau_\varepsilon = 0.05$, costly interpretation of prices can lead to more than 30% of excess return volatility.

Second, return volatility decreases with investor sophistication (i.e., the solid curves are downward sloping). This is because price informativeness increases with τ_η, which implies that sophistication makes the price $\tilde{p}$ closer to the fundamental $\tilde{V}$, driving down the equilibrium return volatility. As explored in Section 5, this return-volatility result has implications for investor welfare.

Return predictability We now examine whether and how asset returns $\tilde{V}-\tilde{p}$ can be predicted by the price $\tilde{p}$. Empirically, one can run a linear regression from $\tilde{V}-\tilde{p}$ on $\tilde{p}$, i.e., $\tilde{V}-\tilde{p} = intercept + m \times \tilde{p} + error$. The regression coefficient is $m = \frac{Cov\left(\tilde{V}-\tilde{p},\tilde{p}\right)}{Var(\tilde{p})}$. In the traditional noisy-REE setting with exogenous noise trading (e.g., Hellwig, 1980), returns exhibit reversals; that is, $m < 0$ (see Banerjee, Kaniel, and Kremer (2009)). This is because exogenous noise demand pushes the price too high and exogenous noisy supply depresses the price too low. In contrast, in our setting with endogenous noise trading due to costly interpretation of prices, returns exhibit momentum: $m > 0$. This provides an explanation for the price momentum documented in the data (e.g., Jegadeesh and Titman, 1993; Moskowitz, Ooi, and Pedersen, 2012).

The price momentum in our model is an underreaction story. When investors are fully sophisticated ($\tau_\eta = \infty$), the price fully aggregates their private

information and there is no return predictability. Formally, by Corollary 1, the price is a martingale ($\tilde{p}^{REE} = E(\tilde{V}|\tilde{p}^{REE})$) and hence the price change is not predictable ($Cov(\tilde{V} - \tilde{p}^{REE}, \tilde{p}^{REE}) = 0$). When investors have limited sophistication, their forecasts do not fully use the information in the price, which in turn causes their trading not to fully incorporate information, thereby making the price underreact to information. Our mechanism shares similarity to Hong and Stein (1999) who generate momentum via "newswatchers." Newswatchers make forecasts based on private information, but do not condition on prices. Since these investors fail to extract information from prices, the slow diffusion generates momentum in Hong and Stein (1999).

In the right two panels of Figure 1, we plot m against τ_η in solid curves, where the dashed lines still indicate the m-values in a standard REE model. In both panels, we observe that m is indeed positive, indicating that there exists price momentum in our economy. In addition, m can be hump-shaped or decreasing in τ_η, depending on the value of the precision τ_ε of private information. It is intuitive that m decreases with τ_η for large values of τ_η, since m eventually degenerates to 0 as τ_η approaches to infinity.

Figure 1 demonstrates that m can also increase with τ_η for small values of τ_η, which is true when investors have coarse private information (i.e., τ_ε is small). The intuition is as follows. When both τ_ε and τ_η are small, investors have little private information and read little information from the price. In equilibrium, the price is close to being a constant since it does not aggregate much information. This means that the price does not have much predictive power for future returns. Now if we increase τ_η, investors start to pay more attention to the price, and thus their trading starts to inject more information into the price, generating more predictability of asset returns.

Proposition 2 (Price informativeness, return volatility, and price momentum)

(a) *Price informativeness*

As investors at date 1 become more sophisticated, the price $\tilde{p}$ conveys more precise information about the asset fundamental $\tilde{V}$. That is, $\frac{\partial \alpha}{\partial \tau_\eta} < 0$.

(b) *Return volatility*

(1) *As $\tau_\eta \to \infty$, return volatility approaches $\tau_\xi^{-1/2}$ (i.e., $\lim_{\tau_\eta \to \infty} \sigma\left(\tilde{V} - \tilde{p}\right) = \tau_\xi^{-1/2}$).*

(2) *As investor sophistication level τ_η increases, return volatility monotonically decreases if investors' fundamental information is sufficiently coarse or sufficiently precise (i.e., $\frac{\partial \sigma\left(\tilde{V}-\tilde{p}\right)}{\partial \tau_\eta} < 0$ if τ_ε is sufficiently small or sufficiently large).*

(c) *Price momentum*

(1) *When investors at date 1 are not fully sophisticated, asset returns exhibit price momentum. When $\tau_\eta \to \infty$, there is no return predictability. That is, $m > 0$ for $\tau_\eta \in (0, \infty)$, and $\lim_{\tau_\eta \to \infty} m = 0$.*

(2) *When investors have sufficiently coarse fundamental information, price momentum m increases with investor sophistication τ_η at low values of τ_η, and price momentum m decreases with investor sophistication τ_η at high values of τ_η. When investors have sufficiently precise fundamental information, price momentum m monotonically decreases in investor sophistication τ_η.*

4.2.2 Investor Disagreement and Trading Volume

Disagreement We define investor disagreement as the dispersion across investors' date-1 expectations about the fundamental $\tilde{V}$, i.e.,

$$Disagreement \equiv \sqrt{Var\left(E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i) - \bar{E}(\tilde{V})\right)}, \tag{17}$$

where

$$\bar{E}(\tilde{V}) \equiv \int_0^1 E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)di = E\left[E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)\,|\tilde{v}, \tilde{u}\right] \tag{18}$$

is the average expectation across investors at date 1.

In the two middle panels of Figure 2, we plot *Disagreement* against investor sophistication τ_η in solid curves. The other parameters take the same values as in Figure 1. The dashed lines still plot the values in the standard REE economy with $\tau_\eta = \infty$. By Corollary 1, when $\tau_\eta = \infty$, the price perfectly reveals the aggregate fundamental information, and so investors agree on asset valuation. As a result, $Disagreement = 0$ for $\tau_\eta = \infty$. Comparing the solid curves to dashed lines, we see that costly interpretation of prices adds disagreement among investors.

Disagreement can change with τ_η non-monotonically, depending on the precision τ_ε of investors' private fundamental information. This is due to two opposite forces. First, investors interpret the price in different ways, and so a higher τ_η means that investors' expectations rely more on their diverse information extracted from the price, thereby leading to a larger belief heterogeneity. Second, a higher τ_η implies that the price conveys more precise information about the asset fundamental (see Part (a) of Proposition 2), which tends to make investors' date-1 belief converge. In addition, since disagreement vanishes when $\tau_\eta = \infty$, it must be the case that the negative effect dominates for sufficiently large τ_η, so that *Disagreement* decreases with τ_η when τ_η is large. Nonetheless, when τ_η is small, the first positive effect can dominate as well. This possibility will

arise when investors' private fundamental information is very coarse (i.e., τ_ε is small). Intuitively, starting from a small τ_ε, before accessing to market data, investors' date-1 beliefs are close to the prior and thus do not differ much from each other; after they see the price and interpret it differently, their opinions start to diverge. Taken together, when τ_ε is small, *Disagreement* is hump-shaped in τ_η. When τ_ε is large, *Disagreement* monotonically decreases with τ_η.

Trading volume To focus on the volume generated solely by different price interpretations, we assume that investors start with a zero initial position of risky assets. Therefore, the trading volume of investor i and the total trading volume are, respectively,

$$\left|D\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)\right| = \left|\frac{E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-\tilde{p}}{\gamma Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)}\right| \text{ and } Volume \equiv \int_0^1 \left|D\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)\right| di. \tag{19}$$

When all investors have the same sophistication level τ_η, they face the same variance in trading the risky asset, i.e.,

$$Risk \equiv Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i) = \left(\tau_v+\tau_\varepsilon+\tau_p\right)^{-1}+\tau_\xi^{-1}, \tag{20}$$

where the second equality follows from equation (15). Hence, by demand function (7) and market-clearing condition (8), the equilibrium price is equal to the average expectation of investors,

$$\tilde{p} = \int_0^1 E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)di = \int_0^1 E\left(\tilde{v}|\,\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)di \equiv \bar{E}\left(\tilde{v}\right). \tag{21}$$

By equations (17)–(21), we can compute

$$Volume = \sqrt{\frac{2}{\pi}}\frac{Disagreement}{\gamma \times Risk}. \tag{22}$$

Thus, the total trading volume is jointly determined by three factors: investors' different date-1 expectations about the asset fundamental $\tilde{v}$, investors' risk aversion coefficient γ, and the risk faced by investors in trading the assets. When

investors disagree more about the future fundamental $\tilde{V}$, they trade more and so the total trading volume is higher. When investors are less risk averse and when they perceive less risk in trading the assets, they also trade more aggressively, leading to a higher total trading volume.

Remark 1 (Hedging-motivated trade) *The assumption that investors start with no risky assets does not affect our result. Suppose instead that investor i is initially endowed with $\tilde{y}_i$ shares of risky asset, where $\tilde{y}_i \sim N\left(0, \sigma_y^2\right)$ is independently and identically distributed across investors. Our baseline model corresponds to a degenerate case of $\sigma_y = 0$. In this extended setting, we can compute that the total trading volume is given by*

$$Volume = \int_0^1 \left| D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) - \tilde{y}_i \right| di = \sqrt{\frac{2}{\pi}} \frac{Disagreement}{\gamma \times Risk} + \sqrt{\frac{2}{\pi}} \sigma_y.$$

This expression differs from equation (22) only by a constant $\sqrt{\frac{2}{\pi}}\sigma_y$ that captures the volume generated by the endowment heterogeneity.

We continue our numerical example of Figure 2 and plot $Volume$ and $Risk$ against τ_η. We observe that $Risk$ decreases with τ_η. This is because more sophisticated investors glean more information from price data for two reasons. First, a higher sophistication level means that they study market data more intensively and can directly get more information from the price. Second, by Part (a) of Proposition 2, when all investors study the price more intensively, the price itself becomes a more informative signal (i.e., α decreases) and thus, each investor at date 1 can infer more information from the price. As $\tau_\eta \to \infty$, the price aggregates perfectly investors' private information and investors' perceived risk declines to $Var(\tilde{V}|\tilde{v}) = \tau_\xi^{-1}$.

The volume pattern mimics the disagreement pattern. First, comparing the solid curves to the dashed lines, we see that costly price interpretation generates excess trading volume. This result is consistent with the empirical evidence

documented in the finance literature (e.g., Odean, 1999; Barber and Odean, 2000). Second, when investors have coarse fundamental information, $Volume$ is hump-shaped in τ_η. When investors have precise fundamental information, $Volume$ monotonically decreases with τ_η.

The literature has long been interested in the tripartite relation among opinion divergence, trading volume, and stock return volatility (e.g., Shalen, 1993). Figures 1 and 2 help to understand some documented empirical findings. First, Garfinkel (2009) constructs an order-based measure for investor opinion divergence and finds that volume is a better proxy for disagreement than return volatility. Garfinkel's disagreement measure is the simple daily standard deviation (across orders) of the distance between each order's requested price and the most recent trade price preceding that order. This measure can be viewed as a close empirical counterpart for our disagreement definition in (17): $E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$ represents the investor's reservation value in the submitted order, and $\bar{E}(\tilde{V})$ is the prevailing price according to equation (21). Our results indeed show that volume mimics disagreement better than return volatility, in particular, when τ_ε is small: both volume and disagreement are hump-shaped in τ_η, while return volatility is decreasing in τ_η.

Second and relatedly, our results also help to reconcile the contradictory empirical findings on the cross-sectional relation between disagreement and return volatility. For instance, Frankel and Foot (1990) and Anderson, Ghysels, and Juergens (2005) document a positive relation, while Garfinkel (2009) documents a negative relation. In our setting, return volatility is always downward sloping in investor sophistication τ_η in Figure 1, but disagreement can exhibit a hump-shape in Figure 2. Thus, to the extent that τ_η is driving the cross-sectional variation, return volatility and disagreement can move in the same or opposite directions.

Some recent studies suggest possible empirical proxies for investor sophistication or attention τ_{η_i}, which therefore facilitates the testing of our model predictions on returns and volume. For instance, Gargano and Rossi (2016) find that males pay more attention than females; that attention is an increasing function of investors' age; and that brokerage account holders with higher invested wealth and higher exposure to small capitalization stocks, growth stocks, momentum stocks, and the overall market, are more attentive. These findings basically connect τ_{η_i} to the observable characteristics of investors or assets.

Proposition 3 (Risk, disagreement, and trading volume)

(a) *Risk*

As investors become more sophisticated at date 1, investors perceive lower risk in trading (i.e., $\frac{\partial Risk}{\partial \tau_\eta} < 0$*). As* $\tau_\eta \to \infty$*, risk approaches* τ_ξ^{-1} *(i.e.,* $\lim_{\tau_\eta \to \infty} Risk = \tau_\xi^{-1}$*).*

(b) *Investor disagreement*

(1) *As* $\tau_\eta \to \infty$*, investor disagreement vanishes (i.e.,* $\lim_{\tau_\eta \to \infty} Disagreement = 0$*).*

(2) *When investors have coarse fundamental information, disagreement is hump-shaped in investor sophistication (i.e., for small values of* τ_ε*,* $\frac{\partial Disagreement}{\partial \tau_\eta} < 0$ *if and only if* τ_η *is sufficiently large). When investors have precise fundamental information, disagreement decreases monotonically with sophistication (i.e., for large values of* τ_ε*,* $\frac{\partial Disagreement}{\partial \tau_\eta} < 0$ *for all values of* τ_η*).*

(c) *Trading volume*

(1) *As $\tau_\eta \to \infty$, the total trading volume vanishes (i.e., $\lim_{\tau_\eta \to \infty} Volume = 0$).*

(2) *When investors have coarse fundamental information, trading volume is hump-shaped in investor sophistication (i.e., for small values of τ_ε, $\frac{\partial Volume}{\partial \tau_\eta} < 0$ if and only if τ_η is sufficiently large). When investors have precise fundamental information, trading volume decreases monotonically with sophistication (i.e., for large values of τ_ε, $\frac{\partial Volume}{\partial \tau_\eta} < 0$ for all values of τ_η).*

5 Sophistication Level Equilibrium

5.1 Sophistication Determination

As we discussed in Section 3.1, the sophistication level is determined by the rational self 0 of each investor at date 0. Inserting the date-1 demand function $D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)$ in (7) into the CARA utility function and taking expectations yield investor i's date-0 payoff, $E\left[-e^{-\gamma\left[(\tilde{V}-\tilde{p})D(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-C\left(\tau_{\eta_i}\right)\right]}\right]$. Note that this expectation is computed under the correct distribution, because self 0 is fully rational in contemplating the sophistication level τ_{η_i} of her future self, which in turn determines how much information the boundedly rational self 1 will read from the asset price $\tilde{p}$ (or, alternatively, the research department knows that the trading desk will add noise but can control its level by investing in training). Formally, τ_{η_i} is determined by

$$\max_{\tau_{\eta_i}} E\left[-\exp\left(-\gamma\left[(\tilde{V}-\tilde{p})D\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)-C\left(\tau_{\eta_i}\right)\right]\right)\right]. \tag{23}$$

Definition 1 *An overall equilibrium of the two-stage game is defined as follows:*

(a) *Financial market equilibrium at date 1, which is characterized by a price function $p(\tilde{v}, \tilde{u})$ and demand functions $D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)$, such that:*

(1) *$D(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$ is given by (7), which maximizes investors' conditional subjective expected utilities given their date-1 beliefs;*

(2) *the market clears almost surely, i.e., equation (8) holds;*

(3) *investors' date-1 beliefs are given by (1), (2), and (3), where $\tilde{s}_p$ in (1) is implied by the equilibrium price function $p(\tilde{v}, \tilde{u})$ and where the sophistication levels $(\tau_{\eta_i})_{i\in[0,1]}$ are determined at date 0.*

(b) *Sophistication level equilibrium at date 0, which is characterized by sophistication levels $(\tau_{\eta_i})_{i\in[0,1]}$, such that τ_{η_i} solves (23), where investors' date-1 beliefs are given by (1)–(3), with $\tilde{s}_p$ in (1) being determined by the price function $p(\tilde{v}, \tilde{u})$ in the date-1 financial market equilibrium.*

5.2 Equilibrium Characterization

Given that investors are ex ante identical, we consider symmetric equilibrium in which all investors choose the same sophistication level. Let $W\left(\tau_{\eta_i}; \tau_\eta\right)$ denote the certainty equivalent of investor i's date-0 payoff when she decides to acquire a sophistication level τ_{η_i} and all the other investors acquire a sophistication level τ_η. That is,

$$\begin{aligned} W\left(\tau_{\eta_i}; \tau_\eta\right) &\equiv -\frac{1}{\gamma} \ln\left(-E\left[-e^{-\gamma\left[(\tilde{V}-\tilde{p})D(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-C\left(\tau_{\eta_i}\right)\right]}\right]\right) \\ &= \frac{1}{2\gamma} \ln\left[\begin{array}{c} [1+\gamma\underbrace{Cov(\tilde{V}-\tilde{p}, D_i)}_{\text{more informed trading}}]^2 \\ -\gamma^2 Var(\tilde{V}-\tilde{p}) \times \underbrace{Var\left(D_i\right)}_{\text{excessive trading}} \end{array}\right] - \underbrace{C\left(\tau_{\eta_i}\right)}_{\text{cost}}, \end{aligned} \tag{24}$$

where the second equality follows from the properties of normal distributions. The optimal sophistication level $\tau^*_{\eta_i}$ is determined by $\tau^*_{\eta_i} = \arg\max_{\tau_{\eta_i}} W\left(\tau_{\eta_i}; \tau_\eta\right)$.

In equation (24), improving the sophistication of future self 1 affects the current self 0's payoff in three ways. The first effect is a direct effect: acquiring

sophistication incurs a cost, $C\left(\tau_{\eta_i}\right)$, which directly harms the investor from self 0's perspective. The other two effects are indirect, which work through affecting the trading in the future financial market. These two indirect effects work in opposite directions.

First, being more sophisticated allows the future self 1 to better read information from the asset price, which therefore makes her trading more aligned with price changes—i.e., buying low and selling high—and therefore benefits the investor at date 0. This positive indirect effect is captured by the term $Cov(\tilde{V}-\tilde{p}, D_i)$. When both the common noise $\tilde{u}$ in investors' date-1 personalized signals $\tilde{s}_{p,i}$ and the residual uncertainty $\tilde{\xi}$ in the asset payoff are sufficiently small, we can compute $Cov(\tilde{V}-\tilde{p}, D_i)$ as follows:

$$\lim_{\tau_u \to \infty, \tau_\xi \to \infty} Cov(\tilde{V}-\tilde{p}, D_i) = \frac{\tau_v\left(\tau_{\eta_i}-\tau_\eta\right)}{\gamma\left(\tau_e+\tau_v+\tau_\varepsilon+\tau_\eta\right)^2}. \tag{25}$$

In (25), $Cov(\tilde{V}-\tilde{p}, D_i)$ is proportional to the difference between investor i's sophistication level τ_{η_i} and the market's average sophistication level τ_η. Intuitively, when $\tau_{\eta_i} > \tau_\eta$, investor i's forecast beats the market, and so her trading improves her ex-ante welfare.

Second, investors engage in speculative trading in the date-1 financial market, because there is no risk-sharing benefits in our setting. In this sense, trading is excessive, and the more an investor's future self trades, the harmful it is from self 0's perspective. Improving the attention level τ_{η_i} allows self 1 to lower her perceived risk and thus she will trade more aggressively, which in turn harms self 0 via the excessive trading channel. This negative effect is captured by the term $Var\left(D_i\right)$ in equation (24), which measures the size of self 1's trading in

the financial market. Again, when both $\tilde{u}$ and $\tilde{\xi}$ are small, we can compute

$$\lim_{\tau_u \to \infty, \tau_\xi \to \infty} Var\left(D_i\right) = \frac{\tau_v \tau_{\eta_i}^2 + \left[\left(\tau_e + \tau_v + \tau_\varepsilon\right)^2 + \tau_\eta^2 + 2\tau_\eta\left(\tau_e + \tau_\varepsilon\right)\right]\tau_{\eta_i}}{\gamma^2\left(\tau_e + \tau_v + \tau_\varepsilon + \tau_\eta\right)^2} + \frac{1}{\gamma^2}\left[\frac{\tau_v \tau_\eta^2}{\left(\tau_e + \tau_v + \tau_\varepsilon + \tau_\eta\right)^2} + \tau_\varepsilon + \tau_e\right], \tag{26}$$

which is increasing in τ_{η_i}.

Panels a1–a3 of Figure 3 respectively plot $W\left(\tau_{\eta_i}; \tau_\eta\right), Cov(\tilde{V} - \tilde{p}, D_i)$, and $Var\left(D_i\right)$ for the parameter configuration of Figure 1. That is, $\tau_v = \tau_\xi = 50, \tau_\varepsilon = 500, \tau_e = \tau_u = 50000, \gamma = 2, \tau_\eta = 1000$, and the sophistication cost function takes a linear form $C\left(\tau_{\eta_i}\right) = k\tau_{\eta_i}$ with $k = 10^{-7}$. In Panel a1 of Figure 3, $W\left(\tau_{\eta_i}; \tau_\eta\right)$ is hump-shaped in τ_{η_i}; it achieves maximum at $\tau_{\eta_i}^* = 423.95$. In Panels a2 and a3, $Cov(\tilde{V} - \tilde{p}, D_i)$ and $Var\left(D_i\right)$ are both increasing in τ_{η_i}, which respectively capture the positive and negative indirect effects of τ_{η_i} on self 0's payoff.

Note that among the three effects of τ_{η_i} on $W\left(\tau_{\eta_i}; \tau_\eta\right)$ in (24) (i.e., $Cov(\tilde{V} - \tilde{p}, D_i)$, $Var\left(D_i\right)$, and $C(\tau_{\eta_i})$), only the effect associated with $Cov(\tilde{V} - \tilde{p}, D_i)$ is positive. Thus, any nonzero values of $\tau_{\eta_i}^*$ is driven by investors' incentive to beat the market in reading prices. However, in equilibrium, no investor beats the market since $\tau_{\eta_i}^* = \tau_\eta^*$ (For instance, $\lim_{\tau_u \to \infty, \tau_\xi \to \infty} Cov(\tilde{V} - \tilde{p}, D_i) = 0$ if $\tau_{\eta_i} = \tau_\eta$ in equation (25)). This implies that the benefit associated with $Cov(\tilde{V} - \tilde{p}, D_i)$ is zero in equilibrium. In this sense, private incentives to improve sophistication are misaligned with social welfare. This force will cause investors to over-invest in sophistication relative to the socially optimal level, a point that we shall revisit shortly in Section 5.4.

The first-order condition of investor i's sophistication determination problem

is:

$$\begin{cases} \left.\frac{\partial W\left(\tau_{\eta_i};\tau_\eta\right)}{\partial \tau_{\eta_i}}\right|_{\tau_{\eta_i}=0} \leq 0, & \text{if } \tau_{\eta_i}=0, \\ \frac{\partial W\left(\tau_{\eta_i};\tau_\eta\right)}{\partial \tau_{\eta_i}}=0, & \text{if } \tau_{\eta_i}>0. \end{cases} \tag{27}$$

In a symmetric equilibrium, we have $\tau_{\eta_i}=\tau_\eta$. Let

$$\phi\left(\tau_\eta\right) \equiv \frac{\partial W\left(\tau_\eta;\tau_\eta\right)}{\partial \tau_{\eta_i}}. \tag{28}$$

The equilibrium sophistication level τ_η^* is determined by the following conditions:

(1) If $\phi\left(0\right)\leq 0$ and $\frac{\partial^2 W\left(\tau_{\eta_i};0\right)}{\partial \tau_{\eta_i}^2}\leq 0$, then $\tau_\eta^*=0$ is an equilibrium sophistication level;

(2) If for some $\tau_\eta^*>0$, $\phi\left(\tau_\eta^*\right)=0$ and $\frac{\partial^2 W\left(\tau_{\eta_i};\tau_\eta^*\right)}{\partial \tau_{\eta_i}^2}\leq 0$, then this value of τ_η^* is an equilibrium sophistication level.

In Panel b1 of Figure 3, we continue the numerical example in Panels a1–a3 and plot $\phi\left(\tau_\eta\right)$ against τ_η. In this example, the equilibrium sophistication level is $\tau_\eta^*=766.34$.

Proposition 4 (Overall equilibrium) *Suppose that τ_u and τ_ξ are sufficiently large and that $C\left(\tau_\eta\right)$ is smooth, increasing, and weakly convex. There exists a unique symmetric overall equilibrium. If $\phi\left(0\right)\leq 0$, then $\tau_\eta^*=0$, and otherwise, τ_η^* is uniquely determined by $\phi\left(\tau_\eta^*\right)=0$. The financial market equilibrium is given by Proposition 1 accordingly at the equilibrium sophistication level τ_η^*.*

5.3 Complementarity, Multiplicity, and Market Fragility

Proposition 4 provides a sufficient condition for the existence and uniqueness of a symmetric overall equilibrium. Nonetheless, our economy admits multiple equilibria. Formally, if $\phi\left(\tau_\eta\right)$ is downward sloping, then the equilibrium is

unique. In contrast, when $\phi(\tau_\eta)$ has an upward sloping segment, multiplicity can arise. It is the complementarity result on price informativeness in Part (a) of Proposition 2 that leads to the possibility of multiplicity.

Intuitively, the root reason that investors acquire sophistication is to better read information from prices and trade on this information. Thus, when the best price signal $\tilde{s}_p$ is more accurate in predicting the fundamental, the benefit from this more informed trading is higher. Recall that, by Part (a) of Proposition 2, as all investors become more sophisticated at date 1, the price conveys more precise information about the fundamental. As a result, each individual's incentive to acquire sophistication can become stronger, leading to complementarity in sophistication acquisition. This complementarity force can be so strong that $\phi(\tau_\eta)$ can be upward sloping at some region, which admits multiple equilibria.

Formally, let $B(\tau_\eta)$ denote the best response of investor i's sophistication-acquisition choice to the sophistication level τ_η chosen by all the other investors. This best response function is implicitly determined by the first order condition (27) of investor i's date-0 problem. Sophistication acquisition is a strategic complement if and only if $B'(\tau_\eta) > 0$ (strategic substitute if and only if $B'(\tau_\eta) < 0$). We know that

$$B'(\tau_\eta) > 0 \Longleftrightarrow \frac{\partial^2 W\left(\tau_{\eta_i}; \tau_\eta\right)}{\partial \tau_{\eta_i} \partial \tau_\eta} > 0. \tag{29}$$

By the definition of $\phi(\tau_\eta)$ in equation (28), we have

$$\phi'(\tau_\eta) = \frac{\partial^2 W\left(\tau_{\eta_i}; \tau_\eta\right)}{\partial \tau_{\eta_i}^2} + \frac{\partial^2 W\left(\tau_{\eta_i}; \tau_\eta\right)}{\partial \tau_{\eta_i} \partial \tau_\eta}. \tag{30}$$

Note that $\frac{\partial^2 W\left(\tau_{\eta_i}; \tau_\eta\right)}{\partial \tau_{\eta_i}^2} \leq 0$ (which is the second order condition of investor i's date-0 problem) and thus, by (29) and (30), we have the following observations:

(1) If $B'(\tau_\eta) \leq 0$, then $\phi'(\tau_\eta) < 0$ and the equilibrium is unique. That is, strategic substitutability in sophistication acquisition leads to the uniqueness of the overall equilibrium;

(2) If $B'(\tau_\eta) > 0$, then the sign of $\phi'(\tau_\eta)$ is indeterminate, and $\phi'(\tau_\eta) > 0$ when the complementarity in sophistication acquisition is sufficiently strong.

Proposition 5 (Multiplicity) *Suppose that τ_ξ is sufficiently large and that $C(\tau_\eta)$ is smooth, increasing, and weakly convex.*

(a) *If τ_ε and $C''(0)$ are sufficiently small, then $\phi(\tau_\eta)$ is upward sloping for small values of τ_η and downward sloping for large values of τ_η. This non-monotonicity can lead to multiple equilibrium levels of sophistication.*

(b) *If τ_ε is sufficiently large, then $\phi(\tau_\eta)$ is downward sloping and there exists a unique equilibrium level τ_η^* of sophistication.*

Proposition 5 suggests that complementarity in sophistication acquisition is particularly strong when investors' fundamental information is coarse. This is because when their own information is not precise, investors will rely more on price information to forecast fundamentals. The requirement of small $C''(0)$ in Part (a) of Proposition 5 is simply a technical condition on the convexity of the cost function, which mechanically drives the shape of $\phi(\tau_\eta)$ (note that $C'(\tau_\eta)$ negatively enters the definition of $\phi(\tau_\eta)$).

In Panel b2 of Figure 3, we provide a numerical example to illustrate the multiplicity of equilibrium. The parameter values in Panel b2 are identical to those in Panel b1, except that $\tau_\varepsilon = 0.05$ and $k = 5 \times 10^{-7}$. We find that in Panel b2 of Figure 3, $\phi(\tau_\eta)$ is hump-shaped, and it crosses zero twice. As a result, there exist three equilibrium levels of τ_η^*: $\{0, 702.21, 4085.24\}$. Among these three equilibria, the middle one is unstable (i.e., $\phi(\tau_\eta)$ crosses zero from below), while the other two equilibria are stable.

This multiplicity result provides a source of market fragility in the sense that a small change in the market environment can cause a significant change

in equilibrium outcomes. To illustrate this point, we use Panel c of Figure 3 to examine the implications of changing the sophistication cost in interpreting market data. Specifically, we continue to use the parameter values in Panel b2 of Figure 3, but now we allow the cost parameter k to continuously change and plot the equilibrium values of τ_η^* against k. When there are multiple equilibria, we use dashed segments to indicate the unstable equilibrium. We see that in Panel c of Figure 3, as k decreases, τ_η^* increases as long as investors coordinate on a particular stable equilibrium (say, the one with a larger value of τ_η^*). This is intuitive: as the cost k of acquiring sophistication becomes lower, investors will choose to become more sophisticated and devote more effort to study the price data.

The multiplicity suggests that a slight change in k can lead to jumps in τ_η^*. For instance, suppose that investors coordinate on a stable equilibrium with a higher value of τ_η^*. Then, when k is close to 8.5×10^{-7}, and when it drops slightly, the equilibrium value of τ_η^* can jump from 0 to 1811.82. This implies that small changes in mental costs can cause individuals to dramatically adjust their sophistication levels. Alternatively, under the institutional interpretation of our setting, the sophistication acquisition cost corresponds to the training cost of trading desks. In this case, a small decrease in training costs can lead to a wave of intensively studying market data, such as development of algorithmic trading in financial markets.

5.4 Sophistication Acquisition and Investor Welfare

In this subsection, we examine the welfare implications of sophistication acquisition by considering two exercises. First, we conduct a comparative statics analysis with respect to sophistication acquisition cost and explore the patterns of equilibrium sophistication and investor welfare. Second, we compare the equi-

librium sophistication level with the sophistication level that maximizes social welfare, and show that investors tend to over-invest in acquiring sophistication in our economy.

5.4.1 The Impact of Sophistication Acquisition Cost

We measure investor welfare from an ex-ante perspective. That is, we define welfare by self 0's equilibrium payoff, $W(\tau^*_{\eta_i}; \tau^*_\eta)$. The following proposition characterizes the investor welfare in a symmetric overall equilibrium.

Proposition 6 (Welfare) In a *symmetric overall equilibrium with sophistication level* τ^*_η*, we have* $Cov(\tilde{V} - \tilde{p}, D_i) = 0$ *and thus, the investor welfare is given by*

$$W\left(\tau^*_\eta; \tau^*_\eta\right) = \frac{1}{2\gamma} \ln\left[1 - \gamma^2 Var(\tilde{V} - \tilde{p}) \times Var\left(D_i\right)\right] - C\left(\tau^*_\eta\right). \qquad (31)$$

As we discussed before, in a symmetric equilibrium, we have $\tau^*_{\eta_i} = \tau^*_\eta$ for $i \in [0, 1]$. Thus, the term $Cov(\tilde{V} - \tilde{p}, D_i)$ vanishes in the definition of $W(\tau_{\eta_i}; \tau_\eta)$ in (24). This is a form of fallacy of composition: each investor tries to acquire sophistication and beat the market, but no one achieves it in equilibrium.

As a result, trading only has a negative effect on equilibrium investor welfare, which is captured by the term $Var(\tilde{V} - \tilde{p}) \times Var\left(D_i\right)$ in equation (31). Intuitively, in our setting, investors trade for speculation purposes, and their speculative positions are proportional to the difference between their forecast of the fundamental and the asset price. After aggregation, the price averages out the idiosyncratic errors in investors' private information and as a result, investors end up holding positions related only to the noises in their information, leading to a "winner's curse" (see Biais, Bossaerts, and Spatt (2010)). Variable $Var\left(D_i\right)$ measures the size of speculative trading; the more investors speculate, the more they lose. Variable $Var(\tilde{V} - \tilde{p})$ is the wealth loss per unit trading:

a higher return variance $Var(\tilde{V}-\tilde{p})$ means that it is more likely for the fundamental $\tilde{V}$ to deviate from the prevailing price $\tilde{p}$ and thus, the winner's curse harms investors more. Taken together, $Var(\tilde{V}-\tilde{p}) \times Var\left(D_i\right)$ captures the welfare loss due to the winner's curse.[8]

The equilibrium value τ_η^* of investor sophistication affects investor welfare $W\left(\tau_\eta^*;\tau_\eta^*\right)$ in three ways (which respectively correspond to the three terms, $C\left(\tau_\eta^*\right)$, $Var\left(D_i\right)$, and $Var(\tilde{V}-\tilde{p})$, in equation (31)). First, a higher τ_η^* will incur a higher cost $C\left(\tau_\eta^*\right)$, which directly harms investor welfare. Second, τ_η^* affects welfare through affecting trading size $Var\left(D_i\right)$. By the definition of $Volume$ in equation (19), we have $Volume = \sqrt{\frac{2}{\pi}Var\left(D_i\right)}$. Thus, Part (c) of Proposition 3 indicates that τ_η^* affects $W\left(\tau_\eta^*;\tau_\eta^*\right)$ via its effect on $Volume$. Note that two similar effects are also present when we discuss how individual sophistication τ_{η_i} affects individual payoff $W\left(\tau_{\eta_i};\tau_\eta\right)$ in Section 5.2.[9]

The third effect of τ_η^* on $W\left(\tau_\eta^*;\tau_\eta^*\right)$ works through return variance $Var(\tilde{V}-\tilde{p})$. That is, increasing τ_η^* can improve equilibrium welfare $W\left(\tau_\eta^*;\tau_\eta^*\right)$ by lowering $Var(\tilde{V}-\tilde{p})$ (see Part (b) of Proposition 2). This effect is absent in the discussions on how τ_{η_i} affects $W\left(\tau_{\eta_i};\tau_\eta\right)$ in Section 5.2. There, from an individual's perspective, $Var(\tilde{V}-\tilde{p})$ is exogenous, and increasing τ_{η_i} increases individual payoff $W\left(\tau_{\eta_i};\tau_\eta\right)$ through increasing $Cov(\tilde{V}-\tilde{p},D_i)$ (i.e., the individual's trading becomes more aligned with price changes). However, in a symmetric equilibrium, $Cov(\tilde{V}-\tilde{p},D_i)$ vanishes and thus it cannot be affected

[8] This result is also related to the idea of "speculative variance" studied by Simsek (2013). In Simsek's setting, investors trade for two purposes, risk sharing and speculation. Speculative variance refers to the part of portfolio risk that is driven by speculation based on heterogeneous beliefs. Speculative variance tends to harm welfare and it is greater when the assets feature greater belief disagreement, both of which are consistent with our model.

[9] Note that the trading-size effects are not identical. Specifically, in Section 5.2, we take τ_η as given and vary τ_{η_i}, while here, the comparative statics requires $\tau_{\eta_i}^* = \tau_\eta^*$. For instance, in equation (26), $Var\left(D_i\right)$ is always increasing in τ_{η_i} for a fixed τ_η, but once we set $\tau_{\eta_i} = \tau_\eta$, $Var\left(D_i\right)$ is decreasing in τ_η for sufficiently large values of τ_η.

by changes in τ_η^*.

Due to the interactions among the three effects of τ_η^* on $W\left(\tau_\eta^*;\tau_\eta^*\right)$, in general there is a non-monotone relation between investor sophistication and investor welfare in equilibrium. Figure 4 illustrates this point for the parameter configuration of Panel b1 of Figure 3. We now allow the sophistication cost parameter k to continuously change and plot τ_η^*, $W\left(\tau_\eta^*;\tau_\eta^*\right)$, $Var(\tilde{V}-\tilde{p}), Var\left(D_i\right)$, and $C\left(\tau_\eta^*\right)$ against k. As illustrated by Panel b1 of Figure 3, the parameter configuration guarantees unique overall equilibrium, so that we do not need to choose a particular equilibrium when conducting comparative statics. But we note that our results are quite robust to parameter configurations.

In Panel a of Figure 4, the equilibrium value τ_η^* of sophistication decreases with the cost k of acquiring sophistication, which is intuitive. In Panel b, the equilibrium investor welfare $W\left(\tau_\eta^*;\tau_\eta^*\right)$ first decreases and then increases with k. This is due to the interactions among the three forces. First, in Panel c1, return variance $Var(\tilde{V}-\tilde{p})$ increases with k, as a result of the decreasing τ_η^* in Panel a. In effect, the combination of Panels a and c1 of Figure 4 is a reflection of the observation that $\sigma(\tilde{V}-\tilde{p})$ decreases in τ_η in the lower middle panel of Figure 1. Second, in Panel c2 of Figure 4, $Var\left(D_i\right)$ also increases with k. Again, combining Panels a and c1 of Figure 4 simply reflects that $Volume$ decreases with τ_η in the lower left panel of Figure 2 (i.e., an increase in k causes a decrease in τ_η, which amplifies disagreement and hence increases trading volume). Third, in Panel c3 of Figure 4, the total cost $C\left(\tau_\eta^*\right)$ of acquiring sophistication is hump-shaped in k, because as k increases, τ_η^* decreases.

5.4.2 Over Investment in Sophistication Acquisition

To examine whether investors over or under invest in sophistication acquisition in our economy, we compare the equilibrium sophistication level τ_η^* with the

level τ_η^{opt} that maximizes social welfare taking the financial market equilibrium as given, where

$$\tau_\eta^{opt} \equiv \arg\max_{\tau_\eta} W\left(\tau_\eta; \tau_\eta\right).$$

We obtain the benchmark sophistication level τ_η^{opt} as follows. At the date-0 sophistication-acquisition stage, we consider symmetric allocations, so that we specify $\tau_{\eta_i} = \tau_\eta$ for $i \in [0, 1]$. Instead, at the date-1 trading stage, we still keep the financial market at its equilibrium characterized by Proposition 1 for a given value of τ_η. By doing so, we isolate the effect of sophistication on welfare, since we do not vary the trading game.[10] The welfare of a representative investor $W\left(\tau_\eta; \tau_\eta\right)$ is then still given by equation (31) and thus the welfare-maximizing sophistication level τ_η^{opt} is chosen to maximize $W\left(\tau_\eta; \tau_\eta\right)$.

In Panels a and b of Figure 4, we use dashed curves to respectively plot the welfare-maximizing sophistication level τ_η^{opt} and the resulting maximal welfare W^{opt} against the cost k of acquiring sophistication. Apparently, by construction, W^{opt} is greater than the equilibrium welfare W^* in Panel b. In Panel a, we see that τ_η^{opt} is lower than the equilibrium level τ_η^* of sophistication, which means that investors tend to over invest in sophistication in our economy. This is due to the misalignment between private and social incentives to acquire sophistication (associated with the term $Cov(\tilde{V} - \tilde{p}, D_i)$ in (24)), that we discussed earlier in Section 5.2: each investor wants to become more sophisticated than others, although in equilibrium, no investor beats the market; this creates an extra private incentive to acquire sophistication. We have conducted analysis with many different parameter values and the main results we highlight in this subsection are robust across them.

[10] This treatment is standard in the literature. For instance, in the normative analysis of Goldstein, Ozdenoren, and Yuan (2013), the authors vary the level of coordination in investors' trading decisions, which is the focus of their paper, while keeping all the other equilibrium features of the economy.

As usual in the literature, one can rely on Pigouvian tax and lump-sum subsidies to implement the socially optimal sophistication level τ_η^{opt}. That is, a benevolent government can impose a tax on sophistication acquisition, which increases appropriately the cost of acquiring sophistication and thus lowers investors' ex-ante incentives to acquire sophistication. The government then returns the collected tax revenue back to investors as subsidy. This tax-subsidy schedule will improve social welfare by adjusting the equilibrium sophistication level τ_η^*.

6 Conclusion

We develop a model to capture the notion that investors cannot costlessly process price data in financial markets. Although investors actively infer information from the price, their information processing is noisy. The more sophisticated are investors, the smaller is this processing noise. After reading price data and form their beliefs, investors hold optimal trading positions according to their own beliefs (and so they are only boundedly rational in extracting information from the price). We find that imperfect price interpretation can inject noise into the price system, which serves as a form of endogenous noise trading in our setting. Compared to the standard REE, our model generates price momentum, excessive return volatility, and excessive trading volume. As investor sophistication increases, return volatility decreases, while disagreement and volume can exhibit a hump shape.

We employ a learning technology to endogenize investors' sophistication levels that in turn determine their bounded rationality at the trading stage. From an individual's perspective, the benefit of sophistication acquisition is to beat the market by reading better information from prices. However, in equilibrium,

this forms a fallacy of composition, because all investors end up with the same equilibrium level of sophistication. The social benefit of sophistication acquisition lies in the fact that more sophisticated traders bring the price closer to the fundamental, which therefore lowers the welfare loss due to speculative trading. The misalignment between private and social incentives to acquire sophistication implies that investors tend to over invest in sophistication acquisition in our economy. Finally, there can exist strategic complementarity in sophistication acquisition, leading to the possibility of multiple equilibria.

References

[1] Addoum, J. M. and J. R. Murfin (2017), "Hidden in Plain Sight: Equity Price Discovery with Informed Private Debt," Working Paper.

[2] Anderson, E. W., E. Ghysels, and J. L. Juergens (2005), "Do Heterogeneous Beliefs Matter for Asset Pricing?" *Review of Financial Studies* 18, 875–924.

[3] Angeletos, G.-M. and J. La'o (2013), "Sentiments," *Econometrica* 81, 739–779.

[4] Baker, M. and J. Wurgler (2007), "Investor Sentiment in the Stock Market," *Journal of Economic Perspectives* 21, 129–151.

[5] Banerjee, S. (2011), "Learning from Prices and the Dispersion in Beliefs," *Review of Financial Studies* 24, 3025–3068.

[6] Banerjee, S., R. Kaniel, and I. Kremer (2009), "Price Drift as an Outcome of Differences in Higher Order Beliefs", *Review of Financial Studies* 22, 3707–3734.

[7] Barber, B. and T. Odean (2000), "Trading Is Hazardous to Your Wealth: The Common Stock Performance of Individual Investors," *Journal of Finance* 55, 773–806.

[8] Barberis, N. and R. Thaler (2003), "A Survey of Behavioral Finance," In Handbook of the Economics of Finance, ed. George Constantinides, Milton Harris and René Stulz, 1053-1128. Amsterdam: Elsevier Academic Press.

[9] Benhabib, J. and P. Wang (2015), "Private Information and Sunspots in Sequential Asset Market," J*ournal of Economic Theory* 148, 558–584.

[10] Benhabib, J., P. Wang, and Y. Wen (2015), "Sentiments and Aggregate Demand Fluctuations," *Econometrica* 83, 549–585.

[11] Berk, J. B. and J. H. van Binsbergen (2016), "Assessing Asset Pricing Models Using Revealed Preference," *Journal of Financial Economics* 119, 1–23.

[12] Biais, B., P. Bossaerts, and C. Spatt (2010), "Equilibrium Asset Pricing and Portfolio Choice under Asymmetric Information," *Review of Financial Studies* 23, 1503–1543.

[13] Blocher, J. and M. Molyboga (2017), "The Revealed Preference of Sophisticated Investors," *European Financial Management*, Forthcoming.

[14] Brunnermeier, M. K., F. Papakonstantinou, and J. A. Parker (2016), "Optimal Time-Inconsistent Beliefs: Misplanning, Procrastination, and Commitment," *Management Science*, http://dx.doi.org/10.1287/mnsc.2015.2360.

[15] Brunnermeier, M. K, and J. A. Parker (2005), "Optimal Expectations," *American Economic Review* 95, 1092–1118.

[16] De Long, J. B., A. Shleifer, L. H. Summers, and R. J. Waldmann (1990) "Noise Trader Risk in Financial Markets," *Journal of Political Economy* 98, 703–738.

[17] Dean, M. and N. Neligh (2017), Experimental Tests of Rational Inattention, Working Paper, Columbia University.

[18] Esponda, I. (2008), "Behavioral Equilibrium in Economies with Adverse Selection," *American Economic Review* 98, 1269–1291.

[19] Esponda, I. and D. Pouzo (2016), "Berk-Nash Equilibrium: A Framework for Modeling Agents with Misspecified Models," *Econometrica* 84, 1093–1130.

[20] Eyster, E. and M. Rabin (2005), "Cursed Equilibrium," *Econometrica* 73, 1623–1672.

[21] Eyster, E., M. Rabin, and D. Vayanos (2017), "Financial Markets where Traders Neglect the Informational Content of Prices," *Journal of Finance*, Forthcoming.

[22] Frankel, J. and K. Froot (1990), "Chartists, Fundamentalists and Trading in the Foreign Exchange Market," *American Economic Review* 80, 181–185.

[23] Garfinkel, J. A. (2009), "Measuring Investors' Opinion Divergence," *Journal of Accounting Research* 47, 1317–1348.

[24] Gargano, A. and A. G. Rossi (2016), "Does it Pay to Pay Attention?", Working Paper.

[25] Gârleanu, N. and L. H. Pedersen (2016), "Efficiently Inefficient Markets for Assets and Asset Management," Working Paper.

[26] Goldstein, I., E. Ozdenoren, and K. Yuan (2013), "Trading Frenzies and Their Impact on Real Investment," *Journal of Financial Economics* 109, 566–582.

[27] Grossman, S. and J. Stiglitz (1980), "On the Impossibility of Informationally Efficient Markets," *American Economic Review* 70, 393–408.

[28] Guesnerie, R. (1992), "An Exploration of the Eductive Justifications of the Rational-Expectations Hypothesis," *American Economic Review* 82, 1254–1278.

[29] Han, J. and F. Sangiorgi (2015), "Searching for Information," Available at SSRN: https://ssrn.com/abstract=2635392.

[30] Hassan. T. A. and T. M. Mertens (2011), "Market Sentiment: A Tragedy of the Commons," *American Economic Review* (Papers & Proceedings) 101, 402–405.

[31] Hassan. T. A. and T. M. Mertens (2017), "The Social Cost of Near-Rational Investment," *American Economic Review* 107, 1059–1103.

[32] Hellwig, M. (1980), "On the Aggregation of Information in Competitive Markets," *Journal of Economic Theory* 22**,** 477–498.

[33] Hellwig, C., S. Kohls, and L. Veldkamp (2012), "Information Choice Technologies," *American Economic Review: Papers & Proceedings* 102, 35–40.

[34] Hong, H. and J. Stein (1999), "A Unified Theory of Underreaction, Momentum Trading and Overreaction in Asset Markets," *Journal of Finance* 54, 2143–2184.

[35] Jegadeesh, N. and S. Titman (1993), "Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency," *Journal of Finance* 48, 65–91.

[36] Kacperczyk, M. T., S. Van Nieuwerburgh, and L. Veldkamp (2016), "A Rational Theory of Mutual Funds' Attention Allocation," *Econometrica* 84, 571–626.

[37] Kahneman, D. (2011), *Thinking, Fast and Slow*. Farrar, Straus and Giroux.

[38] Kim, O. and R. E. Verrecchia (1994), "Market Liquidity and Volume around Earnings Announcements," *Journal of Accounting and Economics* 17, 41–67.

[39] Kovalenkov, A. and X. Vives (2014), "Competitive Rational Expectations Equilibria without Apology," *Journal of Economic Theory* 149, 211–235.

[40] Leahy, J. (2016), "The Economics of Wishful Thinking," http://fordschool.umich.edu/video/2016/john-leahy-economics-wishful-thinking.

[41] LeRoy, S. and R. Porter (1981), "The Present-value Relation: Tests Based on Implied Variance Bounds," *Econometrica* 49, 97–113.

[42] Matějka, Filip and A. McKay (2015), "Rational Inattention to Discrete Choices: A New Foundation for the Multinomial Logit Model," *American Economic Review* 105, 272–298.

[43] Mondria, J. (2010), "Portfolio Choice, Attention Allocation, and Price Comovement," *Journal of Economic Theory* 145, 1837–1864.

[44] Moskowitz, T., Y. H. Ooi, and L. H. Pedersen (2012), "Time Series Momentum," *Journal of Financial Economics* 104, 228–250.

[45] Myatt, D. P. and C. Wallace (2012), "Endogenous Information Acquisition in Coordination Games," *Review of Economic Studies* 79, 340–374.

[46] Odean, T. (1999), "Do Investors Trade Too Much?" *American Economic Review* 89, 1279–1298.

[47] Pavan, A. (2014), "Attention, Coordination, and Bounded Recall," Working Paper, Northwestern University.

[48] Peng, L. and W. Xiong (2006), "Investor Attention, Overconfidence and Category Learning," *Journal of Financial Economics* 80, 563–602.

[49] Peress, J. (2010), "The Tradeoff between Risk Sharing and Information Production in Financial Markets," *Journal of Economic Theory* 145**,** 124–155.

[50] Radner, R. (1979), "Rational Expectations Equilibrium: Generic Existence and the Information Revealed by Prices," *Econometrica* 47, 655–678.

[51] Shalen, C. (1993), "Volume, Volatility, and the Dispersion of Beliefs," *Review of Finandal Studies* 6, 405–434.

[52] Shiller, R. (1981), "Do Stock Prices Move Too Much to Be Justified by Subsequent Changes in Dividends?", *American Economic Review* 71, 421–436.

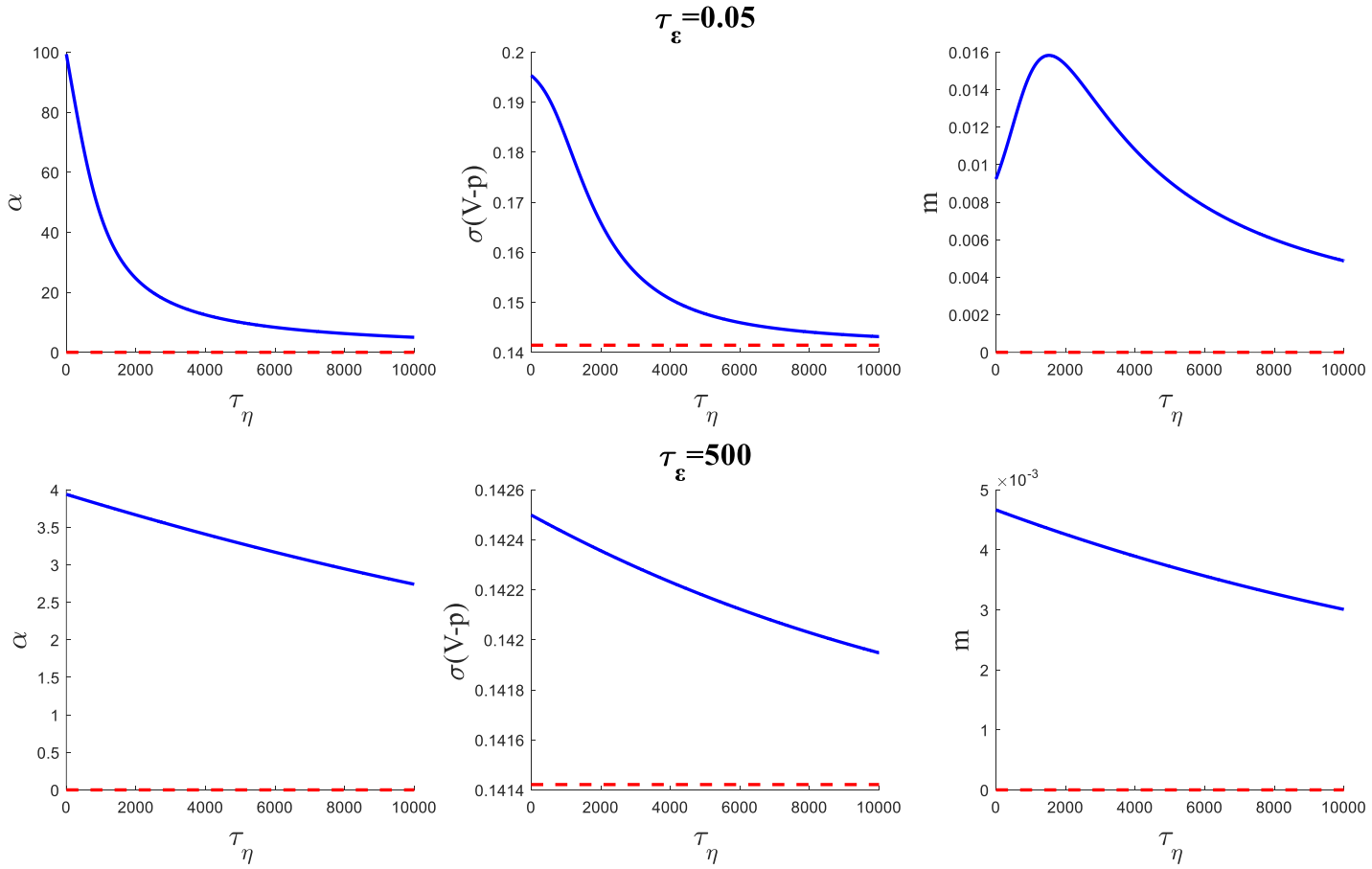
[53] Shleifer, A. (2000), *Inefficient Markets: An Introduction to Behavioral Finance*, 1st ed. Oxford University Press.

[54] Simsek, A. (2013), "Speculation and Risk Sharing with New Financial Assets," *Quarterly Journal of Economics* 128, 1365–1396.

[55] Verrecchia, R. E. (1982), "Information Acquisition in a Noisy Rational Expectations Economy," *Econometrica* 50**,** 1415–1430.

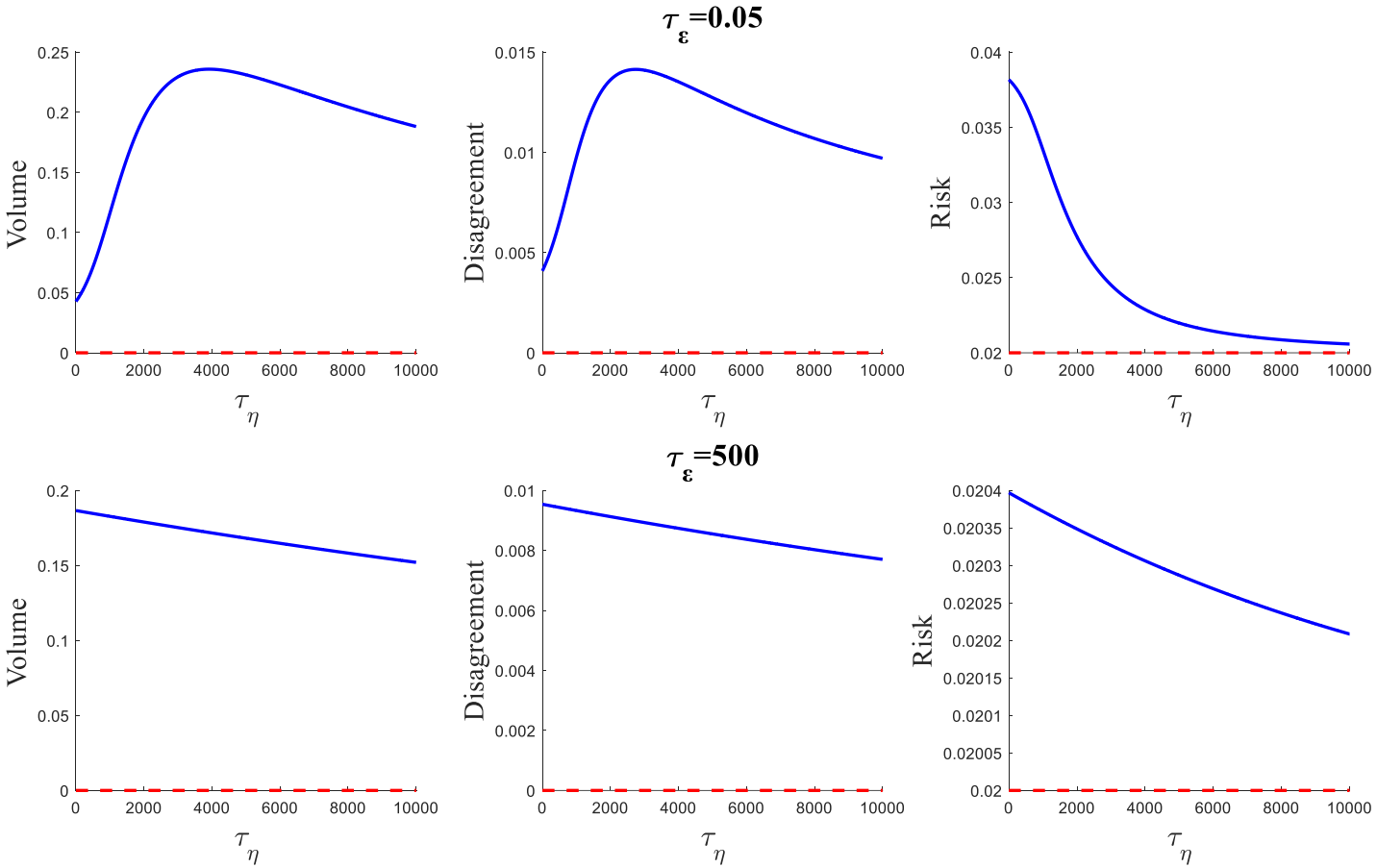
[56] Vives, X. (2008), *Information and Learning in Markets: The Impact of Market Microstructure*. Princeton University Press, Princeton and Oxford.

Figure 1: Price Informativeness, Return Volatilty, and Price Momentum



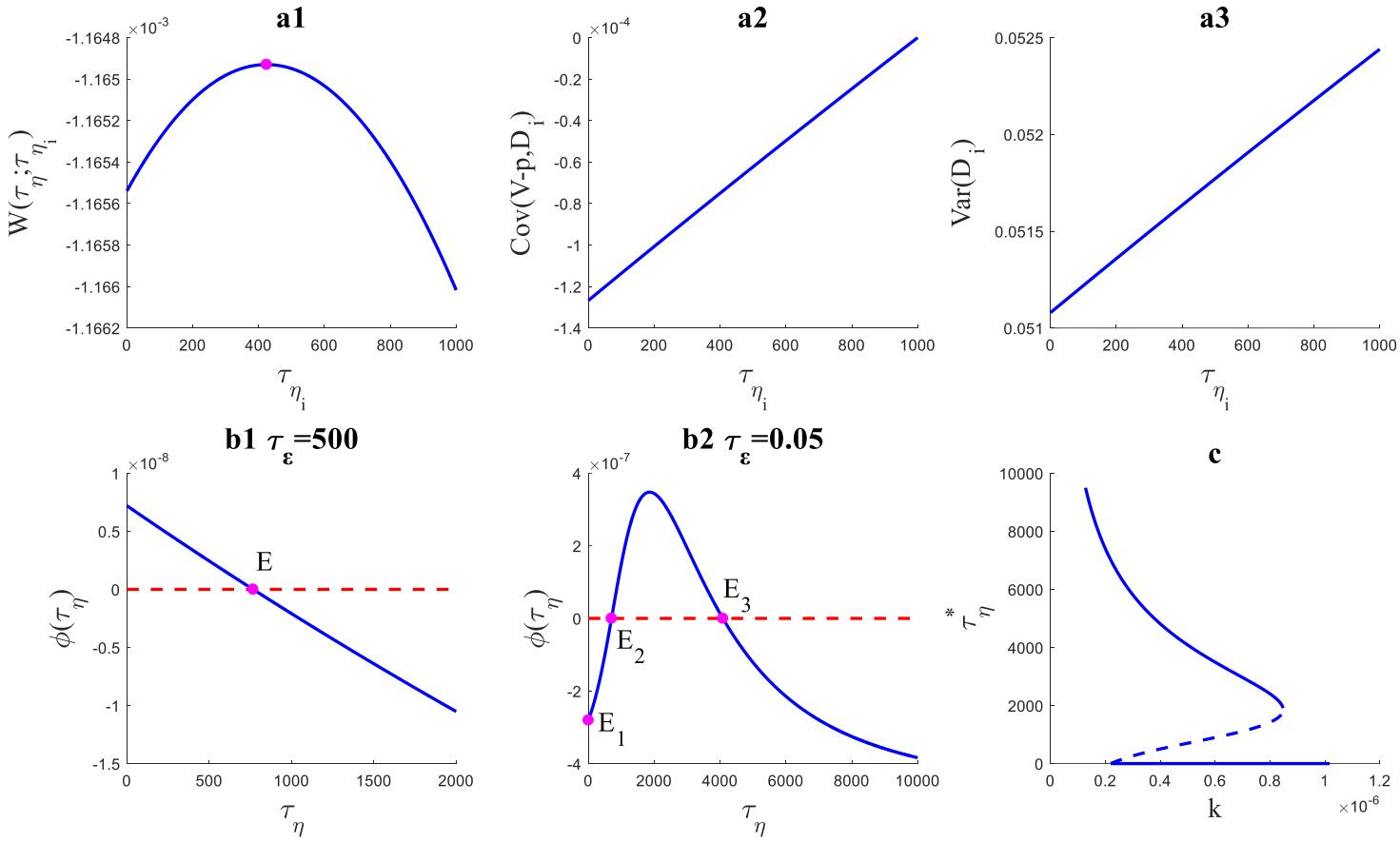
This figure plots price informativeness (negatively measured by α), return volatility ($\sigma(\tilde{V}-\tilde{p})$), and price momentum ($m$) against investors' sophistication level τ_η. In the top panels, we set $\tau_\varepsilon = 0.05$, while in the bottom panels, we set $\tau_\varepsilon = 500$. The other parameters are set as follows: $\tau_v = \tau_\xi = 50$, $\tau_e = \tau_u = 50000$, and $\gamma = 2$. The dashed lines plot the values in a standard REE economy that corresponds to $\tau_\eta = \infty$.

Figure 2: Trading Volume, Disagreement, and Perceived Risk



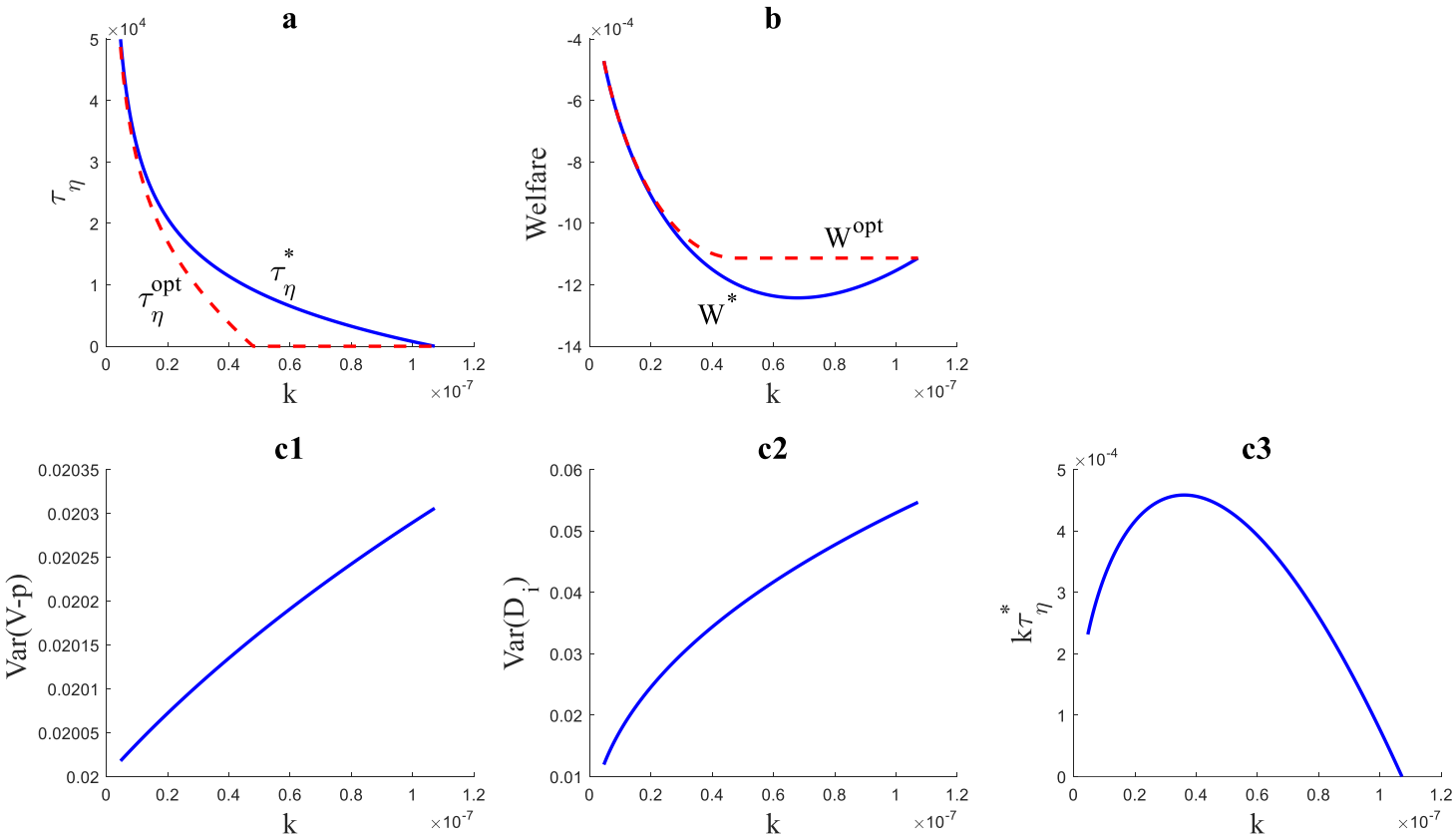
This figure plots trading volume, disagreement, and perceived risk against investors' sophistication level τ_η. In the top panels, we set $\tau_\varepsilon = 0.05$, while in the bottom panels, we set $\tau_\varepsilon = 500$. The other parameters are set as follows: $\tau_v = \tau_\xi = 50$, $\tau_e = \tau_u = 50000$, and $\gamma = 2$. The dashed lines plot the values in a standard REE economy that corresponds to $\tau_\eta = \infty$.

Figure 3: Sophistication Level Equilibrium



Investors' cost function of acquiring sophistication is $C(\tau_{\eta_i}) = k\tau_{\eta_i}$. In all panels, we set $\tau_v = \tau_\xi = 50$, $\tau_e = \tau_u = 50000$, and $\gamma = 2$. Panels a1 – a3 respectively plot $W(\tau_{\eta_i};\tau_\eta)$, $Cov(\tilde{V}-\tilde{p}, D_i)$, and $Var(D_i)$ as functions of τ_{η_i}, where $\tau_\varepsilon = 500$, $\tau_\eta = 1000$, and $k = 10^{-7}$. Panels b1 and b2 plot function $\phi(\tau_\eta) = \frac{\partial W(\tau_\eta;\tau_\eta)}{\partial \tau_\eta}$, where we set $\tau_\varepsilon = 500$ and $k = 10^{-7}$ for Panel b1 and we set $\tau_\varepsilon = 0.05$ and $k = 5 \times 10^{-7}$ for Panel b2. Panel c plots the effect of the sophistication cost k on the equilibrium values of τ^*_η for $\tau_\varepsilon = 0.05$. The dashed segment in Panel c indicates unstable equilibria.

Figure 4: Sophistication Acquisition and Investor Welfare



Investors' cost function of acquiring sophistication is $C(\tau_{\eta_i}) = k\tau_{\eta_i}$. Panels a and b plot the implications of changing sophistication cost parameters k on the equilibrium values of sophistication τ_η^* and welfare W^* (in solid curves) and on the values of welfare-maximizing sophistication τ_η^{opt} and the resulting maximal welfare W^{opt} (in dashed curves). Panels c1 – c3 respectively plot the three terms affecting the equilibrium welfare. The other parameter values are: $\tau_v = 50$, $\tau_\xi = 50$, $\tau_\varepsilon = 500$, $\tau_e = \tau_u = 50000$, and $\gamma = 2$.

Appendices

Appendix A: Lemmas

Lemma 1 *Given* $(\tau_e, \tau_u, \tau_\varepsilon) \in \mathbb{R}^3_{++}$ *and let* $\tau_\eta \to \infty$. *We have:*
(a) $\alpha\tau_\eta^{-1}$ *is bounded; (b)* $\alpha = O\left(\tau_\eta^{-1}\right) \to 0$*; and (c)* $\alpha^2\tau_\eta = O\left(\tau_\eta^{-1}\right) \to 0$.

Proof. By the bounds of α in Proposition 1, we have

$$\begin{aligned} 0 \quad &< \quad \alpha < \frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon} \\ &\Rightarrow \quad 0 < \alpha\tau_\eta < \frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon}\tau_\eta = \tau_u - \frac{\tau_u\tau_\varepsilon\left(\tau_e + \tau_u\right)}{\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon} < \tau_u \\ &\Rightarrow \quad \alpha\tau_\eta = O\left(1\right) \text{ (i.e., } \alpha\tau_\eta \text{ is finite).} \end{aligned}$$

Parts (b) and (c) follow directly from Part (a). ■

Lemma 2 *Given* $(\tau_e, \tau_u, \tau_\eta) \in \mathbb{R}^3_{++}$. *(a)* $\lim_{\tau_\varepsilon \to 0} \alpha = \frac{\tau_u}{\tau_\eta}$ *and* $\lim_{\tau_\varepsilon \to 0} \frac{\partial\alpha}{\partial\tau_\eta} = -\frac{\tau_u}{\tau_\eta^2}$. *(b) As* $\tau_\varepsilon \to \infty$, *we have* $\alpha = O\left(\tau_\varepsilon^{-1}\right) \to 0$ *and* $\frac{\partial\alpha}{\partial\tau_\eta} \propto -\frac{\tau_e\alpha}{(\tau_e+\tau_u)\tau_\varepsilon} \to 0$.

Proof. (a) By the proof for Proposition 1, we know that α is determined by $f\left(\alpha\right) = 0$, where $f\left(\alpha\right)$ crosses 0 from below. As τ_ε increases, $f\left(\alpha\right)$ shifts upward. Since f crosses zero from below, we know that α decreases with τ_ε. So, α is bounded as τ_ε goes to 0. By (16), we know that as $\tau_\varepsilon \to 0$,

$$\left(\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon\right)\alpha \propto \tau_e\tau_u \Rightarrow \alpha \propto \frac{\tau_u}{\tau_\eta}.$$

Inserting $\alpha \propto \frac{\tau_u}{\tau_\eta}$ into the expression of $\frac{\partial\alpha}{\partial\tau_\eta}$ in equation (A1), we can show $\frac{\partial\alpha}{\partial\tau_\eta} \propto -\frac{\tau_e\alpha}{(\tau_e+\tau_u)\tau_\varepsilon}$.

(b) Let $\tau_\varepsilon \to \infty$. By Proposition 1,

$$\begin{aligned} \alpha \quad &\in \quad \left(0, \frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon}\right) \Rightarrow \\ 0 \quad &< \quad \lim_{\tau_\varepsilon\to\infty} \alpha\tau_\varepsilon < \lim_{\tau_\varepsilon\to\infty} \frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon}\tau_\varepsilon = \frac{\tau_e\tau_u}{\tau_e + \tau_u} \\ &\Longrightarrow \quad \lim_{\tau_\varepsilon\to\infty} \alpha\tau_\varepsilon \text{ is finite} \Rightarrow \alpha = O\left(\tau_\varepsilon^{-1}\right). \end{aligned}$$

Inserting $\alpha = O\left(\tau_\varepsilon^{-1}\right)$ into the expression of $\frac{\partial\alpha}{\partial\tau_\eta}$ in equation (A1), we can show $\frac{\partial\alpha}{\partial\tau_\eta} \propto -\frac{\tau_e\alpha}{(\tau_e+\tau_u)\tau_\varepsilon} = -O(\tau_\varepsilon^{-2})$. ■

Lemma 3 *Given* $(\tau_e, \tau_\varepsilon, \tau_\eta) \in \mathbb{R}^3_{++}$ *and let* $\tau_u \to \infty$. *We have:*

$$\begin{aligned}
\alpha &\to \frac{\tau_e}{\tau_\varepsilon}, \tau_p \to \tau_e + \tau_\eta, \\
a_v &\to \frac{\tau_\varepsilon + \tau_e + \tau_\eta}{\tau_v + \tau_\varepsilon + \tau_e + \tau_\eta}, a_u \to \frac{\tau_e\left(\tau_e + \tau_\varepsilon + \tau_\eta\right)}{\tau_\varepsilon\left(\tau_e + \tau_v + \tau_\varepsilon + \tau_\eta\right)}, \\
\tau_{p,i} &\to \tau_e + \tau_{\eta_i}, \beta_{s,i} \to \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon + \tau_e + \tau_{\eta_i}}, \\
\beta_{p,i} &\to \frac{\tau_e + \tau_{\eta_i}}{\tau_v + \tau_\varepsilon + \tau_e + \tau_{\eta_i}}, \text{ and } \beta_{z,i} \to -\frac{\tau_{\eta_i}}{\tau_e + \tau_v + \tau_\varepsilon + \tau_{\eta_i}}.
\end{aligned}$$

Proof. As $\tau_u \to \infty$, the cubic equation in determining α in Proposition 1 degenerates to $(\tau_u\tau_\varepsilon)\,\alpha - \tau_e\tau_u = 0$, which implies $\alpha \to \frac{\tau_e}{\tau_\varepsilon}$. Inserting $\alpha \to \frac{\tau_e}{\tau_\varepsilon}$ into $\tau_p = \frac{\tau_e\tau_u + (\tau_e+\tau_u)\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}$, we have $\tau_p \to \tau_e + \tau_\eta$. Using the expressions of a_v and a_u in Proposition 1, we have $a_v \to \frac{\tau_\varepsilon+\tau_e+\tau_\eta}{\tau_v+\tau_\varepsilon+\tau_e+\tau_\eta}$ and $a_u \to \frac{\tau_e(\tau_e+\tau_\varepsilon+\tau_\eta)}{\tau_\varepsilon(\tau_e+\tau_v+\tau_\varepsilon+\tau_\eta)}$.

Using equation (13) and $\alpha \to \frac{\tau_e}{\tau_\varepsilon}$, we can show $\tau_{p,i} \to \tau_e + \tau_{\eta_i}$. Using $\alpha \to \frac{\tau_e}{\tau_\varepsilon}$, $\tau_{p,i} \to \tau_e + \tau_{\eta_i}$, and the expressions of β's in the proof of Proposition 1, we have the limits of β's in the lemma. ■

Appendix B: Proofs

Proof of Proposition 1

Using Bayes' rule, we can compute

$$E\left(\tilde{v}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) = E(\tilde{v}|\tilde{s}_i, \tilde{s}_{pz,i}) = \beta_{s,i}\tilde{s}_i + \beta_{p,i}\tilde{s}_{p,i} + \beta_{z,i}\tilde{z}_i,$$

where

$$\begin{aligned}
\beta_{s,i} &= \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon + \tau_{p,i}}, \\
\beta_{p,i} &= \frac{\tau_{p,i}}{\tau_v + \tau_\varepsilon + \tau_{p,i}}, \\
\beta_{z,i} &= -\frac{\tau_{\eta_i}\left(\tau_e + \tau_u + \alpha\tau_e\right)}{\tau_e\tau_u + \tau_e\tau_{\eta_i} + \tau_u\tau_{\eta_i}} \frac{\tau_{p,i}}{\tau_v + \tau_\varepsilon + \tau_{p,i}}.
\end{aligned}$$

Note that when $\tau_{\eta_i} = \tau_\eta$, $\tau_{p,i}, \beta_{s,i}, \beta_{p,i}$, and $\beta_{z,i}$ are independent of i, and we denote them by τ_p, β_s, β_p, and β_z. In particular, all investors have the same conditional variance $Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$. Thus, using the demand function and

the market clearing condition, we can show that

$$\tilde{p} = \int_0^1 E(\tilde{v}|\tilde{s}_i, \tilde{s}_{pz,i})di = E\left[E(\tilde{v}|\tilde{s}_i, \tilde{s}_{pz,i})\,|\tilde{v}, \tilde{u}\right].$$

Inserting the expression of $E(\tilde{v}|\tilde{s}_i, \tilde{s}_{pz,i})$ and comparing the coefficients of the conjectured price function (10), we have:

$$a_v = \beta_s + \beta_p \text{ and } a_u = \beta_p(\alpha + 1) + \beta_z.$$

Plugging the expressions of β's into the above two conditions leads to the expressions of a's in Proposition 1.

Inserting the expressions of a's into $\alpha = \frac{a_u}{a_v}$ and simplifying yield to the cubic (16) that determines the value of α. Denote the left-hand side of (16) by $f(\alpha)$. That is,

$$f(\alpha) \equiv (\tau_e\tau_\varepsilon + \tau_\varepsilon\tau_\eta)\alpha^3 + 2\tau_e\tau_\varepsilon\alpha^2 + (\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon)\alpha - \tau_e\tau_u.$$

We can compute $f(0) = -\tau_e\tau_u < 0$ and $f\left(\frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon}\right) > 0$, and thus by the intermediate value theorem, there exists a solution $\alpha \in \left(0, \frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon}\right)$ such that $f(\alpha) = 0$. This result establishes the existence of a financial market equilibrium.

We can compute the discriminant of the cubic $f(\alpha)$ as follows:

$$\Delta = -\tau_\varepsilon \begin{pmatrix} 4\tau_e^3\tau_\eta^4 + 4\tau_e^4\tau_\eta^3 + 4\tau_e\tau_u^3\tau_\varepsilon^3 + 4\tau_e^3\tau_u\tau_\varepsilon^3 + 4\tau_e^4\tau_u\tau_\varepsilon^2 + 27\tau_e^4\tau_u^2\tau_\varepsilon \\ +12\tau_e^3\tau_\varepsilon\tau_\eta^3 + 4\tau_e^3\tau_\varepsilon^3\tau_\eta + 8\tau_e^4\tau_\varepsilon\tau_\eta^2 + 4\tau_e^4\tau_\varepsilon^2\tau_\eta + 4\tau_u^3\tau_\varepsilon^3\tau_\eta + 8\tau_e^2\tau_u^2\tau_\varepsilon^3 \\ +36\tau_e^3\tau_u^2\tau_\varepsilon^2 + 12\tau_e^3\tau_\varepsilon^2\tau_\eta^2 + 12\tau_e\tau_u^2\tau_\varepsilon^2\tau_\eta^2 + 24\tau_e^2\tau_u\tau_\varepsilon^2\tau_\eta^2 + 27\tau_e^2\tau_u^2\tau_\varepsilon\tau_\eta^2 \\ +48\tau_e^2\tau_u^2\tau_\varepsilon^2\tau_\eta + 36\tau_e^4\tau_u\tau_\varepsilon\tau_\eta + 12\tau_e\tau_u^2\tau_\varepsilon^3\tau_\eta + 12\tau_e^2\tau_u\tau_\varepsilon\tau_\eta^3 + 12\tau_e^2\tau_u\tau_\varepsilon^3\tau_\eta \\ +48\tau_e^3\tau_u\tau_\varepsilon\tau_\eta^2 + 52\tau_e^3\tau_u\tau_\varepsilon^2\tau_\eta + 54\tau_e^3\tau_u^2\tau_\varepsilon\tau_\eta \end{pmatrix},$$

which is negative. Thus, there exists a unique real root, which establishes the uniqueness of a financial market equilibrium. QED.

Proof of Corollary 1

By Lemma 1, we have

$$\tau_p = \frac{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta} \propto \tau_\eta.$$

By the expressions of a_v and a_u in Proposition 1, we have

$$a_v \propto \frac{\tau_\varepsilon + \tau_\eta}{\tau_v + \tau_\varepsilon + \tau_\eta} \to 1 \text{ and } a_u = a_v\alpha \to 0.$$

QED.

Proof of Proposition 2

Part (a) Price informativeness By the proof for Proposition 1, we know that α is determined by $f(\alpha) = 0$. Using the implicit function theorem, we can compute:

$$\frac{\partial \alpha}{\partial \tau_\eta} = -\frac{\tau_\varepsilon \alpha^3 + \tau_e \alpha}{3(\tau_e \tau_\varepsilon + \tau_\varepsilon \tau_\eta)\alpha^2 + 4\tau_e \tau_\varepsilon \alpha + (\tau_e \tau_\varepsilon + \tau_e \tau_\eta + \tau_u \tau_\varepsilon)} < 0. \tag{A1}$$

Part (b) Return volatility Using Proposition 1, we can compute

$$\sigma(\tilde{V} - \tilde{p}) = \sqrt{Var(\tilde{v} - \tilde{p}) + \tau_\xi^{-1}},$$

where

$$Var(\tilde{v} - \tilde{p}) = \frac{\tau_v + \left[\frac{(1+\alpha)\tau_e + \alpha\tau_\eta}{\tau_u + (1+\alpha)^2\tau_e + \alpha^2\tau_\eta}\right]^2 \tau_u}{\left(\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta}{\tau_u + (1+\alpha)^2\tau_e + \alpha^2\tau_\eta}\right)^2}. \tag{A2}$$

Part (b1) simply follows combining Lemma 1 and the above expression of $Var(\tilde{v} - \tilde{p})$.

To show Part (b2) of Proposition 2, we first use equation (A2) to directly compute the derivative of $\frac{\partial \log Var(\tilde{v}-\tilde{p})}{\partial \tau_\eta}$, and then combine with Lemma 2 to show that both $\lim_{\tau_\varepsilon \to 0} \frac{\partial \log Var(\tilde{v}-\tilde{p})}{\partial \tau_\eta} < 0$ and $\lim_{\tau_\varepsilon \to \infty} \frac{\partial \log Var(\tilde{v}-\tilde{p})}{\partial \tau_\eta} < 0$. For instance, using Part (a) of Lemma 2, we can compute: $\lim_{\tau_\varepsilon \to 0} \frac{\partial \log Var(\tilde{v}-\tilde{p})}{\partial \tau_\eta} = -\frac{2\tau_\eta\left(\tau_u^2\tau_v + \tau_u\tau_\eta^2 + \tau_v\tau_\eta^2 + 3\tau_u\tau_v\tau_\eta\right)}{\left(\tau_u^2\tau_v + \tau_u\tau_\eta^2 + \tau_v\tau_\eta^2 + 2\tau_u\tau_v\tau_\eta\right)\left(\tau_\eta^2 + \tau_u\tau_v + \tau_v\tau_\eta\right)} < 0$.

Similarly, we can show that $\lim_{\tau_\varepsilon \to \infty} \frac{\partial \log Var(\tilde{v}-\tilde{p})}{\partial \tau_\eta} < 0$.

Part (c) Price momentum Direct computation shows $m = \frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})} - 1$. We can use Proposition 1 to compute:

$$\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})} = \frac{\left(\tau_\varepsilon + \frac{\tau_e\tau_u + (\tau_e+\tau_u)\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)\left(\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u + (\tau_e+\tau_u)\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)}{\left(\tau_\varepsilon + \frac{\tau_e\tau_u + (\tau_e+\tau_u)\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2 + \left(\frac{\tau_e + \alpha\tau_e + \alpha\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2 \tau_v\tau_u}. \tag{A3}$$

Part (c1): Using the above expression, we can show $\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})} > 1$, and so $m = \frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})} - 1 > 0$. Combining expression (A3) and Lemma 1 yields $\lim_{\tau_\eta \to \infty} m = 0$.

Part (c2): Note that $\frac{\partial m}{\partial \tau_\eta}$ and $\frac{\partial}{\partial \tau_\eta} \log\left[\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})}\right]$ have the same sign. So, let us examine $\frac{\partial}{\partial \tau_\eta} \log\left[\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})}\right]$.

Let $\tau_\varepsilon \to 0$. Using Part (a) of Lemma 2 and expression (A3), we can show:

$\frac{\partial}{\partial \tau_\eta} \log\left[\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})}\right] \propto -\left(\tau_e+\tau_u\right)\tau_\eta^4 - \tau_u\left(\tau_e+2\tau_u\right)\tau_\eta^3 + \tau_u\left(2\tau_u^2+\tau_e\tau_v-\tau_u\tau_v\right)\tau_\eta^2 + \tau_e\tau_u^2\tau_v\tau_\eta + 2\tau_u^4\tau_v$.

Thus, $\frac{\partial}{\partial \tau_\eta} \log\left[\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})}\right] < 0$ for large values of τ_η, and $\frac{\partial}{\partial \tau_\eta} \log\left[\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})}\right] > 0$ for small values of τ_η.

Now let $\tau_\varepsilon \to \infty$. Using Part (b) of Lemma 2 and expression (A3), we can show $\frac{\partial}{\partial \tau_\eta} \log\left[\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})}\right] \propto -\frac{\tau_v}{\tau_\varepsilon^2} < 0$. QED.

Proof of Proposition 3

Part (a) Risk By equation (20), $\frac{\partial Risk}{\partial \tau_\eta}$ and $\frac{\partial \tau_p}{\partial \tau_\eta}$ have opposite signs. Direct computation shows

$$\frac{\partial \tau_p}{\partial \tau_\eta} = \frac{\left(\tau_e+\tau_u+\alpha\tau_e\right)^2 - 2\left(\tau_e\tau_u+\tau_e\tau_\eta+\tau_u\tau_\eta\right)\left(\tau_e\left(\alpha+1\right)+\alpha\tau_\eta\right)\frac{\partial \alpha}{\partial \tau_\eta}}{\left(\tau_u+\tau_e\left(\alpha+1\right)^2+\alpha^2\tau_\eta\right)^2} > 0,$$

since $\frac{\partial \alpha}{\partial \tau_\eta} < 0$ by Part (a) of Proposition 2.

Using Lemma 1 and the expression of τ_p, we can show $\lim_{\tau_\eta \to \infty} \tau_p = \infty$, and thus, $\lim_{\tau_\eta \to \infty} Risk = \frac{1}{\tau_\xi}$.

Part (b) Disagreement Direct computation shows:

$$Disagreement = \frac{\sqrt{\tau_\varepsilon + \left(\frac{\tau_u-\alpha\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\right)^2\tau_e + \left(\frac{\tau_e+\tau_u+\alpha\tau_e}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\right)^2\tau_\eta}}{\tau_v+\tau_\varepsilon+\frac{\tau_e\tau_u+(\tau_e+\tau_u)\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}}. \tag{A4}$$

Part (b1) simply follows combining Lemma 1 and the above expression of *Disagreement*.

To prove part (b2), we first use (A4) to compute $\frac{\partial \log Disagreement}{\partial \tau_\varepsilon}$ and then combine with Lemma 2. Specifically:

As $\tau_\varepsilon \to 0$, we have

$$\frac{\partial \log Disagreement}{\partial \tau_\eta} \propto \frac{-\tau_\eta^2+3\tau_u\tau_v+\tau_v\tau_\eta}{2\tau_\eta\left(\tau_\eta^2+\tau_u\tau_v+\tau_v\tau_\eta\right)} \Rightarrow$$

$$\frac{\partial \log Disagreement}{\partial \tau_\eta} > 0 \iff \tau_\eta^2 - \tau_v\tau_\eta - 3\tau_u\tau_v < 0$$

$$\iff \tau_\eta < \frac{\tau_v+\sqrt{\tau_v^2+12\tau_u\tau_v}}{2}.$$

As $\tau_\varepsilon \to \infty$, we have

$$\frac{\partial \log Disagreement}{\partial \tau_\eta} \propto -\frac{1}{2\tau_\varepsilon} < 0.$$

Part (c) Trading volume Direct computation shows

$$Volume = \frac{1}{\gamma}\sqrt{\frac{2}{\pi}}\frac{\sqrt{\tau_\varepsilon + \left(\frac{\tau_u - \alpha\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2 \tau_e + \left(\frac{\tau_e + \tau_u + \alpha\tau_e}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2 \tau_\eta}}{1 + \frac{1}{\tau_\xi}\left(\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u + (\tau_e + \tau_u)\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)}. \tag{A5}$$

Part (c1) simply follows combining Lemma 1 and the above expression of $Volume$.

To prove part (c2), we first use (A5) to compute $\frac{\partial \log Volume}{\partial \tau_\varepsilon}$ and then combine with Lemma 2. Specifically:

As $\tau_\varepsilon \to 0$, we have

$$\begin{aligned}
\frac{\partial \log Volume}{\partial \tau_\eta} &\propto \frac{-\tau_\eta^2 + (\tau_v + \tau_\xi)\tau_\eta + 3\tau_u(\tau_v + \tau_\xi)}{2\tau_\eta\left(\tau_\eta^2 + \tau_u\tau_v + \tau_u\tau_\xi + \tau_v\tau_\eta + \tau_\xi\tau_\eta\right)} \Rightarrow \\
\frac{\partial \log Volume}{\partial \tau_\eta} &> 0 \Longleftrightarrow \tau_\eta^2 - (\tau_v + \tau_\xi)\tau_\eta - 3\tau_u(\tau_v + \tau_\xi) > 0 \\
&\Longleftrightarrow \tau_\eta < \frac{(\tau_v + \tau_\xi) + \sqrt{(\tau_v + \tau_\xi)^2 + 12\tau_u(\tau_v + \tau_\xi)}}{2}.
\end{aligned}$$

As $\tau_\varepsilon \to \infty$, we have

$$\frac{\partial \log Volume}{\partial \tau_\eta} \propto -\frac{1}{2\tau_\varepsilon} < 0.$$

QED.

Proof of Proposition 4

Using demand function (7) and the expression of investor i's date-0 payoff function in equation (24), we can compute:

$$W\left(\tau_{\eta_i}; \tau_\eta\right) = \frac{1}{2\gamma}\ln\left[\begin{array}{c}\left(1 + \frac{Cov\left(\tilde{V} - \tilde{p}, E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i) - \tilde{p}\right)}{Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)}\right)^2 \\ - \frac{Var\left(\tilde{V} - \tilde{p}\right) Var\left(E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i) - \tilde{p}\right)}{Var^2(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)}\end{array}\right] - C\left(\tau_{\eta_i}\right),$$

where

$$\begin{aligned}
Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i) &= (\tau_v + \tau_\varepsilon + \tau_{p,i})^{-1} + \tau_\xi^{-1}, \\
Var(\tilde{V} - \tilde{p}) &= (1 - a_v)^2\tau_v^{-1} + a_u^2\tau_u^{-1} + \tau_\xi^{-1},
\end{aligned}$$

$$
\begin{aligned}
& Var\left(E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-\tilde{p}\right) \\
= \;& \left[\left(\beta_{s,i}+\beta_{p,i}\right)-a_v\right]^2\tau_v^{-1}+\left(\left[\beta_{p,i}\left(\alpha+1\right)+\beta_{z,i}\right]-a_u\right)^2\tau_u^{-1} \\
& +\beta_{s,i}^2\tau_\varepsilon^{-1}+\left(\beta_{p,i}+\beta_{z,i}\right)^2\tau_e^{-1}+\beta_{z,i}^2\tau_{\eta_i}^{-1}, \\
& Cov\left(\tilde{V}-\tilde{p},E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-\tilde{p}\right) \\
= \;& \left(1-a_v\right)\left[\left(\beta_{s,i}+\beta_{p,i}\right)-a_v\right]\tau_v^{-1}-a_u\left(\left[\beta_{p,i}\left(\alpha+1\right)+\beta_{z,i}\right]-a_u\right)\tau_u^{-1}.
\end{aligned}
$$

Let $\tau_u\to\infty$ and $\tau_\xi\to\infty$. Using Lemma 3 and the above equations, we can compute:

$$
\lim_{\tau_u\to\infty,\tau_\xi\to\infty}W\left(\tau_{\eta_i};\tau_\eta\right)=\frac{1}{2\gamma}\ln\left[\frac{\tau_v\tau_{\eta_i}+\left(\tau_e+\tau_\varepsilon+\tau_\eta\right)^2+\tau_v\left(\tau_e+\tau_v+\tau_\varepsilon\right)}{\left(\tau_e+\tau_v+\tau_\varepsilon+\tau_\eta\right)^2}\right]-C\left(\tau_{\eta_i}\right).
$$

Denote the right-hand side of the above equation by $W_\infty\left(\tau_{\eta_i};\tau_\eta\right)$. When $C\left(\tau_{\eta_i}\right)$ is smooth, increasing, and weakly convex, $W_\infty\left(\tau_{\eta_i};\tau_\eta\right)$ is smooth and strictly concave in τ_{η_i}. Thus, for any given τ_η, there exists a unique $\tau_{\eta_i}^*$ that maximizes $W_\infty\left(\tau_{\eta_i};\tau_\eta\right)$.

Direct computations show

$$
\phi\left(\tau_\eta\right)\equiv\frac{\partial W_\infty\left(\tau_\eta;\tau_\eta\right)}{\partial\tau_{\eta_i}}=\frac{1}{2\gamma}\frac{\tau_v}{\tau_v\tau_\eta+\left(\tau_e+\tau_\varepsilon+\tau_\eta\right)^2+\tau_v\left(\tau_e+\tau_v+\tau_\varepsilon\right)}-C'\left(\tau_\eta\right).
$$

Given that $C\left(\tau_{\eta_i}\right)$ is weakly convex, we know that $\phi\left(\tau_\eta\right)$ is strictly decreasing in τ_η and that $\lim_{\tau_\eta\to\infty}\phi\left(\tau_\eta\right)<0$. Thus, if $\phi\left(0\right)\leq 0$, then $\tau_\eta^*=0$ is the unique equilibrium sophistication level. If $\phi\left(0\right)>0$, then there exists a unique $\tau_\eta^*\in(0,\infty)$ satisfying $\phi\left(\tau_\eta^*\right)=0$, which determines the unique equilibrium sophistication level. QED.

Proof of Proposition 5

Part (a) Let us define

$$
\Phi\left(\tau_\eta\right)\equiv\phi\left(\tau_\eta\right)+C'\left(\tau_\eta\right).
$$

Now suppose $\tau_\varepsilon\to 0$. Using Part (a) of Lemma 2, after massive algebra, we can compute

$$
\Phi\left(\tau_\eta\right)\propto\frac{2\frac{\partial\sigma_{rd}}{\partial\tau_\eta}-\sigma_r^2\frac{\partial\sigma_d^2}{\partial\tau_\eta}}{2\gamma\left(1-\sigma_r^2\sigma_d^2\right)},
$$

where $\frac{\partial \sigma_{rd}}{\partial \tau_\eta} = \frac{\tau_\eta^2 \frac{\tau_e \tau_u^2 \tau_v + \tau_e \tau_u \tau_\eta^2 + \tau_e \tau_v \tau_\eta^2 + \tau_u \tau_v \tau_\eta^2 + 2\tau_e \tau_u \tau_v \tau_\eta}{\left(\tau_\eta^2 + \tau_u \tau_v + \tau_v \tau_\eta\right)^3 \left(\tau_e \tau_u + \tau_e \tau_\eta + \tau_u \tau_\eta\right)}}{\left(\tau_v + \frac{\tau_\eta^2}{\tau_u + \tau_\eta}\right)^{-1} + \tau_\xi^{-1}}$,

$\sigma_r^2 = \frac{\tau_u^2 \tau_v + \tau_u \tau_\eta^2 + \tau_v \tau_\eta^2 + 2\tau_u \tau_v \tau_\eta}{\left(\tau_\eta^2 + \tau_u \tau_v + \tau_v \tau_\eta\right)^2} + \frac{1}{\tau_\xi}$,

$$\frac{\partial \sigma_d^2}{\partial \tau_\eta} = \frac{\begin{bmatrix} -\frac{\tau_\eta^2}{\left(\tau_\eta^2 + \tau_u \tau_v + \tau_v \tau_\eta\right)^2} \left(\left(\tau_v + \frac{\tau_\eta^2}{\tau_u + \tau_\eta}\right)^{-1} + \tau_\xi^{-1} \right) \\ -2 \frac{\tau_\eta^3}{\left(\tau_\eta^2 + \tau_u \tau_v + \tau_v \tau_\eta\right)^2} \left(\tau_v + \frac{\tau_\eta^2}{\tau_u + \tau_\eta}\right)^{-2} \left(\frac{\tau_\eta}{\tau_u + \tau_\eta}\right)^2 \end{bmatrix}}{\left(\left(\tau_v + \frac{\tau_\eta^2}{\tau_u + \tau_\eta}\right)^{-1} + \tau_\xi^{-1} \right)^3},$$

and $\sigma_d^2 \propto \frac{\frac{\tau_\eta^3}{\left(\tau_\eta^2 + \tau_u \tau_v + \tau_v \tau_\eta\right)^2}}{\left(\left(\tau_v + \frac{\tau_\eta^2}{\tau_u + \tau_\eta}\right)^{-1} + \tau_\xi^{-1} \right)^2}$.

Setting $\tau_\xi^{-1} = 0$ and computing, we can show that the sign of $\Phi'(\tau_\eta)$ is the same as the sign of the following 16th order polynomial of τ_η:

$$Q(\tau_\eta) = q_{16}\tau_\eta^{16} + q_{15}\tau_\eta^{15} + q_{14}\tau_\eta^{14} + ... + q_2\tau_\eta^2 + q_1\tau_\eta + q_0,$$

where the coefficients are:

$$q_{16} = -2(\tau_e + \tau_u)(3\tau_u^2 + 5\tau_e\tau_u + 5\tau_e\tau_v + 5\tau_u\tau_v),$$

$$q_{15} = -\begin{pmatrix} 9\tau_u^4 + 39\tau_e^2\tau_u^2 + 19\tau_e^2\tau_v^2 + 19\tau_u^2\tau_v^2 + 46\tau_e\tau_u^3 \\ +52\tau_u^3\tau_v + 38\tau_e\tau_u\tau_v^2 + 140\tau_e\tau_u^2\tau_v + 88\tau_e^2\tau_u\tau_v \end{pmatrix},$$

$$q_{14} = -2\begin{pmatrix} 2\tau_u^5 + 30\tau_e^2\tau_u^3 + 6\tau_e^2\tau_v^3 + 6\tau_u^2\tau_v^3 + 50\tau_u^3\tau_v^2 + 25\tau_e\tau_u^4 + 44\tau_u^4\tau_v \\ +12\tau_e\tau_u\tau_v^3 + 180\tau_e\tau_u^3\tau_v + 131\tau_e\tau_u^2\tau_v^2 + 81\tau_e^2\tau_u\tau_v^2 + 145\tau_e^2\tau_u^2\tau_v \end{pmatrix},$$

$$q_{13} = \tau_u^6 - 44\tau_e^2\tau_u^4 - \tau_e^2\tau_v^4 - \tau_u^2\tau_v^4 - 50\tau_u^3\tau_v^3 - 180\tau_u^4\tau_v^2 - 24\tau_e\tau_u^5 - 52\tau_u^5\tau_v - 2$$
$$\tau_e\tau_u\tau_v^4 - 438\tau_e\tau_u^4\tau_v - 142\tau_e\tau_u^2\tau_v^3 - 702\tau_e\tau_u^3\tau_v^2 - 92\tau_e^2\tau_u\tau_v^3 - 474\tau_e^2\tau_u^3\tau_v - 551\tau_e^2\tau_u^2\tau_v^2,$$

$$q_{12} = -2\tau_u\begin{pmatrix} 7\tau_e^2\tau_u^4 - 5\tau_e^2\tau_v^4 - 10\tau_u^2\tau_v^4 + 8\tau_u^3\tau_v^3 + 44\tau_u^4\tau_v^2 + 3\tau_e\tau_u^5 \\ -4\tau_u^5\tau_v - 15\tau_e\tau_u\tau_v^4 + 124\tau_e\tau_u^4\tau_v + 126\tau_e\tau_u^2\tau_v^3 \\ +440\tau_e\tau_u^3\tau_v^2 + 126\tau_e^2\tau_u\tau_v^3 + 198\tau_e^2\tau_u^3\tau_v + 468\tau_e^2\tau_u^2\tau_v^2 \end{pmatrix},$$

$$q_{11} = -\tau_u\begin{pmatrix} \tau_e^2\tau_u^5 - 6\tau_e^2\tau_v^5 - 8\tau_u^2\tau_v^5 - 178\tau_u^3\tau_v^4 - 260\tau_u^4\tau_v^3 \\ -109\tau_u^5\tau_v^2 + 2\tau_e\tau_u^6 - 16\tau_u^6\tau_v - 14\tau_e\tau_u\tau_v^5 \\ +44\tau_e\tau_u^5\tau_v - 338\tau_e\tau_u^2\tau_v^4 - 154\tau_e\tau_u^3\tau_v^3 + 398\tau_e\tau_u^4\tau_v^2 \\ -159\tau_e^2\tau_u\tau_v^4 + 128\tau_e^2\tau_u^4\tau_v + 160\tau_e^2\tau_u^2\tau_v^3 + 731\tau_e^2\tau_u^3\tau_v^2 \end{pmatrix},$$

$$q_{10} = 2\tau_u^2\tau_v\begin{pmatrix} \tau_u^6 + 17\tau_e^2\tau_u^4 + 33\tau_e^2\tau_v^4 + 33\tau_u^2\tau_v^4 + 300\tau_u^3\tau_v^3 + 316\tau_u^4\tau_v^2 \\ +2\tau_e\tau_u^5 + 78\tau_u^5\tau_v + 66\tau_e\tau_u\tau_v^4 + 105\tau_e\tau_u^4\tau_v + 690\tau_e\tau_u^2\tau_v^3 \\ +644\tau_e\tau_u^3\tau_v^2 + 408\tau_e^2\tau_u\tau_v^3 + 23\tau_e^2\tau_u^3\tau_v + 364\tau_e^2\tau_u^2\tau_v^2 \end{pmatrix},$$

$$q_9 = \tau_u^3\tau_v \begin{pmatrix} 34\tau_e^2\tau_u^4 + 330\tau_e^2\tau_v^4 + 240\tau_u^2\tau_v^4 + 1108\tau_u^3\tau_v^3 + 686\tau_u^4\tau_v^2 \\ -2\tau_e\tau_u^5 + 66\tau_u^5\tau_v + 558\tau_e\tau_u\tau_v^4 + 310\tau_e\tau_u^4\tau_v + 3136\tau_e\tau_u^2\tau_v^3 \\ +2226\tau_e\tau_u^3\tau_v^2 + 2334\tau_e^2\tau_u\tau_v^3 + 583\tau_e^2\tau_u^3\tau_v + 2184\tau_e^2\tau_u^2\tau_v^2 \end{pmatrix},$$

$$q_8 = 2\tau_u^4\tau_v \begin{pmatrix} 3\tau_e^2\tau_u^4 + 495\tau_e^2\tau_v^4 + 252\tau_u^2\tau_v^4 + 622\tau_u^3\tau_v^3 + 196\tau_u^4\tau_v^2 \\ +4\tau_u^5\tau_v + 696\tau_e\tau_u\tau_v^4 + 58\tau_e\tau_u^4\tau_v + 2240\tau_e\tau_u^2\tau_v^3 + 994\tau_e\tau_u^3\tau_v^2 \\ +2142\tau_e^2\tau_u\tau_v^3 + 242\tau_e^2\tau_u^3\tau_v + 1484\tau_e^2\tau_u^2\tau_v^2 \end{pmatrix},$$

$$q_7 = 2\tau_u^5\tau_v^2 \begin{pmatrix} 87\tau_e^2\tau_u^3 + 990\tau_e^2\tau_v^3 + 336\tau_u^2\tau_v^3 + 435\tau_u^3\tau_v^2 + 6\tau_e\tau_u^4 \\ +56\tau_u^4\tau_v + 1134\tau_e\tau_u\tau_v^3 + 499\tau_e\tau_u^3\tau_v + 2100\tau_e\tau_u^2\tau_v^2 \\ +2667\tau_e^2\tau_u\tau_v^2 + 1184\tau_e^2\tau_u^2\tau_v \end{pmatrix},$$

$$q_6 = 4\tau_u^6\tau_v^2 \begin{pmatrix} 6\tau_e^2\tau_u^3 + 693\tau_e^2\tau_v^3 + 147\tau_u^2\tau_v^3 + 92\tau_u^3\tau_v^2 + 3\tau_u^4\tau_v + 630\tau_e\tau_u\tau_v^3 \\ +66\tau_e\tau_u^3\tau_v + 647\tau_e\tau_u^2\tau_v^2 + 1146\tau_e^2\tau_u\tau_v^2 + 285\tau_e^2\tau_u^2\tau_v \end{pmatrix},$$

$$q_5 = \tau_u^7\tau_v^3 \begin{pmatrix} 308\tau_e^2\tau_u^2 + 2772\tau_e^2\tau_v^2 + 336\tau_u^2\tau_v^2 + 28\tau_e\tau_u^3 \\ +85\tau_u^3\tau_v + 1932\tau_e\tau_u\tau_v^2 + 1010\tau_e\tau_u^2\tau_v + 2691\tau_e^2\tau_u\tau_v \end{pmatrix},$$

$$q_4 = 2\tau_u^8\tau_v^3\left(18\tau_e^2\tau_u^2 + 990\tau_e^2\tau_v^2 + 60\tau_u^2\tau_v^2 + 4\tau_u^3\tau_v + 504\tau_e\tau_u\tau_v^2 + 113\tau_e\tau_u^2\tau_v + 517\tau_e^2\tau_u\tau_v\right),$$

$$q_3 = \tau_u^9\tau_v^4\left(22\tau_e\tau_u^2 + 235\tau_e^2\tau_u + 990\tau_e^2\tau_v + 24\tau_u^2\tau_v + 342\tau_e\tau_u\tau_v\right),$$

$$q_2 = 2\tau_u^{10}\tau_v^4\left(12\tau_e^2\tau_u + 165\tau_e^2\tau_v + \tau_u^2\tau_v + 34\tau_e\tau_u\tau_v\right),$$

$$q_1 = 6\tau_e\tau_u^{11}\tau_v^5\left(11\tau_e + \tau_u\right),$$

and $q_0 = 6\tau_e^2\tau_u^{12}\tau_v^5$.

Since $q_{16} < 0$, we have $\Phi'(\tau_\eta) < 0$ for large values of τ_η. Note that $C(\cdot)$ is weakly convex, and thus $\phi'(\tau_\eta) = \Phi'(\tau_\eta) - C''(\tau_\eta) < 0$ for large values of τ_η. Similarly, the condition of $q_0 > 0$ implies that $\Phi'(\tau_\eta) > 0$ for small values of τ_η. If $C''(0)$ is sufficiently small, then $\phi'(0) = \Phi'(0) - C''(0) > 0$, and thus $\phi'(\tau_\eta) > 0$ for sufficiently small values of τ_η.

In the main text, we have already constructed a numerical example to show the possibility of multiple equilibrium levels of sophistication. This completes the proof of Part (a) of Proposition 5.

Part (b) Suppose $\tau_\varepsilon \to \infty$. Also set $\tau_\xi^{-1} = 0$. Similar to the proof of Part (a), we use Part (b) of Lemma 2, and after massive algebra, we can show that

$$\Phi(\tau_\eta) \propto \frac{\tau_e+\tau_u}{2\gamma\left(\tau_e^2\tau_u+2\tau_e\tau_u^2+\tau_u^3\right)\tau_\varepsilon^2} \times\frac{3\tau_u\left(\tau_e+\tau_u\right)^2\alpha^2\tau_\varepsilon^3+2\tau_e\tau_u\left(\tau_e+\tau_u\right)\alpha\tau_\varepsilon^2+5\tau_u\tau_v\left(\tau_e+\tau_u\right)^2\tau_\varepsilon}{\left(\tau_e+\tau_u\right)\tau_\varepsilon+\tau_e\tau_u+\tau_e\tau_v+\tau_u\tau_v+\tau_e\tau_\eta+\tau_u\tau_\eta}.$$

Therefore, the sign of $\Phi'(\tau_\eta)$ is determined by the sign of

$$\frac{3\tau_\varepsilon^3\tau_u\left(\tau_e+\tau_u\right)^2 2\alpha+2\tau_\varepsilon^2\tau_e\tau_u\left(\tau_e+\tau_u\right)}{3\tau_u\left(\tau_e+\tau_u\right)^2\alpha^2\tau_\varepsilon^3+2\tau_e\tau_u\left(\tau_e+\tau_u\right)\alpha\tau_\varepsilon^2+5\tau_u\tau_v\left(\tau_e+\tau_u\right)^2\tau_\varepsilon}\frac{\partial\alpha}{\partial\tau_\eta} -\frac{\tau_e+\tau_u}{\left(\tau_e+\tau_u\right)\tau_\varepsilon+\tau_e\tau_u+\tau_e\tau_v+\tau_u\tau_v+\tau_e\tau_\eta+\tau_u\tau_\eta}.$$

Note that by Part (b) of Lemma 2, we have $\frac{\partial\alpha}{\partial\tau_\eta}\propto-\frac{\tau_e\alpha}{(\tau_e+\tau_u)\tau_\varepsilon}<0$ as $\tau_\varepsilon\to\infty$. Thus, $\Phi'(\tau_\eta)<0$, which implies $\phi'(\tau_\eta)=\Phi'(\tau_\eta)-C''(\tau_\eta)<0$ given that $C(\cdot)$ is weakly convex. This in turn means that the equilibrium level τ_η^* of sophistication is unique in economies with large values of τ_ε. QED.

Proof of Proposition 6

When τ_{η_i} is identical across investors, we have

$$\begin{aligned} & E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-\tilde{p}=\beta_{s,i}\tilde{\varepsilon}_i+\left(\beta_{p,i}+\beta_{z,i}\right)\tilde{e}_i+\beta_{z,i}\tilde{\eta}_i\Rightarrow \\ & Cov(\tilde{V}-\tilde{p},D_i) \\ = \;& Cov\left(\tilde{V}-\tilde{p},\frac{E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-\tilde{p}}{\gamma Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)}\right) \\ = \;& \frac{Cov\left(\tilde{\xi}+(1-a_v)\tilde{v}-a_u\tilde{u},\beta_{s,i}\tilde{\varepsilon}_i+\left(\beta_{p,i}+\beta_{z,i}\right)\tilde{e}_i+\beta_{z,i}\tilde{\eta}_i\right)}{\gamma Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)} \\ = \;& 0. \end{aligned}$$

The expression of $W\left(\tau_\eta^*;\tau_\eta^*\right)$ in (31) follows from inserting $Cov(\tilde{V}-\tilde{p},D_i)=0$ into equation (24). QED.

Appendix C: Two Types of Investors

In this appendix, we consider a variation of the baseline model and show that our main results are robust.

Setup In the date-1 financial market, investors are still endowed with a private fundamental signal $\tilde{s}_i$, and their initial interpretation about the price is still represented by signal $\tilde{s}_{p,i}$. Now we divide investors into two groups: sophisticated (with an endogenous measure μ) and unsophisticated (with an endogenous measure $1-\mu$). Sophisticated investors can completely purge out the receiver noise $\tilde{x}_i$ and thus have access to the best price signal $\tilde{s}_p$. Unsophisticated investors still keep interpreting price information as $\tilde{s}_{p,i}$. That is, sophisticated investors end up with an infinite sophistication level, while unsophisticated investors end up with a zero sophistication level (i.e., $\tau_{\eta_i} = \infty$ for $i \in [0, \mu]$, and $\tau_{\eta_i} = 0$ for $i \in (\mu, 1]$). Whether an investor is sophisticated or not is determined by her self 0 at date 0. Becoming sophisticated incurs a fixed cost $c > 0$. All the other features of the model remain unchanged. This two-type setting matches well the original DSSW (1990) setup with noise traders (unsophisticated investors) and arbitrageurs (unsophisticated investors).

Financial market equilibrium at date 1 The price function in the date-1 financial market is still given by equation (10). The CARA-normal setting implies that a sophisticated investor's demand for the risky asset is

$$D_S\left(\tilde{p}; \tilde{s}_i, \tilde{s}_p\right) = \frac{E(\tilde{V}|\tilde{s}_i, \tilde{s}_p) - \tilde{p}}{\gamma Var(\tilde{V}|\tilde{s}_i, \tilde{s}_p)},$$

where by Bayes' rule, we have

$$\begin{aligned} E(\tilde{V}|\tilde{s}_i, \tilde{s}_p) &= \beta_{s,S}\tilde{s}_i + \beta_{p,S}\tilde{s}_p, \\ Var(\tilde{V}|\tilde{s}_i, \tilde{s}_p) &= \frac{1}{\tau_v + \tau_\varepsilon + \alpha^{-2}\tau_u} + \frac{1}{\tau_\xi}, \end{aligned}$$

with

$$\beta_{s,S} = \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon + \alpha^{-2}\tau_u} \text{ and } \beta_{p,S} = \frac{\alpha^{-2}\tau_u}{\tau_v + \tau_\varepsilon + \alpha^{-2}\tau_u}.$$

Similarly, we can compute the demand for the risky asset of an unsophisticated investor as follows:

$$D_U\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}\right) = \frac{E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}) - \tilde{p}}{\gamma Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i})},$$

where

$$\begin{aligned} E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}) &= \beta_{s,U}\tilde{s}_i + \beta_{p,U}\tilde{s}_{p,i}, \\ Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}) &= \frac{1}{\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u}{\tau_u+\tau_e(\alpha+1)^2}} + \frac{1}{\tau_\xi}, \end{aligned}$$

with

$$\beta_{s,U} = \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u}{\tau_u+\tau_e(\alpha+1)^2}} \text{ and } \beta_{p,U} = \frac{\frac{\tau_e\tau_u}{\tau_u+\tau_e(\alpha+1)^2}}{\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u}{\tau_u+\tau_e(\alpha+1)^2}}.$$

Inserting the above demand functions into the following market clearing condition,

$$\int_0^{\mu} D_S\left(\tilde{p}; \tilde{s}_i, \tilde{s}_p\right) di + \int_0^{1-\mu} D_U\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}\right) di = 0,$$

we can compute the implied price function, which is in turn compared with the conjectured price function, yielding the following fifth-order polynomial of α that determines the financial market equilibrium:

$$F\left(\alpha; \mu\right) = A_5\alpha^5 + A_4\alpha^4 + A_3\alpha^3 + A_2\alpha^2 + A_1\alpha + A_0 = 0,$$

where

$$\begin{aligned} A_5 &= -\tau_e\tau_\varepsilon\left(\tau_v + \tau_\xi + \tau_\varepsilon\right), A_4 = -2\tau_e\tau_\varepsilon\left(\tau_v + \tau_\xi + \tau_\varepsilon\right), \\ A_3 &= -\tau_\varepsilon\left(\tau_e\tau_u + \tau_e\tau_v + \tau_u\tau_v + \tau_e\tau_\xi + \tau_e\tau_\varepsilon + \tau_u\tau_\xi + \tau_u\tau_\varepsilon\right), \\ A_2 &= \left(1-\mu\right)\tau_e\tau_u\left(\tau_v + \tau_\xi - \tau_\varepsilon\right), \\ A_1 &= -\left(1-\mu\right)\tau_u\tau_\varepsilon\left(\tau_e + \tau_u\right), \text{ and } A_0 = \left(1-\mu\right)\tau_e\tau_u^2. \end{aligned}$$

It is clear that there always exists a solution to $F\left(\alpha; \mu\right) = 0$, and hence existence is established. After we compute α, the price coefficients are as follows:

$$\begin{aligned} a_v &= \frac{\mu\frac{\beta_{s,S}+\beta_{p,S}}{Var(\tilde{V}|\tilde{s}_i,\tilde{s}_p)} + \left(1-\mu\right)\frac{\beta_{s,U}+\beta_{p,U}}{Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i})}}{\frac{\mu}{Var(\tilde{V}|\tilde{s}_i,\tilde{s}_p)} + \frac{1-\mu}{Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i})}}, \\ a_u &= \frac{\mu\frac{\beta_{p,S}\alpha}{Var(\tilde{V}|\tilde{s}_i,\tilde{s}_p)} + \left(1-\mu\right)\frac{\beta_{p,U}(\alpha+1)}{Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i})}}{\frac{\mu}{Var(\tilde{V}|\tilde{s}_i,\tilde{s}_p)} + \frac{1-\mu}{Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i})}}. \end{aligned}$$

Sophistication level equilibrium at date 0 At date 0, self 0 of each investor considers whether to spend cost c to achieve full sophistication in the future financial market. Again, self 0 is fully rational in making this decision, and we can view c as a commitment cost to ensure that the future self completely removes bounded rationality. Consider a representative investor i.

Suppose that she rationally expects that a fraction μ of investors will choose to achieve full sophistication. Let $W_S(\mu)$ and $W_U(\mu)$ respectively denote the ex-ante expected utilities of a sophisticated self 1 and an unsophisticated self 1. We need to compare $W_S(\mu)$ and $W_U(\mu)$ to determine investor i's choice of sophistication at date 0.

Direct computation shows

$$\begin{aligned} W_S(\mu) &\equiv -\frac{1}{\gamma}\ln\left(-E\left[-e^{-\gamma\left[(\tilde{V}-\tilde{p})D_S-c\right]}\right]\right) \\ &= \frac{1}{2\gamma}\ln\left[[1+\gamma Cov(\tilde{V}-\tilde{p},D_S)]^2-\gamma^2 Var(\tilde{V}-\tilde{p})\times Var(D_S)\right]-c \\ &= \frac{1}{2\gamma}\ln\left[\frac{Var(\tilde{V}-\tilde{p})}{Var(\tilde{V}|\tilde{s}_i,\tilde{s}_p)}\right]-c, \\ W_U(\mu) &\equiv -\frac{1}{\gamma}\ln\left(-E\left[-e^{-\gamma(\tilde{V}-\tilde{p})D_U}\right]\right) \\ &= \frac{1}{2\gamma}\ln\left[[1+\gamma Cov(\tilde{V}-\tilde{p},D_U)]^2-\gamma^2 Var(\tilde{V}-\tilde{p})\times Var(D_U)\right]. \end{aligned}$$

Thus, the net benefit of becoming sophisticated is

$$\Phi(\mu)\equiv W_S(\mu)-W_U(\mu).$$

In this setup, the fraction μ of sophisticated investors serves the same role as the sophistication level τ_η in the baseline model. Function $\Phi(\mu)$ corresponds to function $\phi(\tau_\eta)$ in Section 5. The equilibrium fraction μ^* is defined by the following three conditions:

(a) If $\Phi(0)\leq 0$, then $\mu^*=0$ is an equilibrium fraction of sophisticated investors;

(b) If $\Phi(1)\geq 0$, then $\mu^*=1$ is an equilibrium fraction of sophisticated investors; and

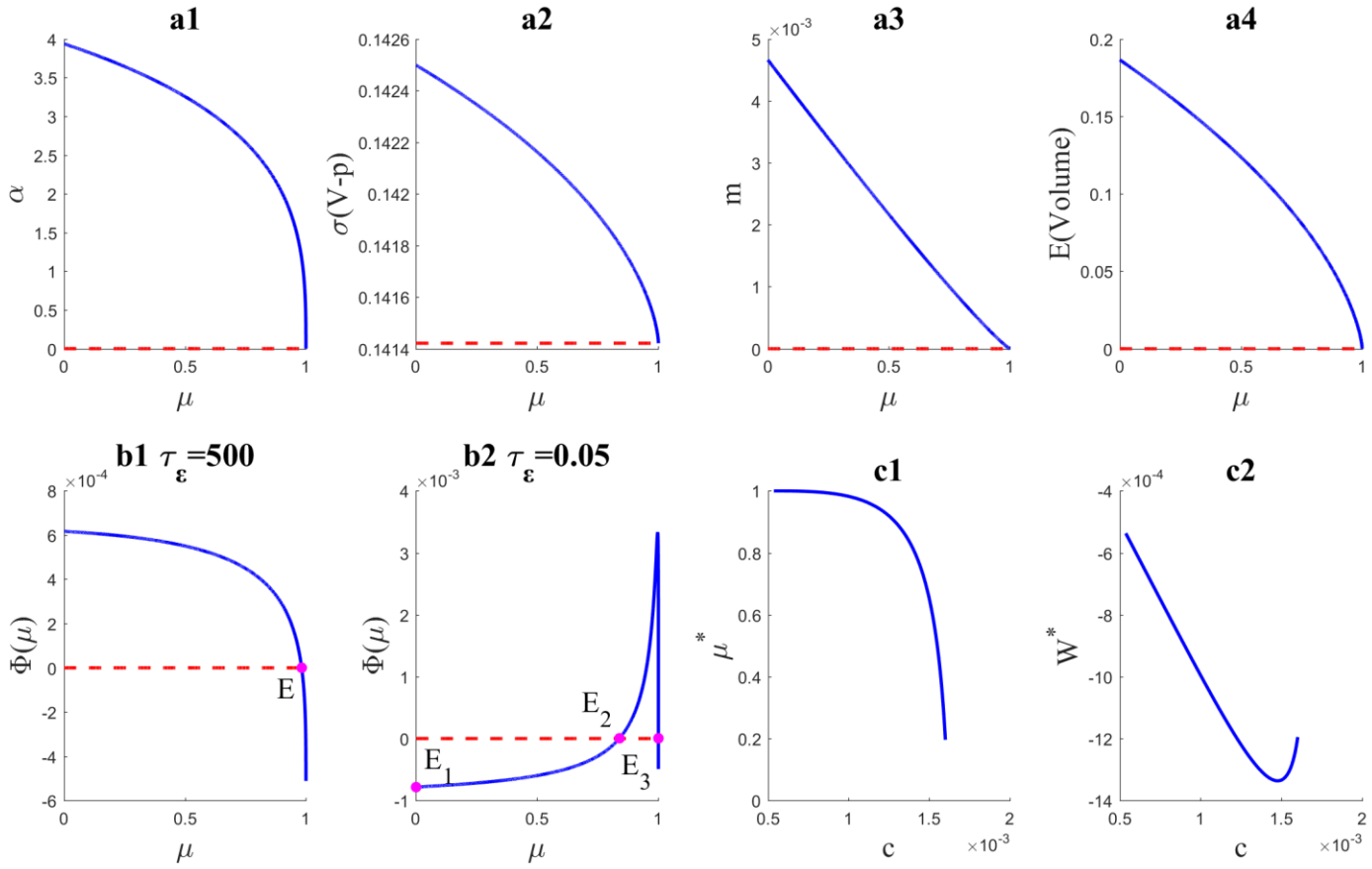
(c) If $\Phi(\mu^*)=0$ for some $\mu^*\in(0,1)$, then μ^* constitutes an interior equilibrium fraction of sophisticated investors.

Results We now show that our main qualitative results continue to hold in this variant model. In Panels a1–a4 of Figure C1, we plot price informativeness measure α, return volatility $\sigma(\tilde{V}-\tilde{p})$, price momentum $m=\frac{Cov\left(\tilde{V}-\tilde{p},\tilde{p}\right)}{Var(\tilde{p})}$, average aggregate trading volume $E(Volume)$ against the fraction μ of sophisticated investors. This exercise corresponds to Figures 1 and 2 in the baseline model.

In Panels b1 and b2 of Figure C1, we plot the benefit of acquiring sophistication, $\Phi(\mu)$. This corresponds to Panels b1–b2 of Figure 3. In Panels c1 and c2 of Figure C1, we plot the equilibrium fraction μ^* of sophisticated investors and the equilibrium investor welfare W^* against the cost c of acquiring sophistication, which respectively correspond to Panels a and b of Figure 4. In Panels a1–a4 and Panels c1–c2 of Figure C1, we set $\tau_v = \tau_\xi = 50, \tau_\varepsilon = 500, \tau_e = \tau_u = 50000$, and $\gamma = 2$. In Panels b1 and b2 of Figure C1, we $\tau_v = \tau_\xi = 50, \tau_e = \tau_u = 50000$, $\gamma = 2$, and $c = 10^{-3}$.

We make the following observations that confirm the main results in our baseline model. First, in Panels a1–a4 of Figure C1, compared to the standard REE (with $\mu = 1$), costly price interpretation injects noise into the price (i.e., $\alpha > 0$ for $\mu < 1$), generates excess return volatility and trading volume (i.e., $\sigma(\tilde{V} - \tilde{p}) > \sqrt{\frac{1}{\tau_\xi}}$ and $E(Volume) > 0$ for $\mu < 1$), and leads to price momentum (i.e., $m > 0$ for $\mu < 1$). Second, there can exhibit strategic complementarity in sophistication acquisition, leading to multiple equilibrium values of μ^*. Specifically, in Panel b2 of Figure C1, we see that $\Phi(\mu)$ can be upward sloping for a certain range of μ, and there are three equilibrium values of μ^*: $0, 0.84$, and 0.99. Third, in Panel c2 of Figure C1, equilibrium welfare W^* first decreases and then increases with c, which is similar to Panel b of Figure 4.

Figure C1: The Economy with Two Types of Investors



Panels a1 – a4 plot price informativeness (negatively measured by α), return volatility, price momentum, and trading volume against the fraction μ of sophisticated investors. The dashed lines plot the values in a standard REE economy (i.e., μ =1). Panels b1 and b2 plot the net benefit of sophistication acquisition. Panels c1 and c2 plot the equilibrium mass μ^* and welfare W^* against the cost c of sophistication acquisition. The parameter values are: $\tau_v = \tau_\xi = 50$, $\tau_e = \tau_u = 50000$, and $\gamma = 2$. Except Panel b2, we also set $\tau_\varepsilon = 500$ in all other panels. In Panels b1 and b2, the sophistication acquisition cost is $c = 10^{-3}$.