# COORDINATED AUTONOMY: PURSUIT FORMATIONS OF MULTIVEHICLE SYSTEMS 

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy Graduate Department of Electrical and Computer Engineering University of Toronto

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# Abstract <br> Coordinated Autonomy: Pursuit Formations of Multivehicle Systems 

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Inspired by pursuit problems found in the mathematics and physics literature, this thesis studies the geometric formations of autonomous agent systems consisting of individuals programmed to pursue one another. More generally, autonomous yet coordinated multivehicle systems might find application as distributed sensor arrays in terrestrial, space, or oceanic exploration, enhance the efficiency of surveillance, search, and rescue missions, or even contribute to the development of automated transportation systems. Therefore, from an engineering standpoint, the question of how to prescribe desired global behaviours through the application of only simple and local interactions is of significant and practical interest.

In the first part of this thesis, the notion of pursuit is introduced by examining a system of identical linear agents in the plane. This idea is then extended to a system of wheeled vehicles, each subject to a single nonholonomic constraint, which is the principal focus of the thesis. It is revealed that the equilibrium formations of vehicles in cyclic pursuit are generalized regular polygons, and it is exposed how the multivehicle system's global behaviour can be shaped through appropriate controller gain assignments. The local stability of these equilibrium polygons is subsequently examined, revealing which formations are asymptotically stable. Finally, the results of multivehicle coordination experiments are reported. These experiments serve to demonstrate the practicality of the aforementioned distributed control strategy. The findings of this work not only bode well for continuing research on pursuit-based coordination techniques, but also for other
cooperative control strategies employing similar local interactions.
The last part of the thesis explores how structure in the interconnection topology among individuals of a multiagent system influences symmetry in its trajectories. More specifically, it is revealed how circulant connectivity preserves cyclic group symmetries in a formation of simple planar integrators, and to what extent circulant connectivity is necessary to achieve symmetry invariance.

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## List of Symbols

| $\bar{\mu}$ | Points of intersection in Figure 3.6, page 64 |
| :---: | :---: |
| $\bar{c}$ | Complex conjugate of $c \in \mathbb{C}$ |
| $\checkmark$ | Denotes the end of a remark |
| $\square$ | Denotes the end of a proof |
| 4 | Denotes the end of an example |
| (B) | Registered trade mark |
| $\operatorname{circ}\left(c_{1}, c_{2}, \ldots, c_{n}\right)$ | An $n \times n$ circulant matrix with elements $c_{1}, c_{2}, \ldots, c_{n}$, page 29 |
| $\mathbb{C}$ | The set of complex numbers |
| $\operatorname{diag}\left(d_{1}, d_{2}, \ldots, d_{n}\right)$ | An $n \times n$ diagonal matrix with elements $d_{1}, d_{2}, \ldots, d_{n}$ |
| $\Gamma(A)$ | Digraph associated with adjacency matrix $A$, page 19 |
| $\Gamma(A, z(t))$ | Formation graph for formation $z(t) \in \mathbb{C}^{n}$, page 136 |
| $\operatorname{gcd}(n, q)$ | Denotes the greatest common divisor of integers $n$ and $q$ |
| $\operatorname{Img}(A)$ | Image of a $A$ |
| $\kappa$ | Degree of coupling between vertices of a digraph, page 134 |
| $\operatorname{Ker}(A)$ | Kernel of $A$ |
| $\lambda$ | Typically used to denote an eigenvalue |
| $\lceil x\rceil$ | Denotes the smallest integer greater than or equal to $x$ |
| $\mathbb{S}^{n}$ | The $n$ dimensional unit sphere |
| $\mathbb{Z}$ | The set of integers |
| $\mathcal{C}^{1}$ | The space of continuously differentiable functions |
| $\mathcal{C}^{\infty}$ | The space of smooth functions |
| $\mathcal{E}$ | Finite set of edges $\left\{e_{i j}\right\}$ of a graph, page 18 |
| $\mathcal{E}_{t}$ | Set of edge vectors of $\Gamma(A, z(t))$ at time $t$, page 136 |
| $\mathcal{F}_{\text {d }}$ | Formation subspace associated with $\{n / d\}$, page 89 |
| $\mathcal{M}$ | Often used to denote a smooth submanifold of $\mathbb{R}^{n}$, page 48 |
| $\mathcal{N}$ | The set $\{1,2, \ldots, n\}$ |
| $\mathcal{V}$ | Finite set of vertices $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ of a graph, page 18 |
| $\mathcal{V}_{t}$ | Set of vertex locations of $\Gamma(A, z(t))$ at time $t$, page 136 |


| $\mu$ | Represents $q=d / n$ on a continuum, page 59 |
| :---: | :---: |
| $\\|\cdot\\|_{2}$ | Euclidean norm |
| $\omega$ | Often used to represent $e^{j 2 \pi / n}$ |
| $\bar{w}(\cdot), \underline{w}(\cdot)$ | Boundary describing functions, page 61 |
| $\partial \mathcal{S}$ | Denotes the boundary of the set $\mathcal{S}$ |
| $\Phi$ | Often used to denote a transformation of coordinates, page 32 |
| $\phi$ | Denotes rover steering angle in Chapter 5, page 111 |
| $\Pi_{n}$ | Fundamental $n \times n$ permutation matrix, page 28 |
| $\mathbb{R}$ | The set of real numbers |
| $\mathbb{R}^{n}$ | An $n$-dimensional vector space over $\mathbb{R}$ |
| $\mathbb{R}_{+}^{n}$ | The set of $n$-tuples for which all components belong to $\mathbb{R}_{+}$ |
| $\mathbb{R}^{m \times n}$ | The set of all $m \times n$ matrices with elements in $\mathbb{R}$ |
| $\mathbb{R}_{+}$ | The set of nonnegative real numbers |
| $\sigma$ | In Chapter 6, denotes a permutation of $\mathcal{N}$, page 135 |
| $\underline{\text { A }}$ | Incidence matrix associated with the matrix $A$, page 19 |
| $\{n / d\}$ | Notation used to denote a generalized regular polygon with $n$ vertices and density $d$, page 44 |
| $A \otimes B$ | Kronecker product of matrices $A$ and $B$, page 30 |
| $A^{*}$ | Conjugate transpose of $A$ |
| $A_{\mathcal{S}}^{\star}$ | Linear transformation induced by $A$ in the quotient space of a subspace $\mathcal{S}$, page 31 |
| $A^{\top}$ | Transpose of $A$ |
| $b_{i}$ | The $i$-th natural basis vector in $\mathbb{R}^{n}$ |
| $C_{m}$ | Denotes the (cyclic) rotation group of order $m$, page 133 |
| $d$ | Frequently used to denote the density of a polygon, page 44 |
| $d g(x)$ | The differential of a real function $g(s)$, page 48 |
| $F_{n}$ | The $n \times n$ Fourier matrix, page 29 |
| $h_{1}, h_{2}, h_{3}$ | Leading principal minors of a $3 \times 3$ matrix $H$ |
| $I\left(\mathbb{R}^{2}\right)$ | Group of isometries in $\mathbb{R}^{2}$, page 133 |
| $I_{n}$ | The $n \times n$ identity matrix |
| j | When not used as an index, represents $\sqrt{-1}$ |
| $k$ | Controller gain, used in Chapters 3 and 4 |
| $k^{\star}$ | Ratio $k_{r}$ : $k_{\alpha}$ for equilibrium in Chapter 4, page 79 |
| $k_{r}, k_{\alpha}$ | Controller gains used in Chapter 4 |
| $n$ | Typically used to denote the number of agents or vehicles |


| $P_{\sigma}$ | Permutation matrix corresponding to permutation $\sigma$, page 136 |
| :--- | :--- |
| $q$ | Used to represent the ratio $d / n$ in Chapters 3 and 4, page 47 |
| $r_{i}, \alpha_{i}, \beta_{i}$ | Relative coordinates for unicycle $i$, page 39 |
| $T_{\bar{x}} \mathcal{M}$ | Tangent space to manifold $\mathcal{M}$ at $\bar{x} \in \mathcal{M}$, page 48 |
| $v_{i}, \omega_{i}$ | Linear and angular speed inputs to unicycle $i$, page 35 |
| $v_{R}$ | Unicycle fixed reference speed, page 40 |
| $w$ | Represents $w_{i}$ on a continuum, page 59 |
| $w_{i}$ | Real part of $\omega^{i-1}$, page 57 |
| $x_{i}, y_{i}, \theta_{i}$ | Position and orientation (state) of unicycle $i$, page 35 |
| $z_{i}$ | Frequently the position vector $\left(x_{i}, y_{i}\right)$ of an agent, page 21 |
| AS | Asymptotically stable |
| CCD | Charge-coupled device |
| FOV | Field-of-view |
| GAS | Globally asymptotically stable |
| GPS | Global positioning system |
| LPF | Low-pass filter |
| NiMH | Nickel-metal hydride |
| ODE | Ordinary differential equation |
| PCMCIA | Personal Computer Memory Card International Association |
| RAM | Random access memory |
| RGB | Red-green-blue |
| USB | Universal serial bus |
| UTIAS | University of Toronto Institute for Aerospace Studies |
|  |  |

## Chapter 1

## Introduction

Multiagent systems and cooperative control are nowadays becoming research topics of increasing popularity, especially within the systems and control community - and with good reason. There exist numerous potential engineering applications. For example, teams of cooperating autonomous mobile robots might find application in terrestrial, space, and oceanic exploration, military surveillance and rescue missions, or even automated transportation systems, to suggest only a few possibilities. More generally, an agent may not only be a robot or vehicle, but might be a space satellite, a computer program, or perhaps even a living organism, cell, or molecule. In fact, one could argue that such systems pervade almost all of science. Hence, the question of how to prescribe desired global behaviours for a system of interconnected agents through the application of only simple and local interactions is of significant and practical interest.

Consider, for example, a group of autonomous mobile robots, initially placed at random in a room, and suppose it is desired that they exit the room in a graceful, orderly fashion. One approach to solving this toy problem involves two steps: (i) have a supervisor issue a command to all the robots, requesting them to form a circular arrangement; then, after the robots have accomplished the first task, (ii) have the supervisor command a robot, which is close to the door, to exit, whereupon the others should follow in sequence. This example of coordinated autonomy is illustrated in Figure 1.1 and begs the subproblem: How should the robots individually behave in order that the group as a whole self-organizes to form a circle, or other formation?

From a systems engineering perspective, the ultimate objective is that of synthesis. In other words, given an interconnected collection of agents, how might one design and construct decentralized controllers, with possibly limited and/or


Figure 1.1: A group of autonomous robots, gracefully exiting a room
dynamic information flow between the agents, that together generate a predictable and desirable global outcome for the network as a whole? To answer this question, it must first be understood what makes these kinds of systems work. Given an interagent coupling architecture and individual agent dynamics, how does a network of interacting dynamical systems collectively behave? This thesis places emphasis only on particular instances of the latter question, the principal result being a detailed analysis of the possible emergent steady-state behaviours for a network of interacting wheeled-vehicles that are subject to a restricted information flow structure called cyclic pursuit.

Of course, growing interest in the study of multiple agent systems insinuates they should possess distinct advantages over non-interacting individuals. The absolute best one might hope for is that a level of synergy is achieved in that the system as a whole is, in some meaningful way, greater than the sum of its individual agents' abilities. For example, it is not hard to imagine how several mobile robots deployed in a remote and hazardous environment (e.g., in the ocean, outer space, or on Mars) might, on information retrieval missions, act as an itinerant and reconfigurable sensor array, able to collect simultaneous data readings from multiple geographic locations. This information could then be used to influence the network's global actions, for instance, to optimize the resolution of measurements by formation reconfiguration. Moreover, if the system is also robust to the failure of individuals, then the entire mission may not be jeopardized should a certain acceptable number of agents malfunction.

Indeed, one needs to look no further than the natural environment for inspi-
ration. Perhaps the most widely cited archetype is that of social insects. Ants are singly simple creatures, yet as a collective they can perform remarkably complicated tasks. They are able to create intricate systems of roadways leading to food sources and build living bridges to cross gaps in their way-and all in a decentralized fashion. The study of such systems might not only spawn new ideas for control, but also an understanding of how to systematically mimic the talents of certain sociobiological systems. An ability such as this could, at least in some cases, be tremendously advantageous. However, the question of how to do so brings one back to the problem of understanding the fundamental mechanisms that govern these types of systems.

As was already implied, the research of this thesis is motivated by the problem of coordinating, in a decentralized and leaderless way, the motions of multiple autonomous mobile robots. Studied, in particular, are formation strategies for wheeled-vehicle systems based on the basic idea that individual vehicles should locally pursue one-another. Motivated by the above introduction, an extensive review of the literature on multiple agent systems, which serves to contextualize the contributions of this thesis, is provided in the section that immediately follows. Finally, an overview of the thesis is given, which introduces more explicitly the types of multivehicle formation problems studied in the main body of this thesis.

### 1.1 Literature Synopsis

Multiagent systems, or networks of systems, are pervasive in a great number of scientific disciplines, from the study of macro- and micro-biological interactions to complex physical phenomena, from internet software to networked power stations, and to the focus of this thesis: multirobot control. So, are there unifying principles? Surely, there must be; although a grand unifying theory is for certain a lofty aspiration (and is not within the scope of this thesis). On the other hand, the following literature synopsis illustrates how researchers from various scientific fields are now beginning to unravel some fundamental results, many of which show promise for practical engineering application. Rather than attempt to be encyclopedic, only a select number of references are provided, the intent being to represent each discipline through example. A more technical introduction to a few of the most relevant concepts is provided in Chapter 2.

### 1.1.1 Inspiration from Biology

Aggregate patterns in the spatial distribution of individuals within groups of living creatures is, according to researchers in the biological sciences, a phenomenon that occurs in a number of organisms, from bacteria to the higher vertebrates (Parrish, Viscido, and Grünbaum, 2002). Therefore, what the specific behavioural traits are of individuals that give rise to these patterns is apparently a topic of particular and growing interest to those in a number of scientific disciplines.

Flocks and related synchronized group behaviours such as schools of fish or herds of land animals are both beautiful to watch and intriguing to contemplate. A flock exhibits many contrasts. It is made up of discrete birds, yet overall motion seems fluid; it is simple in concept yet is so visually complex; it seems randomly arrayed, and yet is magnificently synchronized. Perhaps most puzzling is the strong impression of intentional, centralized control. Yet, all evidence indicates that flock motion must be merely the aggregate result of the actions of individual animals, each acting solely on the basis of its own local perception of the world (Reynolds, 1987, p. 25).

For some time now, numerous researchers have contended that mathematical and/or computational models are required in order to make a connection between individual and group characteristics, the purpose being to distinguish "behavioural cause from organizational effect by studying the consequences of various hypothetical social interaction rules" (Parrish et al., 2002, p. 298). For example, Breder, Jr. (1954) promoted the view that a typical fish school shows clear evidence of both attraction and repulsion between individuals. Moreover, he suggested that repulsion is evidently more effective than attraction at short distances, otherwise individuals would collide, and that attraction must be larger than repulsion at greater distances, otherwise a cohesive group would never result. Breder, Jr. went on to propose that this attractive-repulsive relationship between individuals could be hypothetically modelled by the equation $c=a / d^{m}-r / d^{n}$ (with $m$ much smaller than $n$ ), where $a$ is an attractive force, $r$ is a repelling force, $d$ denotes the distance between individuals, and $c$ represents a measure of the group cohesiveness (combination of attractive and repulsive forces). Thus, in keeping with his observations of schooling fish, the cohesiveness $c<0$ when the distance $d$ is small and $c>0$ when $d$ is large.

Progress has been made since Breder, Jr.'s 1954 article, and according to Parrish et al., the most recent analyses have focused on models involving three basic parameter categories: behavioural matching, positional reference, and numerical preference. Behavioural matching refers to the tendency for individuals to match their behaviour with others nearby (e.g., match speed and/or orientation). Breder, Jr.'s attractive-repulsive model falls under the category of positional preference, which is most often expressed as a preferred distance to neighbouring individuals. Finally, numerical preference describes the propensity for individuals to observe only a subset of the individuals that form a group. The cardinality of this subset is sometimes called the rule size (see Parrish et al., 2002, for illustrative examples).

In 1987, Reynolds developed his so-called distributed behavioural model, which is undoubtedly one of the most widely recognized examples of artificially created decentralized and self-organizing group behaviour (Reynolds, 1987). His so-called boids (or bird-oids) each obey the same local rules of navigation, which are based on each individual's local perception of the environment, a strategy that collectively results in natural and visually appealing aggregate motions ${ }^{1}$. Reynolds built his simulated flock by assigning the following behavioural rules to individuals, stated in order of decreasing precedence: (i) avoid collisions with nearby flock-mates; (ii) attempt to match the velocity of nearby flock-mates, and; (iii) attempt to stay close to nearby flock-mates. Notice the similarities between Reynolds' rules and the model parameter categories described in Parrish et al. (2002). Since 1987, extensions consistent with Reynolds' approach have been developed for simulated life and computer animation purposes, incorporating detailed artificial world models and reinforcement learning algorithms for locomotion and other tasks (e.g., Terzopoulous, Tu, and Grzeszczuk, 1994).

Artificial models such as those described by Reynolds (1987) have, in turn, been recognized as a useful tool for researchers in the biological sciences. By way of individual-based computer simulations "it has been possible to demonstrate that group leadership, hierarchical control, and global information are not necessary for collective behaviour" (Couzin, Krause, James, Ruxton, and Franks, 2002, p. 1). For example, Couzin et al. attempted to simulate aggregate behaviour in a more "biologically realistic" fashion, drawing on their own observations

[^0]and expertise, and on studies reported in the biology literature. In their model, each individual attempts to maintain a minimum distance from other individuals within a spherical zone of repulsion, located around the individual. If there are no neighbouring individuals within this zone of repulsion, then the individual attempts to align itself with neighbours in its so-called zone of orientation and is attracted to neighbours in its zone of attraction. These zones were designed so as to reflect more accurately the true sensory capabilities of real animals (e.g., fish, birds, ants). As the parameters of their model were varied (e.g., the width of the behavioural zones), Couzin et al. observed noticeable transitions in the collective dynamical behaviour of the individuals in their model. They reported three ${ }^{2}$ apparently distinct global behaviours: (i) swarming occurs when there is little or no alignment of the individuals (i.e., the individuals are principally influenced by attractive-repulsive behaviours); (ii) a torus occurs when individuals perpetually travel around an empty centre, and; (iii) parallel motion occurs when individuals align themselves and travel in an almost rectilinear fashion. The significance of these characteristic global traits is in the fact that they also appear in research described in the literature of various other scientific disciplines.

Couzin et al. (2002) also reported observing a hysteresis phenomenon in the parameter space. They noted that a change in model parameters that initiates a particular transition from one global behaviour to another does not necessarily result in the original collective behaviour when reversed. As a result, they conjectured that the animal group must have, in some sense, a collective memory, despite the fact that individuals within the group have no explicit knowledge of the group's history. Although this is an interesting conjecture, from a systems theoretic viewpoint, it is possible this result is simply a consequence of dependence on the initial conditions of the aggregate dynamical system.

For a broad treatment of self organization in biology, one which describes many examples of collective behaviour observed in nature, the reader is also referred to the text by Camazine et al. (2001). Of course, biology is also a promising source of inspiration for engineers.

From [a systems and control] engineering perspective, the high level of coordination achieved by these [natural] groups, and the idea that they are the result of a lengthy optimization process (natural selec-

[^1]tion) makes social grouping behaviours interesting candidates for biomimetic, or at least biologically inspired, algorithms that confer on robotic systems some desirable traits of natural groups (Grünbaum, Viscido, and Parrish, 2004, p. 103).

In fact, analytical studies of biologically inspired aggregate dynamical system models have recently appeared in the systems and control literature. Gazi and Passino (2002a, 2002b) investigated the stability properties of swarms of autonomous agents in an $m$-dimensional Euclidean space under the assumption that at each instant every agent knows the exact position of every other agent in the group. For $n$ agents, they modelled the kinematic behaviour of each agent $i \in\{1,2, \ldots, n\}$, denoted $z_{i} \in \mathbb{R}^{m}$, by the ordinary differential equation (ODE)

$$
\dot{z}_{i}=\sum_{j=1}^{n} f\left(z_{j}-z_{i}\right)
$$

where the function $f: \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ was given by the attractive-repulsive function

$$
f(\zeta)=\zeta\left[a-b \exp \left(-\frac{\|\zeta\|_{2}^{2}}{c}\right)\right]
$$

for some constants $a, b, c>0, b>a$. Thus, if the distance between individuals is large (respectively, small), the agents are attracted (respectively, repelled) to one another. They proved, using Lyapunov techniques, that the collection of individuals will form a cohesive swarm in finite time, showed that its centroid remains stationary, and explicitly computed a bound for the swarm size that depends only on the model parameters. Gazi and Passino (2002b) extended their initial swarm model by further assuming that the individuals evolve in an attractant/repellant profile (i.e., the agents are attracted/repelled by certain regions in space in order to simulate, for example, the natural presence of nutrients or toxic substances).

Relevant biologically inspired multiple agent research has also appeared in the mathematics literature. For example, Bruckstein's (1993) curiosity about the evolution of ant trails led him to an interesting mathematical discovery, the outcome being his answer to "why the ant trails look so straight and nice."

As a final note, Grünbaum et al. (2004, pp. 103-104) remarked that, while there exist mathematical methods for characterizing the behaviour of engineered multiple agent systems, the converse is typically not possible. Evidently, there are
relatively few techniques for deducing the underlying algorithm(s) responsible for any specific set of observed animal movements.

### 1.1.2 Coupled Oscillator Science

Engineers and biologists are not the only scientists interested in understanding how to characterize the behaviour of potentially large networks of interconnected systems. Owing to its generality, mathematicians and physicists continue to shed light on this important topic. According to Strogatz (2001), empirical studies have contributed to the enhanced comprehension of "food webs, electrical power grids, cellular and metabolic networks, the World-wide Web,..., telephone call graphs, co-authorship and citation networks of scientists, and the quintessential 'old-boy' network: the overlapping boards of directors of the largest companies in the United States" (Strogatz, 2001, p. 268).

Consider, for instance, an array of semiconductor lasers, where synchronizing the lasing elements in phase is important to achieve a large output power that is also appropriately concentrated (Kozyreff, Vladimirov, and Mandel, 2000). Such an array can be modelled as a system consisting of several coupled oscillator equations. Similar networks of coupled oscillators occur in biology; for example in cardiac pacemaker cells, in the intestine, and in the nervous system (Strogatz, 2001). Cole (1991) studied an ant colony as a population of coupled oscillators, where individual ants oscillate between activity and inactivity and coupling occurs when individuals activate their inactive neighbours. Similarly, Boi, Couzin, Buono, Franks, and Britton (1999) considered the interaction between spatial groups of worker ants as a network of coupled oscillators. As a result of its universality, synchrony in arrays of nonlinear coupled oscillators has been for some time a subject of particular interest to physicists. One of the most notable works is by Kuramoto $(1975,1984)$, who proposed a solvable model for collective synchronization. "Twenty-five years later, the Kuramoto model continues to surprise us" (Strogatz, 2001, p. 272). For a detailed review, see Strogatz (2000).

The archetypal Kuramoto model describes the behaviour of $n$ oscillators $\theta_{i}$ with natural frequencies $\omega_{i}$. Each oscillator is modelled by the ODE

$$
\begin{equation*}
\dot{\theta}_{i}=\omega_{i}+\frac{k}{n} \sum_{j=1}^{n} \sin \left(\theta_{j}-\theta_{i}\right), \tag{1.1}
\end{equation*}
$$

where the gain $k$ prescribes the coupling intensity. Notice that the coupling topology between oscillators is all-to-all (i.e., every oscillator is influenced by every other oscillator). Kuramoto's principal result showed that as $n \rightarrow \infty$, there exists a critical gain $k_{1}$ below which the oscillators are completely unsynchronized and that there is another gain $k_{2} \geq k_{1}$ above which all the oscillators are synchronized. When the gain $k$ is between $k_{1}$ and $k_{2}$, the oscillators exhibit partially synchronized behaviour. It turns out that the Kuramoto model has wide-ranging applicability. For example, Watanabe and Strogatz (1994, p. 212) studied a model for large arrays of superconducting Josephson junctions ${ }^{3}$, which when averaged is remarkably similar to the basic coupled oscillator model (1.1). For a finite number $n$ of coupled oscillators, Watanabe and Strogatz were able to provide a global stability analysis, proving that the oscillators either synchronize or converge to a manifold of incoherent (i.e., unsynchronized) states. In other words, for finite $n$ there is no partial synchronization phenomenon.

New research in the systems and control engineering literature indicates that this notion of oscillator synchronization may have practical implications in cooperative control engineering. For instance, motivated by a desire to coordinate the motions of individuals in groups of underwater vehicles, Sepulchre, Paley, and Leonard (2004); Paley, Leonard, and Sepulchre (2004) applied the results of Watanabe and Strogatz (1994) to the development of control laws that generate either synchronized circular trajectories or parallel motion; which behaviour ensues effectively depends on the gain $k$ in (1.1). In only a few words, what Sepulchre et al. did is identify and exploit the connections between "phase models of coupled oscillators and kinematic models of groups of self-propelled particles." Yet a significant limitation of their result is the all-to-all coupling assumption, which in many engineering applications would require a prohibitively demanding communication topology. Towards addressing this issue, Jadbabaie, Motee, and Barahona (2004) studied the finite $n$ Kuramoto model in the case of arbitrary connectivity, in essence rewriting (1.1) in terms of the incidence matrix of an undirected graph that describes the oscillator interconnection topology. Their work shows that, for connected graphs (i.e., those in which there is a path from every vertex to every other vertex), the oscillators either synchronize or converge to a manifold of incoherent states; the limiting state, of course, depends on the gain $k$. The success

[^2]of their analysis technique undeniably hints at an "advantageous marriage of systems and control theory and graph theory" (Jadbabaie et al., 2004, p. 4301) when studying networks of interacting dynamical systems.

As the above examples suggest, this phenomenon of (oscillator) synchrony has engineering potential and gives the impression of being ubiquitous among many physical and living systems. The latter idea forms the thesis for a very recent book by Strogatz (2003), wherein Strogatz explores, in wonderfully colourful narrative, what he simply calls sync. His text provides a vast number of examples and chronicles the development of coupled oscillator science in the mathematical, biological, and physics literature. However, Strogatz struggles with the question of "whether the models describe reality faithfully" (Strogatz, 2003, p. 65). Notice, this is the very same issue remarked by Grünbaum et al. (2004) (see the end of Section 1.1.1) concerning the challenge of deducing the underlying algorithm(s) responsible for the behaviour of real networks of interacting systems.

### 1.1.3 Heuristic Techniques

Reynolds' (1987) distributed behavioural model is an example of heuristic or behaviour-based design, since the emergent flocking behaviour results from the interaction of multiple active component behaviours (i.e., Reynolds' rules). Brooks (1986) popularized this type of hierarchical behaviour-based approach to robot control, defining a new paradigm in artificial intelligence now known as the subsumption architecture (Arkin, 1998, p. 130-141). Although his immediate application was to the control of a single mobile robot, its potential for extension to robot collections is obvious. In fact, until very recently, much of the multiple agent robotics research had focused on the use of behaviour-based algorithms. For example, Balch and Arkin (1998) evaluated a reactive strategy designed to implement multivehicle formations in combination with behaviours for collision avoidance and other navigational goals. Since this thesis does not focus on hierarchical algorithms of this sort, for more information the reader is referred to Matarić (1995); Cao, Fukunaga, and Kahng (1997); Arkin (1998); Barfoot, Earon, and D'Eleuterio (1999); Barfoot (2002); and references therein.

Regardless of its practical engineering appeal, the behaviour-based paradigm is "thin when it comes to providing controllers with guarantees; and engineers and theorists want guarantees" (Balch, 2003). Corresponding mathematical re-
sults are rare, as noted by Matarić (1995); Sugihara and Suzuki (1990). It is perhaps this very shortcoming that has inspired recent and developing interest in formally proving the technical completeness of algorithms that might be viewed as originally heuristic in nature.

### 1.1.4 Formations and Cooperative Control

Some have gone so far as to suggest that, despite its ability to exhibit complex outcomes, rigorous analysis of even the most simple behaviour-based algorithms might be an impractical task (Braitenberg, 1984). However, this has not deterred researchers in the analysis of certain nearest-neighbour techniques, where individual agents are designed to move based on the motions of their nearest neighbours. For example, Wang (1989) analyzed certain formation stability properties for the case when one mobile robot is provided a reference trajectory and designated the group leader. Early work by Sugihara and Suzuki (1990) investigated a set of heuristic algorithms for the generation of geometric patterns in the plane (e.g., lines, circles, or polygons). Suzuki and Yamashita (1999) later stressed the need for rigorous proof of the correctness of these sorts of algorithms.

Leonard and Fiorelli (2001) presented a method for distributed control of multiple autonomous agents by using artificial potential functions and so-called virtual leaders. The technique is intuitive in that individual agents behave according to interaction forces generated by sensing the positions of neighbouring agents. They were able to prove the stability of certain formations based on the construction of an appropriate Lyapunov function. Beard, Lawton, and Hadaegh (2001) developed an architecture for the coordination of multiple spacecraft in formation, which allows for both centralized and decentralized control. Egerstedt and Hu (2001) proposed a method for stabilizing rigidly constrained vehicle formations while moving along a desired common path for the formation. Desai, Ostrowski, and Kumar (2001) developed a technique for transitioning from one rigid formation to another while at the same time following a global trajectory for the group. Tabuada, Pappas, and Lima (2001) constructed a framework for studying the feasibility of vehicle formations that must satisfy both the formation and vehicle kinematic constraints. Justh and Krishnaprasad (2002, 2003, 2004) studied local unicycle steering laws for generating both rectilinear and circular formations in the plane, which they have shown are the only possible (relative) equilibrium
formations for identical fixed speed vehicles. Their approach uses alignment and separation control terms to determine the formation, which is also based on the pose of all other vehicles in the group (i.e., it is all-to-all). Tanner, Pappas, and Kumar (2004) recently introduced the notion of leader-to-state stability, a concept for vehicle formations that is similar in spirit to the notion of string stability in vehicle platoons (Swaroop and Hedrick, 1996). Recently, Ögren, Fiorelli, and Leonard (2004) presented a control strategy for a network of autonomous vehicles to reconfigure cooperatively in response to their sensed environment. Their approach allows the vehicles to act collectively as, for example, a sensor array that adapts its resolution in order to optimize measurements. Belta and Kumar (2004) addressed the problem of controlling a large number of robots required to move as a group by abstracting the group behaviour as a lower dimensional system. This abstracted information is subsequently used as feedback for individuals.

As has already been established, a useful and thus increasingly common approach to the analysis of algorithms similar to Reynolds' distributed behavioural model is to employ techniques from mathematical graph theory. Typically, a graph is used to track the influence of neighbouring agents on an individual. The vertices of the graph usually represent agents while the edges, which are possibly directed and/or dynamic, represent the existence (or non-existence if no edge is present) of explicit or implicit communication between agents. For example, Fax and Murray (2004) investigated the effect of information flow topology between agents, modelled using algebraic graph theory, on formation stability. Inspired by Reynolds' approach, Jadbabaie, Lin, and Morse (2003) proved convergence results for a nearest-neighbour type problem, guaranteeing that all $n$ agents eventually move in an identical fashion, despite the distributed nature of the coordination law. In the cooperative control systems literature, this result has become commonly known as consensus or agreement and is analogous to synchronization (as discussed in Section 1.1.2). Jadbabaie et al. employed a family of graphs with $n$ vertices to track the interactions between neighbouring agents. Olfati-Saber and Murray (2003b) developed a graph theoretic framework for generating similar flocking behaviours in the presence of obstacles. In fact, Reynolds' model has spawned a flurry of papers, too numerous to list, claiming rigorous analysis. See in addition Tanner, Jadbabaie, and Pappas (2003b, 2003a); Beard and Stepanyan (2003); Z. Lin, Broucke, and Francis (2004); Olfati-Saber and Murray (2004); Moreau (2004); Z. Lin, Francis, and Maggiore (2005), and references therein.

### 1.2 Overview of the Thesis

This thesis studies the geometric formations of multivehicle systems under a novel reconfiguration strategy inspired by the mathematical notion of pursuit. Of particular interest are the stability, invariance, and symmetry of formations. A brief overview of the thesis is given as follows.

It has already been remarked that this work is principally motivated by the fundamental question of how to prescribe desired global behaviours for a system of interconnected agents through the application of only simple and local interactions. The current chapter serves to demonstrate the ubiquity of this problem. It reveals that relevant work is ongoing in a number of scientific fields, especially in the biological sciences, in physics, as well as in robotics and control engineering.

Subsequently, to place into context the contributions of this thesis, Chapter 2 introduces the general problem framework by way of a more technical review of select concepts from the developing theories of multiple agent systems and cooperative control. To this end, the notions of connectivity and consensus are studied as an introduction to the remaining central chapters. Furthermore, a unique historical perspective on the pursuit approach taken here is disclosed.

Chapters 3 and 4 study a particularly intuitive control law for multivehicle systems that achieves circular pursuit patterns in the plane. By extending the problem of traditional cyclic pursuit to vehicles that are subject to a single nonholonomic constraint (i.e., a unicycles), it is shown that the multivehicle system's equilibrium formations correspond to stationary generalized regular polygons and that the system's global behaviour can be changed through appropriate controller gain assignments. This type of formation strategy might have, in particular, potential application in the deployment of distributed sensor arrays, enabling scientists to collect simultaneous seismological, meteorological, or other pertinent environmental data on planetary exploration missions (e.g., as described in Earon, Barfoot, and D'Eleuterio, 2001). In each of Chapters 3 and 4, the details of local stability analyses reveal exactly which equilibrium formations are asymptotically stable, the results of which are surprisingly nonintuitive.

Because the theory of Chapters 3 and 4 is based solely on vehicles modelled as ideal kinematic unicycles, it is important to experimentally validate the proposed multivehicle pursuit strategy. Chapter 5 summarizes the apparatus and outcomes of multivehicle coordination experiments conducted at the University of Toronto's

Institute for Aerospace Studies. The robots used in these experiments are in many ways significantly different from kinematic unicycles.

Chapter 6 returns to the linear pursuit problem initially introduced in Chapter 2. In contrast to a current trend in the literature towards arbitrary connectivity, this chapter studies how structure in the interconnection topology among agents relates to invariance of symmetries in the overall system's trajectories. Symmetric formations are often required to solve practical tasks; for example, as in the multivehicle target tracking problem described by Aranda, Martínez, and Bullo (2005). In Chapter 6, it is demonstrated that circulant structure plays a fundamental role when the task is preserving symmetry in multiagent formations.

A detailed summary of contributions can be found in Section 7.1.

## Chapter 2

## Autonomous Agent Systems

THE purpose of this chapter is to provide the reader with a more technical introduction to certain relevant concepts from the developing theories of multiple agent systems and cooperative control. At the same time, as a preface to the remaining central chapters of this thesis, the notion of pursuit is formally introduced as a multivehicle coordination strategy.

### 2.1 Agents and Interconnections

This thesis focuses solely on agents that move in two dimensions; for example, recall the group of mobile robots depicted in Figure 1.1. Consider a finite number $n>1$ of agents whose positions are denoted by the vectors $z_{i}(t)=\left(x_{i}(t), y_{i}(t)\right) \in$ $\mathbb{R}^{2}, i=1,2, \ldots, n$, at time $t \geq 0$. Suppose, for now, that the agents are identical and that each one can be modelled as a simple kinematic integrator

$$
\begin{equation*}
\dot{z}_{i}=u_{i}, \tag{2.1}
\end{equation*}
$$

where the $u_{i} \in \mathbb{R}^{2}$ are inputs. Consider, for the purpose of illustration, the inputs

$$
\begin{equation*}
u_{i}=a\left(z_{i+1}-z_{i}\right), \tag{2.2}
\end{equation*}
$$

where $a>0$ is some constant and the agent indices $i+1$ are evaluated modulo $n$. Henceforth, all indices $i+j$, with $i, j \in \mathbb{Z}$, should be evaluated modulo $n$. At each instant, agent $i$ moves directly towards agent $i+1$ at a speed proportional to the distance separating the two agents. What trajectories result? Figure 2.1 shows the trajectories of $n=6$ agents, generated by numerical integration, initially located at random and subject to the control law (2.2) with $a=1$. As $t \rightarrow \infty$, the
agents spiral into a fixed point. By repeating the experiment, it can be deduced that there is always exactly one fixed point, whose location depends on the initial conditions but not on the positive constant $a$.


Figure 2.1: Six agents subject to control law (2.2)
Now consider a second example, where the inputs $u_{i}$ of (2.1) are given by

$$
\begin{equation*}
u_{i}=a\left(z_{i+2}-z_{i}\right), \tag{2.3}
\end{equation*}
$$

with $a>0$. By reusing the initial conditions from Figure 2.1, Figure 2.2 shows the trajectories of six agents subject to the control law (2.3). In this case, half of the agents spiral into a fixed point, while the other half spiral into a different fixed point. Again, by repeating the experiment it can be deduced that there is (almost) always exactly two fixed points, the locations of which depend on the initial conditions but not on the positive constant $a$.

It is not hard to intuitively explain the behavioural differences observed between the trajectories of Figures 2.1 and 2.2. In the case of Figure 2.2, note how the agents can be divided into exactly two disjoint subsets, $\{1,3,5\}$ and $\{2,4,6\}$. Each subset has the property that no agent in that subset influences, by way of


Figure 2.2: Six agents subject to control law (2.3)
the inputs $u_{i}$, the motions of any agent in the other subset. Therefore, the transfer of information between agents occurs only within subsets. No such partitioning of the set $\{1,2, \ldots, 6\}$ is possible for the control inputs (2.2).

Interagent information flow, whether by way of local sensing or direct communication, is a topic of fundamental importance in the analysis and design of distributed multiple agent systems. Numerous researchers have come to realize that algebraic graph theory might serve as an effective tool for modelling the flow of information between agents (Jadbabaie et al., 2003; Olfati-Saber and Murray, 2003a; Z. Lin et al., 2004). If the flow is directed (i.e., not necessarily pairwise), then it can be modelled as a digraph (short for directed graph), denoted $\Gamma=(\mathcal{V}, \mathcal{E})$. The digraph $\Gamma$ consists of a finite set $\mathcal{V}$ of $|\mathcal{V}|=n$ vertices, one for each agent, along with a set $\mathcal{E}$ of $|\mathcal{E}| \geq 0$ directed edges $e_{i j}=\left(v_{i}, v_{j}\right) \in \mathcal{E}$, where $v_{i}, v_{j} \in \mathcal{V}$. The existence of an edge $e_{i j}$ indicates that the $i$-th agent receives information, either directly or indirectly, about the $j$-th agent. If information is passed between agents by way of local sensing, then $\Gamma$ is sometimes referred to as a sensor graph (Z. Lin et al., 2004). For example, Figure 2.3 depicts the sensor graphs for $n=6$ agents subject to (2.2) and (2.3). Alternatively, one might say
that agent $i$ is influenced by agent $j$. In this case, the arrows are reversed and $\Gamma$ is referred to as an influence graph (Farina and Rinaldi, 2000).

(a) Control law (2.2)

(b) Control law (2.3)

Figure 2.3: Example sensor graphs for $n=6$ agents

The next two sections supply some relevant terminology and mathematical background relating to digraphs and their associated matrix theory, further details of which can be found in Horn and Johnson (1985) and Fiedler (1986).

### 2.1.1 Some Digraph Terminology

A digraph $\Gamma$ is called weighted if along with every edge in $\mathcal{E}$ there is an associated number $a_{i j} \neq 0$, called the edge weight. The adjacency matrix $A$ of an $n$-vertex weighted digraph is an $n \times n$ nonnegative matrix whose $i j$-th entry is the weight $a_{i j}$ associated with the edge $e_{i j} \in \mathcal{E}$ and is otherwise zero. Alternatively, given an adjacency matrix $A$, one can define an associated digraph, denoted $\Gamma(A)$. Consider, for example, (2.3) and the digraph of Figure 2.3b. In this case, the adjacency matrix $A$ corresponding to the graph $\Gamma(A)$ is given by

$$
A=\left[\begin{array}{llllll}
0 & 0 & a & 0 & 0 & 0  \tag{2.4}\\
0 & 0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & 0 & a \\
a & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0
\end{array}\right]
$$

The out-degree of a particular vertex is the sum of the weights of the edges
in $\mathcal{E}$ exiting the vertex. Similarly, the in-degree of a particular vertex is the sum of the edges in $\mathcal{E}$ entering the vertex. A graph is called balanced if the in-degree and out-degree of all its vertices are equal (Olfati-Saber and Murray, 2003a). Consequently, the out-degree matrix (respectively, in-degree matrix) of $\Gamma(A)$ is a diagonal matrix $D=\operatorname{diag}\left(d_{1}, \ldots, d_{n}\right)$ with diagonal elements corresponding to either the out-degree (respectively, in-degree) of each vertex. The Laplacian of a digraph is defined as $L=D-A$. A directed path in the digraph is a sequence of contiguous edges $\left\{e_{i_{1} i_{2}}, e_{i_{2} i_{3}}, e_{i_{3} i_{4}}, \ldots\right\}$ in $\Gamma(A)$. The digraph is said to be strongly connected if between every pair of vertices $\left(v_{i}, v_{j}\right)$ there is a directed path of finite length that begins at $v_{i}$ and ends at $v_{j}$. Thus, the associated adjacency matrix $A=\left[a_{i j}\right]$ of a strongly connected digraph $\Gamma(A)$ has the property that for every pair of distinct integers $(p, q)$, with $1 \leq p, q \leq n$, there exists a sequence of distinct integers $\left\{p, i_{1}, i_{2}, \ldots, i_{m}, q\right\}$, with $1 \leq m \leq n$, such that the entries $\left\{a_{p i_{1}}, a_{i_{1}, i_{2}}, \ldots, a_{i_{m} q}\right\}$ are nonzero. In particular, the sensor graph of Figure 2.3a is strongly connected, while the sensor graph of Figure 2.3b is not. This accounts for the difference in behaviours observed in Figures 2.1 and 2.2, a fact that will be made more precise later in this chapter.

### 2.1.2 Digraphs and Nonnegative Matrices

Digraphs have an important connection to nonnegative matrices. Let $\mathcal{N}:=$ $\{1,2, \ldots, n\}$. Given two real $n \times n$ matrices $A=\left[a_{i j}\right]$ and $B=\left[b_{i j}\right]$, we write that (i) $A \geq B$ if $a_{i j} \geq b_{i j}$ for all $i, j \in \mathcal{N}$; (ii) $A>B$ if $A \geq B$ and $A \neq B$; and (iii) $A \gg B$ if $a_{i j}>b_{i j}$ for all $i, j \in \mathcal{N}$. An analogous notation applies to vectors. The matrix $A$ is called nonnegative if $A \geq 0$. The incidence matrix of a real $n \times n$ matrix $A$ is defined as the matrix $\underline{A}=\left[\underline{a}_{i j}\right]$, where $\underline{a}_{i j}=1$ if $a_{i j} \neq 0$, and $\underline{a}_{i j}=0$ if $a_{i j}=0$. Notice that $\underline{A}$ is nonnegative.

Given a digraph $\Gamma(A)$, its connectivity can be algebraically related to its adjacency matrix $A$. The following lemma has been adapted from Theorem 3 of Farina and Rinaldi (2000, p. 21),

Theorem 2.1: The number of directed paths of length $m$ from vertex $v_{i}$ to vertex $v_{j}$ in $\Gamma(A)$ is equal to the ij-th element of the matrix $\underline{A}^{m}$.

Proof (after Farina and Rinaldi, 2000, p. 21): Note that the assertion holds, by definition, when $m=1$. Assume that the $i j$-th element of $\underline{A}^{m}$, denoted $\underline{a}_{i j}^{m}$,
coincides with the number of directed paths of length $m$ from $v_{i}$ to $v_{j}$ in $\Gamma(A)$, denoted $p_{i j}^{m}$. Therefore, the number of directed paths $p_{i j}^{m+1}$ of length $m+1$ from $v_{i}$ to $v_{j}$ can be found by considering directed paths that are concatenations of an edge $e_{k j}$ with a directed path of length $m$ from $v_{i}$ to $v_{k}$. In other words,

$$
p_{i j}^{m+1}=\sum_{k=1}^{n} p_{i k}^{m} \underline{a}_{k j}=\sum_{k=1}^{n} \underline{a}_{i k}^{m} \underline{a}_{k j}=\underline{a}_{i j}^{m+1},
$$

concluding the proof by induction.

For example, the incidence matrix $\underline{A}$ for the digraph $\Gamma(A)$ of Figure 2.3b is given by (2.4) with $a=1$. The number of directed paths of length $m=2$ from vertex $v_{3}$ to vertex $v_{1}$ is given by the $(3,1)$ entry of

$$
\underline{A}^{2}=\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{array}\right] .
$$

This is easily seen in Figure 2.3b.
For the case when the adjacency matrix happens to be nonnegative, Fiedler (1986, Theorem 4.4) offers the following theorem.

Theorem 2.2: If $A$ is nonnegative, then the $i j$-th element of $A^{m}$ is nonzero if and only if there is a directed path of length $m$ from vertex $v_{i}$ to vertex $v_{j}$ in $\Gamma(A)$.

A square matrix $A$ is said to be reducible if there exists a permutation of its rows and columns that takes it into the form

$$
\left[\begin{array}{cc}
A_{1} & 0 \\
A_{2} & A_{3}
\end{array}\right]
$$

If not, the matrix is said to be irreducible. The following is a standard result (Berman and Plemmons, 1994, combining Theorem 2.1.3 and Theorem 2.2.7).

Theorem 2.3: Let $A \geq 0$ be an $n \times n$ matrix. Then, the following are equivalent:
(i) $A$ is irreducible;
(ii) $\Gamma(A)$ is strongly connected; and,
(iii) $\left(I_{n}+A\right)^{n-1} \gg 0$.

### 2.1.3 Connectivity, Pursuit, and Consensus

From the point of view of interagent sensing, the agents subject to (2.2) require no common coordinate system since the information passed between them is only relative (not absolute). Yet, in Figure 2.1, they eventually come to agree on a common point by merely having arrived at one. This type of result is frequently called consensus, and has broad relevance. For instance, achieving parallel motion in a school of fish, as described in Section 1.1.1, is analogous to the individuals having reached a consensus with respect to their heading angles. The synchronization of coupled oscillators, talked about in Section 1.1.2, constitutes a consensus among the oscillators with regards to their phase and frequency. As a result, there has been recent and marked interest in the consensus problem and, in particular, conditions for the achievability of consensus. Therefore, a short discussion of this subject is warranted. In this section, a fundamental result about the achievability of consensus is related to the existing theory of positive systems (see Luenberger, 1979; Farina and Rinaldi, 2000).

Let $z_{i}=\left(x_{i}, y_{i}\right)$ such that $z=\left(z_{1}, z_{2}, \ldots, z_{n}\right)$ is the aggregate vector of agent positions. Consider a group of kinematic integrators (2.1) subject to linear interconnection control laws such that the multiagent system has the form

$$
\begin{align*}
& \dot{x}=M x  \tag{2.5}\\
& \dot{y}=M y, \tag{2.6}
\end{align*}
$$

where $M$ is a real $n \times n$ matrix. Each row $i$ of $M$ specifies the local control strategy to be followed by agent $i$. Clearly, the $x_{i}$ and $y_{i}$ components of $z_{i}=\left(x_{i}, y_{i}\right)$ evolve independently. Thus, it is sufficient to look at only one coordinate, say (2.5). Assume that the individual agents are capable of sensing only relative position information about each other (e.g., as in the examples (2.2) and (2.3)). Thus, there is no need for a global positioning system nor explicit communication among agents. By this assumption, the aggregate system matrix $M$ must have zero row-sums; equivalently, $M[11 \cdots 1]^{\top}=0$.

Now, imagine a potential strategy for achieving consensus based on the simple notion that individual agents should pursue one-another. Given the observations regarding (2.2) depicted in Figure 2.1, this seems like a reasonable approach. Note how this pursuit strategy is equivalent to allowing only attractive relationships between agents. More specifically, it implies that the off-diagonal entries of the aggregate system matrix $M$ are all nonnegative. Such matrices are called Metzler and are intrinsic to the theory of positive linear systems.

Let $\mathbb{R}_{+}^{n}$ denote the set of $n$-tuples for which all components belong to $[0, \infty)$. Generally, with respect to (2.5), a set $\mathcal{D}$ is said to be positively invariant if $x(0) \in$ $\mathcal{D}$ implies that $x(t) \in \mathcal{D}$ for all $t \geq 0$.

Definition 2.1 (Positive System): The linear system (2.5) is called positive if the set $\mathbb{R}_{+}^{n}$ is positively invariant.

This definition applies equally to nonlinear systems. The following is a well known result (Luenberger, 1979, pp. 204-205).

Proposition 2.1: The linear system (2.5) is positive if and only if $M$ is Metzler.
Consequently, the proposed pursuit strategy implies that the multiagent system (2.5) forms a positive system. The following theorem, given without proof, presents an existing result from the literature on multiagent systems and the problem of consensus (cf. Moreau, 2003, 2004; Beard and Stepanyan, 2003; Ren, Beard, and McLain, 2004; Z. Lin et al., 2005). The result holds irrespective of whether the multiagent system is positive.

Theorem 2.4: A consensus among the agents (2.5) is achievable if and only if:
(C) there exists at least one vertex $v_{j}, j \in \mathcal{N}$, with the property that there is a directed path to vertex $v_{j}$ from every other vertex $v_{i}, i \in \mathcal{N}, i \neq j$, in $\Gamma(M)$.

Finally, the simulation results of Figures 2.1 and 2.2 can be explained: The strongly connected digraph of Figure 2.3a satisfies condition (C), while the digraph of Figure 2.3b does not. This graphical condition (C) has a convenient algebraic equivalent, which is captured by the following proposition.

Proposition 2.2: Let $M$ be an $n \times n$ matrix. There exists at least one directed path to vertex $v_{j}$ from every other vertex $v_{i}, i \in \mathcal{N}, i \neq j$, in $\Gamma(M)$ if and only if $\left(I_{n}+\underline{M}\right)^{n-1} b_{j} \gg 0$, where $b_{j}$ is the $j$-th natural basis vector.

Proof: $(\Rightarrow)$ According to Theorem 2.1, the number of directed paths of length $m$ from the vertex $v_{i}$ to another vertex $v_{j}$ is equal to the $i j$-th element of $\underline{M}^{m}$. Let $b_{j}$ denote the $j$-th natural basis vector. Then, the number of directed paths of length $m$ from $v_{i}$ to $v_{j}$ is simply the $i$-th component of $\underline{M}^{m} b_{j}$. Among all the possible directed paths from $v_{i}$ to $v_{j}$ there must exist at least one directed path of length not greater than $n-1$. Therefore, since there exists at least one directed path to $v_{j}$ from every other vertex $v_{i}, i \in \mathcal{N}, i \neq j$, the sum of the vectors $b_{j}$, $\underline{M} b_{j}, \underline{M}^{2} b_{j}, \ldots, \underline{M}^{n-1} b_{j}$ must be strictly positive. Equivalently, for all constants $\alpha_{i}>0, i=1,2, \ldots, n$, the sum $\left(\alpha_{1} I_{n}+\alpha_{2} \underline{M}+\alpha_{3} \underline{M}^{2}+\ldots+\alpha_{n} \underline{M}^{n-1}\right) b_{j} \gg 0$. In particular, the binomial theorem yields

$$
\left(I_{n}+\underline{M}\right)^{n-1} b_{j}=\sum_{m=0}^{n-1}\binom{n-1}{m} I_{n}^{n-m-1} \underline{M}^{m} b_{j} \gg 0
$$

$(\Leftarrow)$ By noting the above construction, the condition $\left(I_{n}+\underline{M}\right)^{n-1} b_{j} \gg 0$ is equivalent to $b_{j}+\underline{M} b_{j}+\underline{M}^{2} b_{j}+\cdots+\underline{M}^{n-1} b_{j} \gg 0$. However, by Theorem 2.1, this implies that there exists at least one path of length not greater than $n-1$ to $v_{j}$ from every other vertex $v_{i}, i \in \mathcal{N}, i \neq j$, concluding the proof.

In other words, the necessary and sufficient condition (C) of Theorem 2.4 is equivalent to the existence of at least one natural basis vector $b_{j}, j \in \mathcal{N}$, such that $\left(I_{n}+\underline{M}\right)^{n-1} b_{j} \gg 0$. Note that if (C) holds for all $n$ vertices, then one has that $\left(I_{n}+\underline{M}\right)^{n-1} \gg 0$, which is the well-known condition for irreducibility of a nonnegative matrix $M$ (cf. Theorem 2.3). Equivalently, the digraph $\Gamma(M)$ is strongly connected, as per Theorem 2.3. In this thesis, as in the research of Z. Lin et al. (2005) and Moreau (2003), sensor graph edges are assigned in the opposite direction to influence graphs. As a result, consensus is achievable if and only if there is at least one state that is reachable from all other states. In Z. Lin et al. (2005), such states are called globally reachable. In Moreau (2003, 2005), such a digraph is called weakly connected, terminology which does not fit with the commonly used definition of a weakly connected digraph being one that is strongly connected when the sense of edge directions is ignored. In Beard and Stepanyan (2003) and Ren et al. (2004), the graph is said to have a spanning tree. Ren et al. (2004) and Z. Lin et al. (2005) also report that (C) holds if and only if zero is a simple eigenvalue of $M$. In the context of Markov chains and discrete-time systems, Luenberger (1979, Chapter 7) uses the term accessible to describe this
very same idea. Similarly, in nonnegative matrix theory (Berman and Plemmons, 1994, cf. Definition 2.3.7), a state $x_{i}$ is said to have access to state $x_{j}$ if there is a directed path in $\Gamma(M)$ from $v_{i}$ to $v_{j}$. If state $x_{j}$ also has access to state $x_{i}$, then the states are said to communicate. Thus, for irreducible matrices, every state communicates with every other state.

The condition (C) is used in the context of positive systems with an input (i.e., $\dot{x}=A x+b u, u \in \mathbb{R}$ ) to distinguish whether the system's input influences every state component (Muratori and Rinaldi, 1991; Farina and Rinaldi, 2000; Piccardi and Rinaldi, 2002). In this case, the system (with $A$ Metzler and vector $b>0)$ is said to be excitable if the condition holds for the incidence pair $(\underline{A}, \underline{b})$; specifically, $\left(I_{n}+\underline{A}\right)^{n-1} \underline{b} \gg 0$ (cf. Proposition 2.2). For the system (2.5), without inputs, one might analogously say that (2.5) is excitable if there is at least one state component that influences every other state component. In the theory of decentralized systems, the concept known as input reachability describes a similar system property (Siljak, 1991).

Although this excitability (accessible, globally reachable, spanning tree) property can be easily checked graphically, Proposition 2.2 offers a quick and simple computational algorithm for checking this property, which could very easily be implemented on a computer. For each $j \in \mathcal{N}$, one simply has to check whether $\left(I_{n}+\underline{M}\right)^{n-1} b_{j} \gg 0$. If the condition holds, then the $j$-th agent influences all the other agents (equivalently, vertex $v_{j}$ is globally reachable), otherwise not.

### 2.2 Agents in Cyclic Pursuit

The condition (C) of Theorem 2.4 accounts for all possible (time invariant) interconnection topologies that achieve consensus among a group of autonomous agents, and one might imagine generalizations of this such as the inclusion of sensing/communication delays (e.g., Moreau, 2005) or dynamically changing graphs (e.g., Tanner et al., 2003b; Ren and Beard, 2005). Contrary to an apparent trend in the literature towards generalization, the majority of this thesis instead focuses specifically on agents in cyclic pursuit, for example, as in the control law (2.2) and as depicted in Figure 2.3a for $n=6$ agents. A principal objective of this thesis is to study pursuit strategies for autonomous agent systems subject to motion constraints (i.e., wheeled vehicles).

### 2.2.1 The History of Pursuit

Problems based on the notion of pursuit have appealed to the curiosity of mathematicians and scientists over a period spanning centuries. These ideas apparently originated in the mathematics of pursuit curves (c. 1732), first studied by French scientist Pierre Bouguer (Bernhart, 1959). Simply put, if a point $q$ in space moves along a known curve, then another point $p$ describes a pursuit curve if the motion of $p$ is always directed towards $q$ and the two points move with equal speeds. More than a century later, in 1877, mathematician Edouard Lucas asked, what trajectories would be generated if three dogs, initially placed at the vertices of an equilateral triangle, were to run one-after-the-other? In 1880, Henri Brocard replied with the answer that each dog's pursuit curve would be that of a logarithmic spiral and that the dogs would eventually meet at a common point, known now as the Brocard point of a triangle (Bernhart, 1959).

In one of his several Scripta Mathematica articles on the subject, Bernhart (1959) revealed an intriguing history of cyclic pursuit, beginning with Brocard's response to Lucas in 1880. Among his findings, Bernhart reported on a Pi Mu Epsilon talk given by a man named Peterson, who apparently extended the original three dogs problem to $n$ ordered "bugs" that start at the vertices of a regular $n$-polygon. He is said to have illustrated his results for the square using four "cannibalistic spiders." Thus, if each bug pursues the next modulo $n$ (i.e., cyclic pursuit) at fixed speed, the bugs will trace out logarithmic spirals and eventually meet at the polygon's centre. Watton and Kydon (1969) provided their own solution to this regular $n$-bugs problem, also noting that the constant speed assumption is not necessary. Interestingly, the bugs problem has been used for artistic design. For instance, plotting the line-of-sight for each bug at regular intervals while tracing out the pursuit curves (see Figure 2.4) generates pleasing geometric patterns (Peterson, 2001).

Now, suppose the $n$ bugs do not start at the vertices of a regular $n$-polygon. Klamkin and Newman (1971) showed that, for three bugs, so long as the bugs are not initially arranged so that they are collinear, they will meet at a common point and this meeting will be mutual. For $n$ bugs, this problem was later examined by Behroozi and Gagnon (1979), who proved that "a bug cannot capture a bug which is not capturing another bug [i.e., mutual capture], except by head-on collision." They used their result to show that, specifically for the 4-bugs problem,


Figure 2.4: Pursuit patterns for the regular 3- and 4-bugs problem
the terminal capture is indeed mutual. Recently, Richardson (2001a) resolved this issue for the general $n$-bugs problem, showing that "it is possible for bugs to capture their prey without all bugs simultaneously doing so, even for non-collinear initial positions." However, he proved that, if the initial conditions are chosen at random, then the probability of a non-mutual capture is zero.

Other variations on the traditional cyclic pursuit problem have also been investigated. For example, Bruckstein, Cohen, and Efrat (1991) studied both continuous (ants) and discrete (crickets and frogs) pursuit problems, as well as both constant and varying speed scenarios. Although intrinsically simple, it is evident that the cyclic pursuit problem has many rich and interesting facets.

### 2.2.2 Linear Cyclic Pursuit

A particular version of the classical $n$-bugs problem has $n$ agents in cyclic pursuit such that agent $i$ pursues agent $i+1$ modulo $n$ according to the control law (2.2). This differential equation model of the $n$-bugs problem appeared in Bruckstein et al. (1991). Recall that, since each coordinate of the agent location $z_{i}(t)=$ $(x(t), y(t)) \in \mathbb{R}^{2}$ evolves independently, the $n$-agent system (2.1)-(2.2) decouples into two identical linear systems of the form (2.5).

Bruckstein et al. (1991) proved that for every initial condition, the agents exponentially converge to a single point. Moreover, they showed that this limit point is computable from the initial conditions of the agents. A similar version of the following theorem can also be found in Z. Lin et al. (2004).

Theorem 2.5: Consider $n$ planar agents with kinematics (2.1)-(2.2). For every initial condition, the centroid of the agents $z_{1}(t), z_{2}(t), \ldots, z_{n}(t)$ remains stationary and every agent $z_{i}(t), i=1,2, \ldots, n$, exponentially converges to this centroid.

Proof (after Bruckstein et al., 1991, pp. 16-18): As previously noted, one needs only to study the linear system (2.5), where

$$
M=\left[\begin{array}{cccccc}
-a & a & 0 & \cdots & \cdots & 0 \\
0 & -a & a & 0 & \cdots & 0 \\
\vdots & & \ddots & \ddots & & \vdots \\
0 & \cdots & 0 & -a & a & 0 \\
0 & \cdots & \cdots & 0 & -a & a \\
a & 0 & \cdots & \cdots & 0 & -a
\end{array}\right]=a\left(\Pi-I_{n}\right)
$$

$I_{n}$ is $n \times n$ identity matrix, and $\Pi$ contains the off-diagonal elements. This matrix $M$ has rank $n-1$, and thus has a zero eigenvalue corresponding to the eigenvector $e=(1,1, \ldots, 1)$. Its characteristic polynomial is (Bruckstein et al., 1991, p. 17)

$$
p_{M}(\lambda)=(\lambda+a)^{n}-a^{n} .
$$

Since $a>0$, this polynomial cannot vanish on the closed right-half complex plane, excluding the origin, implying that all the nonzero eigenvalues must have negative real parts. Let $\left\{e, v_{1}, v_{2}, \ldots, v_{n-1}\right\}$ be a Jordan basis for $M$ and note that $a \Pi v_{i}=a v_{i}+\lambda v_{i}$, for $\lambda$ a nonzero eigenvector of $M$. Premultiplying by $e^{\top}$ and using the fact that $e^{\top} \Pi=e^{\top}$, one obtains

$$
a e^{\top} v_{i}=a e^{\top} v_{i}+\lambda e^{\top} v_{i},
$$

implying that $e$ is orthogonal to the vectors $\left\{v_{1}, v_{2}, \ldots, v_{n-1}\right\}$. Factor the initial state as $x(0)=\alpha e+\sum_{i=1}^{n-1} \beta_{i} v_{i}$, where $\alpha$ is a constant. Owing to stability of the nonzero eigenvalues, $x(t) \rightarrow \alpha e$ as $t \rightarrow \infty$. On the other hand $e^{\top} x(t)=\alpha e^{\top} e=\alpha \times n$, meaning that $\alpha$ is the centroid of the agents and that this centroid remains stationary for all $t \geq 0$.

### 2.2.3 Circulant Matrices

Prior to offering an alternate proof of Theorem 2.5, this section summarizes some basic results from the theory of circulant matrices, which will also become fundamentally important in subsequent chapters of this thesis. For a detailed treatise, the reader is referred to the authoritative text by Davis (1994).

A circulant matrix (or circulant for short) of order $n$ is a matrix of the form

$$
C=\left[\begin{array}{cccc}
c_{1} & c_{2} & \cdots & c_{n} \\
c_{n} & c_{1} & \cdots & c_{n-1} \\
\vdots & \vdots & & \vdots \\
c_{2} & c_{3} & \cdots & c_{1}
\end{array}\right]=: \operatorname{circ}\left(c_{1}, c_{2}, \ldots, c_{n}\right)
$$

Each row is simply the row above shifted one element to the right (and wrapped around; i.e., modulo $n$ ). The entire matrix is determined by the first row.

Let $\Pi_{n}$ denote the fundamental $n \times n$ permutation matrix

$$
\Pi_{n}=\left[\begin{array}{cccccc}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
1 & 0 & 0 & 0 & \cdots & 0
\end{array}\right]=\operatorname{circ}(0,1,0, \ldots, 0),
$$

which plays a special role in the theory of circulants. One can then "push forward" the matrix $\Pi_{n}$ to form subsequent permutation matrices; for example

$$
\Pi_{n}^{2}=\left[\begin{array}{cccccc}
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 1 & 0 & 0 & \cdots & 0
\end{array}\right]=\operatorname{circ}(0,0,1,0, \ldots, 0)
$$

and subsequently $\Pi_{n}^{3}, \Pi_{n}^{4}, \ldots$, and so on. Note that $\Pi_{n}$ is itself circulant. Let $I_{n}=$ $\Pi_{n}^{0}$ denote the $n \times n$ identity matrix. By using the structure of the permutation matrices $\Pi_{n}^{k}$, with $k=0,1, \ldots, n-1$, every circulant $C$ can be represented by

$$
\begin{equation*}
C=\operatorname{circ}\left(c_{1}, c_{2}, \ldots, c_{n}\right)=c_{1} I_{n}+c_{2} \Pi_{n}+c_{3} \Pi_{n}^{2}+\cdots+c_{n} \Pi_{n}^{n-1} . \tag{2.7}
\end{equation*}
$$

Thus, the polynomial

$$
p_{C}(\lambda)=c_{1}+c_{2} \lambda+c_{3} \lambda^{2}+\cdots+c_{n} \lambda^{n-1}
$$

is called the circulant's representer, since $C=p_{C}\left(\Pi_{n}\right)$. Finally, a matrix $C$ is circulant if and only if it commutes with the fundamental permutation matrix.

Theorem 2.6 (Davis, 1994, Theorem 3.1.1): Let $C$ be an $n \times n$ matrix. Then $C$ is a circulant matrix if and only if $\Pi_{n} C=C \Pi_{n}$.

## Diagonalization of Circulants

Define $\omega:=e^{j 2 \pi / n}$ where $j=\sqrt{-1}$ and let $\Omega_{n}=\operatorname{diag}\left(1, \omega, \omega^{2}, \ldots, \omega^{n-1}\right)$, which are the $n$ roots of unity. Let $F_{n}$ denote the $n \times n$ Fourier matrix given via

$$
F_{n}^{*}=\frac{1}{\sqrt{n}}\left[\begin{array}{ccccc}
1 & 1 & 1 & \cdots & 1  \tag{2.8}\\
1 & \omega & \omega^{2} & \cdots & \omega^{n-1} \\
1 & \omega^{2} & \omega^{4} & \cdots & \omega^{2(n-1)} \\
\vdots & \vdots & \vdots & & \vdots \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)}
\end{array}\right]
$$

and note that $F_{n}^{*}=\left(F_{n}^{*}\right)^{\top}$, and so $F_{n}=F_{n}^{\top}$. Also, $F_{n} F_{n}^{*}=I_{n}$ (i.e., it is unitary). It is possible to verify the following diagonalization formula for $\Pi_{n}$.

Theorem 2.7 (Theorem 3.2.1 of Davis, 1994): $\Pi_{n}=F_{n}^{*} \Omega_{n} F_{n}$.
Theorem 2.8 (after Theorem 3.2.2 of Davis, 1994): If $C$ is an $n \times n$ circulant matrix, then it is diagonalizable by the Fourier matrix $F_{n}$. More precisely, the circulant $C=F_{n}^{*} \Lambda_{C} F_{n}$, where $\Lambda_{C}=\operatorname{diag}\left(p_{C}(1), p_{C}(\omega), \ldots, p_{C}\left(\omega^{n-1}\right)\right)$.

This result is easy to check by using Theorem 2.7 and computing

$$
\begin{aligned}
F_{n} C F_{n}^{*} & =F_{n}\left(p_{C}\left(\Pi_{n}\right)\right) F_{n}^{*} \\
& =c_{1} F_{n} F_{n}^{*}+c_{2} F_{n} \Pi_{n} F_{n}^{*}+c_{3} F_{n} \Pi_{n}^{2} F_{n}^{*}+\cdots+c_{n} F_{n} \Pi_{n}^{n-1} F_{n}^{*} \\
& =c_{1} I_{n}+c_{2} \Omega_{n}+c_{3} \Omega_{n}^{2}+\cdots+c_{n} \Omega_{n}^{n-1} \\
& =\operatorname{diag}\left(p_{C}(1), p_{C}(\omega), p_{C}\left(\omega^{2}\right), \ldots, p_{C}\left(\omega^{n-1}\right)\right) .
\end{aligned}
$$

Corollary 2.1: The eigenvalues of $C$ are $\lambda_{i}=p_{C}\left(\omega^{i-1}\right)$, where $i=1,2, \ldots, n$.

## Block Circulant Matrices

First, let $A=\left[a_{i j}\right]$ and $B=\left[b_{i j}\right]$ be $m \times n$ and $p \times q$ matrices, respectively. Then, the Kronecker product of $A$ and $B$ is the $m p \times n q$ matrix

$$
A \otimes B:=\left[\begin{array}{ccc}
a_{11} B & \cdots & a_{1 n} B \\
\vdots & & \vdots \\
a_{m 1} B & \cdots & a_{m n} B
\end{array}\right]
$$

A useful property of the Kronecker product is that $A C \otimes B D=(A \otimes B)(C \otimes D)$.
Let $A_{1}, A_{2}, \ldots, A_{n}$ be $m \times m$ matrices. A block circulant matrix of type $(m, n)$ is a $m n \times m n$ matrix of the form

$$
A=\left[\begin{array}{cccc}
A_{1} & A_{2} & \cdots & A_{n} \\
A_{n} & A_{1} & \cdots & A_{n-1} \\
\vdots & \vdots & & \vdots \\
A_{2} & A_{3} & \cdots & A_{1}
\end{array}\right]=: \operatorname{circ}\left(A_{1}, A_{2}, \ldots, A_{n}\right) .
$$

Note that $A$ is not necessarily circulant (only block circulant). Let $\mathcal{B C}(m, n)$ designate the set of block circulant matrices of type ( $m, n$ ). Similar to circulant matrices, every block circulant $A$ can be represented by

$$
\begin{equation*}
A=\operatorname{circ}\left(A_{1}, A_{2}, \ldots, A_{n}\right)=\sum_{k=0}^{n-1}\left(\Pi_{n}^{k} \otimes A_{k+1}\right) \tag{2.9}
\end{equation*}
$$

Theorem 2.9 (after Theorem 5.6.4 of Davis, 1994): If $A \in \mathcal{B C}(m, n)$, then it has the form $A=\left(F_{n} \otimes I_{m}\right)^{*} \operatorname{diag}\left(D_{1}, D_{2}, \ldots, D_{n}\right)\left(F_{n} \otimes I_{m}\right)$, where

$$
\left[\begin{array}{c}
D_{1}  \tag{2.10}\\
D_{2} \\
\vdots \\
D_{n}
\end{array}\right]=\left(\sqrt{n} F_{n}^{*} \otimes I_{m}\right)\left[\begin{array}{c}
A_{1} \\
A_{2} \\
\vdots \\
A_{n}
\end{array}\right]
$$

Proof (sketch): We know that $A \in \mathcal{B C}(m, n)$ if and only if it is of the form
(2.9). Each element of this sum can be written as

$$
\begin{aligned}
\Pi_{n}^{k} \otimes A_{k+1} & =\left(F_{n}^{*} \Omega_{n}^{k} F_{n}\right) \otimes I_{m}^{*} A_{k+1} I_{m} \\
& =\left(F_{n} \otimes I_{m}\right)^{*}\left(\Omega_{n}^{k} \otimes A_{k+1}\right)\left(F_{n} \otimes I_{m}\right)
\end{aligned}
$$

which yields

$$
A=\left(F_{n} \otimes I_{m}\right)^{*}\left(\sum_{k=0}^{n-1} \Omega_{n}^{k} \otimes A_{k+1}\right)\left(F_{n} \otimes I_{m}\right)
$$

It can be shown by explicit computation (Davis, 1994, p. 180) that

$$
\sum_{k=0}^{n-1} \Omega_{n}^{k} \otimes A_{k+1}=\operatorname{diag}\left(D_{1}, D_{2}, \ldots, D_{n}\right)
$$

where the diagonal blocks are given by (2.10).

Theorem 2.9 furnishes a way to block diagonalize block circulant matrices. By using the fact that $\left(F_{n} \otimes I_{m}\right)^{*}\left(F_{n} \otimes I_{m}\right)=I_{m n}$, one obtains

$$
\Lambda_{M}=\operatorname{diag}\left(D_{1}, D_{2}, \ldots, D_{n}\right)=\left(F_{n} \otimes I_{m}\right) A\left(F_{n} \otimes I_{m}\right)^{*}
$$

### 2.2.4 Invariant Subspaces

With reference to Halmos (1958), this section reviews the notion of an invariant subspace. Consider a linear transformation $A: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$. Then a subspace $\mathcal{S} \subset$ $\mathbb{R}^{n}$ is said to be invariant under $A$ (or $A$-invariant for short) if $x \in \mathcal{S}$ implies that $A x \in \mathcal{S}$. An $A$-invariant subspace $\mathcal{S}$ induces a linear transformation $A_{\mathcal{S}}: \mathcal{S} \rightarrow \mathcal{S}$ called the restriction of $A$ to $\mathcal{S}$. Moreover, the same subspace $\mathcal{S}$ induces a linear transformation in the quotient space $\mathbb{R}^{n} / \mathcal{S}$, denoted $A_{\mathcal{S}}^{\star}: \mathbb{R}^{n} / \mathcal{S} \rightarrow \mathbb{R}^{n} / \mathcal{S}$.

Suppose $\mathcal{S} \subset \mathbb{R}^{n}$ has dimension $m<n$. Then, there exists a canonical basis that gives $A$ the upper-triangular matrix form

$$
\left[\begin{array}{cc}
A_{\mathcal{S}} & * \\
0_{(n-m) \times m} & A_{\mathcal{S}}^{\star}
\end{array}\right] .
$$

This form can be obtained by choosing a basis $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ for $\mathbb{R}^{n}$ so that the
elements $\left\{x_{1}, x_{2}, \ldots, x_{m}\right\}$ are in $\mathcal{S}$, while $\left\{x_{m+1}, x_{m+2}, \ldots, x_{n}\right\}$ are not.

### 2.2.5 Proper Changes of Coordinates

Let $\xi$ and $\varphi$ be vectors in $\mathbb{R}^{n}$. Consider a proper change of coordinates given by $\varphi=\Phi(\xi)$, which transforms a set of state equations from $\xi$-coordinates to $\varphi$-coordinates. By the phrase "proper change of coordinates" it is meant that the map $\Phi: \mathcal{D} \rightarrow \mathbb{R}^{n}$, where $\mathcal{D}$ is an open set of $\mathbb{R}^{n}$, is a diffeomorphism (i.e., a continuously differentiable map with a continuously differentiable inverse; or more explicitly, $\Phi$ is bijective and both $\Phi$ and $\Phi^{-1}$ are of class $\mathcal{C}^{1}$ ). The following standard theorem gives a sufficient condition for a map $\Phi$ to be a diffeomorphism (this version has been adapted from Isidori, 1995, Appendix A).

Theorem 2.10 (Inverse Function Theorem): Let $\mathcal{D}$ be an open set of $\mathbb{R}^{n}$ and $\Phi: \mathcal{D} \rightarrow \mathbb{R}^{n}$ be a $\mathcal{C}^{\infty}$ mapping. If the Jacobian $[\partial \Phi / \partial \xi]_{\bar{\xi}}$ is nonsingular at some $\bar{\xi} \in \mathcal{D}$, then there exists an open neighbourhood $\mathcal{U}$ of $\bar{\xi} \in \mathcal{D}$ such that $\mathcal{V}=\Phi(\mathcal{U})$ is open in $\mathbb{R}^{n}$ and the restriction of $\Phi$ to $\mathcal{U}$ is a diffeomorphism onto $\mathcal{V}$.

### 2.2.6 Alternate Proof of Theorem 2.5

Finally, this section offers an alternate proof of Theorem 2.5. This alternate proof may not be as direct as the one of Section 2.2.2, originally given by Bruckstein et al. (1991). Moreover, there may be other approaches that are more efficient than the one offered here. Rather, the purpose of this section is to introduce a specific perspective that will prove useful in Chapters 3 and 4.

As previously noted, one needs only to study the linear system (2.5). Notice that $M$ is a circulant matrix of the form

$$
M=\operatorname{circ}(-a, a, 0, \ldots, 0)
$$

Therefore, the system's equilibrium point $\bar{x}=\left(\bar{x}_{1}, \bar{x}_{2}, \ldots, \bar{x}_{n}\right)$ must satisfy $\bar{x}_{1}=$ $\bar{x}_{2}=\cdots=\bar{x}_{n}$. Moreover, due to cyclic pursuit

$$
\begin{equation*}
\sum_{i=1}^{n} \dot{x}_{i}(t)=0 \Longrightarrow \sum_{i=1}^{n} x_{i}(t) \equiv c \text { for all } t \geq 0 \tag{2.11}
\end{equation*}
$$

where the constant $c$ is determined by the initial locations via $c=\sum_{i=1}^{n} x_{i}(0)$. In other words, the equilibrium point $\bar{x}$ must be the centroid; that is

$$
\bar{x}_{i}=\frac{1}{n} \sum_{i=1}^{n} x_{i}(0) .
$$

The centroid is also stationary, by (2.11).
Hence, for every initial condition $x(0)$, the system (2.5) is constrained to evolve on an $M$-invariant affine subspace $\mathcal{S}_{c} \subset \mathbb{R}^{n}$ defined by

$$
\mathcal{S}_{c}=\left\{x \in \mathbb{R}^{n}:\left[\begin{array}{llll}
1 & 1 & \cdots & 1
\end{array}\right] x=c\right\} .
$$

This affine subspace $\mathcal{S}_{c}$ has dimension $n-1$. Consider a change of coordinates given by $\hat{x}=x-c / n \times(1,1, \ldots, 1)$. In these coordinates, the dynamics (2.5) remain the same but the centroid of the points $\hat{x}_{1}(0), \hat{x}_{2}(0), \ldots, \hat{x}_{n}(0)$ is the origin. Moreover, in these coordinates $\mathcal{S}_{c}$ is a subspace of $\mathbb{R}^{n}$. Therefore, there exists an induced linear transformation in the quotient space $M_{\mathcal{S}_{c}}^{\star}: \mathbb{R}^{n} / \mathcal{S}_{c} \rightarrow \mathbb{R}^{n} / \mathcal{S}_{c}$ whose eigenvalues do not influence the stability of the origin for $\hat{x}(0) \in \mathcal{S}_{c}$. Hence, there exists a change of basis that transforms $M$ into the form

$$
\left[\begin{array}{cc}
M_{\mathcal{S}_{c}} & * \\
0_{1 \times(n-1)} & M_{\mathcal{S}_{c}}^{\star}
\end{array}\right] .
$$

Consider another change of coordinates $\tilde{x}=P \hat{x}$, given by

$$
\tilde{x}_{1}=\hat{x}_{1}, \tilde{x}_{2}=\hat{x}_{2}, \ldots, \tilde{x}_{n-1}=\hat{x}_{n-1}, \tilde{x}_{n}=\sum_{i=1}^{n} \hat{x}_{i}
$$

which yields

$$
\dot{\tilde{x}}=\left[\begin{array}{ccccc}
1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \ddots & 0 & 0 \\
0 & \cdots & 0 & 1 & 0 \\
1 & 1 & \cdots & \cdots & 1
\end{array}\right] M P^{-1} \tilde{x}
$$

and subsequently

$$
\begin{aligned}
\dot{\tilde{x}} & =\left[\begin{array}{cccccc}
-a & a & 0 & \cdots & \cdots & 0 \\
0 & -a & a & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & 0 & 0 \\
0 & \cdots & 0 & -a & a & 0 \\
0 & \cdots & \cdots & 0 & -a & a \\
0 & \cdots & \cdots & \cdots & \cdots & 0
\end{array}\right]\left[\begin{array}{ccccc}
1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \ddots & 0 & 0 \\
0 & \cdots & 0 & 1 & 0 \\
-1 & -1 & \cdots & -1 & 1
\end{array}\right] \tilde{x} \\
& =\left[\begin{array}{ccccc|c}
-a & a & 0 & \cdots & \cdots & 0 \\
0 & -a & a & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & 0 & 0 \\
0 & \cdots & 0 & -a & a & 0 \\
-a & \cdots & \cdots & -a & -2 a & a \\
\hline 0 & \cdots & \cdots & \cdots & 0 & 0
\end{array}\right] \tilde{x}=\left[\begin{array}{cc}
M_{\mathcal{S}_{c}} & * \\
0_{1 \times(n-1)} & M_{\mathcal{S}_{c}}^{\star}
\end{array}\right] \tilde{x} .
\end{aligned}
$$

Therefore, when determining the stability of $\bar{x}$ we can disregard exactly one zero eigenvalue and conclude stability based on the remaining $n-1$ eigenvalues of $M$.
$M$ is circulant and its representer is $p_{M}(\lambda)=a(\lambda-1)$. And so, by Corollary 2.1, the eigenvalues of $M$ must be given by $\lambda_{i}=p_{M}\left(\omega^{i-1}\right)$; in particular

$$
\begin{aligned}
\lambda_{1} & =a\left(\omega^{0}-1\right)=0 \\
\lambda_{2} & =a\left(\omega^{1}-1\right)=a\left(e^{j 2 \pi / n}-1\right) \\
\lambda_{3} & =a\left(\omega^{2}-1\right)=a\left(e^{j 4 \pi / n}-1\right) \\
& \vdots \\
\lambda_{n} & =a\left(\omega^{n-1}-1\right)=a\left(e^{j 2(n-1) \pi / n}-1\right) .
\end{aligned}
$$

Alternatively, in complex number form

$$
\lambda_{i}=a\left[\cos \left(\frac{2 \pi(i-1)}{n}\right)-1\right]+j a \sin \left(\frac{2 \pi(i-1)}{n}\right),
$$

with $i=1,2, \ldots, n$. Thus, for all $a>0, M$ always has exactly one zero eigenvalue, while the remaining $n-1$ eigenvalues lie strictly in the left-half complex plane.

Indeed, the fixed point in Figure 2.1 to which the six agents converge corresponds to the centroid of the agents $z_{1}(t), z_{2}(t), \ldots, z_{6}(t)$, for all $t \geq 0$.

### 2.3 Vehicles in Pursuit

The linear pursuit problem is itself quite interesting and has some beautiful extensions (e.g., see Bruckstein et al., 1991; Bruckstein, Sapiro, and Shaked, 1995). However, the principal focus of this thesis is on a nonlinear analog involving wheeled vehicles. Suppose the above linear cyclic pursuit scenario is extended to one in which each agent is instead a single-wheeled vehicle called a kinematic unicycle, as illustrated in Figure 2.5, with nonlinear state model

$$
\left[\begin{array}{c}
\dot{x}_{i}  \tag{2.12}\\
\dot{y}_{i} \\
\dot{\theta}_{i}
\end{array}\right]=\left[\begin{array}{cc}
\cos \theta_{i} & 0 \\
\sin \theta_{i} & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{c}
v_{i} \\
\omega_{i}
\end{array}\right]=G\left(\theta_{i}\right) u_{i}
$$

where $\left(x_{i}, y_{i}\right) \in \mathbb{R}^{2}$ denotes the $i$-th vehicle's position, $\theta_{i} \in \mathbb{S}^{1}$ is the vehicle's orientation, and $u_{i}=\left(v_{i}, \omega_{i}\right) \in \mathbb{R}^{2}$ are control inputs.


Figure 2.5: Top view of a unicycle

The unicycle in Figure 2.5 is constrained to move in the direction $\theta$ (i.e., there is no lateral slipping of the wheel). This constraint happens to be nonholonomic in nature, meaning that the associated constraint equation cannot be integrated so as to reduce the dimension of the state-space. A more detailed review of nonholonomic systems can be found in Appendix B.1.

In the aforementioned scenario, the unicycles will not generally be able to head towards their designated targets at each instant. Instead, depending on the allowed control energy, each vehicle will require some finite time to steer itself towards its preassigned leader. What trajectories can be generated?

Let $r_{i}$ denote the distance between vehicle $i$ and vehicle $i+1$, and let $\alpha_{i}$ be the difference between the $i$-th vehicle's heading and the heading that would take it directly towards its prey, vehicle $i+1$ (see Figure 3.1 on page 38). In analogy
with the linear control law (2.2), an intuitive pursuit law for (2.12) is to assign vehicle $i$ 's forward speed $v_{i}$ in proportion to the distance error $r_{i}$, while at the same time assigning its angular speed $\omega_{i}$ in proportion to the heading error $\alpha_{i}$. In the next three central chapters of this thesis, the possible equilibrium formations for multivehicle systems of this sort are studied in detail.

## Chapter 3

## Fixed-speed Pursuit Formations

Following the introduction given in Section 2.3 to vehicles in cyclic pursuit, the simplest case is perhaps when the vehicles all travel at the same constant speed. Hence, the purpose of this chapter is to study the achievable behaviours for fixedspeed unicycles in cyclic pursuit. The chapter begins by revealing the complete set of possible equilibrium formations and their geometry. Next, a technique for local stability analysis is presented that works for any number of vehicles. Finally, limit cycle behaviours are briefly discussed.

### 3.1 Nonlinear Equations of Pursuit

With reference to the kinematic unicycle model (2.12) on page 35 , recall that $r_{i}$ denotes the distance between vehicle $i$ and vehicle $i+1$, and that $\alpha_{i}$ is the difference between the $i$-th vehicle's heading and the one that would take it directly towards its prey, vehicle $i+1$. These variables are graphically depicted in Figure 3.1.

As in the previous chapter, and henceforth, all indices $i+j$, with $i, j \in \mathbb{Z}$, should be evaluated modulo $n$.

### 3.1.1 Transformation to Relative Coordinates

Firstly, it is useful to consider a transformation to coordinates involving the variables $r_{i}$ and $\alpha_{i}$, which shall be referred to as relative coordinates. Let $q_{i}=$ $\left(x_{i}, y_{i}, \theta_{i}\right)$ and define $\tilde{q}_{i}:=R\left(\theta_{i+1}\right)\left(q_{i}-q_{i+1}\right)$, where $R(\theta)$ is the rotation matrix

$$
R(\theta)=\left[\begin{array}{ccc}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{array}\right]
$$



Figure 3.1: Relative coordinates with vehicle $i$ in pursuit of $i+1$; see (3.2)
which yields dynamics

$$
\dot{\tilde{q}}_{i}=G\left(\tilde{\theta}_{i}\right) u_{i}-\left[\begin{array}{ll}
1 & 0  \tag{3.1}\\
0 & 0 \\
0 & 1
\end{array}\right] u_{i+1}+\left[\begin{array}{c}
\omega_{i+1} \tilde{y}_{i} \\
-\omega_{i+1} \tilde{x}_{i} \\
0
\end{array}\right]
$$

In these coordinates, vehicle $i$ views itself in a coordinate frame centred at vehicle $i+1$ and aligned with vehicle $i+1$ 's heading. As per Figure 3.1, define

$$
\begin{align*}
r_{i} & =\sqrt{\tilde{x}_{i}^{2}+\tilde{y}_{i}^{2}}  \tag{3.2a}\\
\alpha_{i} & =\arctan \left(\frac{\tilde{y}_{i}}{\tilde{x}_{i}}\right)+\pi-\tilde{\theta}_{i}  \tag{3.2b}\\
\beta_{i} & =\tilde{\theta}_{i}-\pi, \tag{3.2c}
\end{align*}
$$

with $r_{i} \in \mathbb{R}_{+}-\{0\}$ and $\alpha_{i}, \beta_{i} \in \mathbb{R}$. When $\tilde{x}_{i}=0$, it is assumed that $\arctan \left(\tilde{y}_{i} / 0\right)$ evaluates to $\pm \pi / 2$, depending on the sign of $\tilde{y}_{i}$. By taking the derivative of (3.2a),

$$
\dot{r}_{i}=\frac{\tilde{x}_{i} \dot{\tilde{x}}_{i}+\tilde{y}_{i} \dot{\tilde{y}}_{i}}{\sqrt{\tilde{x}_{i}^{2}+\tilde{y}_{i}^{2}}}
$$

By substituting (3.1), this yields

$$
\dot{r}_{i}=\frac{1}{r_{i}}\left(v_{i}\left(\tilde{x}_{i} \cos \tilde{\theta}_{i}+\tilde{y}_{i} \sin \tilde{\theta}_{i}\right)-\tilde{x}_{i} v_{i+1}\right) .
$$

But, from Figure 3.1 and (3.2c) one has that $\tilde{x}_{i} / r_{i}=\cos \left(\alpha_{i}+\beta_{i}\right), \tilde{y}_{i} / r_{i}=\sin \left(\alpha_{i}+\right.$
$\beta_{i}$ ), and $\tilde{\theta}_{i}=\beta_{i}+\pi$. Thus, one obtains

$$
\begin{aligned}
\dot{r}_{i} & =-v_{i}\left(\cos \left(\alpha_{i}+\beta_{i}\right) \cos \beta_{i}+\sin \left(\alpha_{i}+\beta_{i}\right) \sin \beta_{i}\right)-v_{i+1} \cos \left(\alpha_{i}+\beta_{i}\right) \\
& =-v_{i} \cos \alpha_{i}-v_{i+1} \cos \left(\alpha_{i}+\beta_{i}\right) .
\end{aligned}
$$

The derivative with respect to time of (3.2c) is $\dot{\beta}_{i}=\omega_{i}-\omega_{i+1}$. Finally, by taking the derivative with respect to time of $(3.2 \mathrm{~b})$, it holds that

$$
\dot{\alpha}_{i}=\frac{\tilde{x}_{i} \dot{\tilde{y}}_{i}-\tilde{y}_{i} \dot{\tilde{x}}_{i}}{\tilde{x}_{i}^{2}+\tilde{y}_{i}^{2}}-\dot{\tilde{\theta}}_{i} .
$$

By substituting (3.1) and the derivative in time of (3.2c), this yields

$$
\begin{aligned}
\dot{\alpha}_{i} & =\frac{1}{r_{i}^{2}}\left(v_{i}\left(\tilde{x}_{i} \sin \tilde{\theta}_{i}-\tilde{y}_{i} \cos \tilde{\theta}_{i}\right)+v_{i+1} \tilde{y}_{i}-\omega_{i+1}\left(\tilde{x}_{i}^{2}+\tilde{y}_{i}^{2}\right)\right)-\dot{\beta}_{i} \\
& =\frac{1}{r_{i}}\left(v_{i}\left(\sin \left(\alpha_{i}+\beta_{i}\right) \cos \beta_{i}-\cos \left(\alpha_{i}+\beta_{i}\right) \sin \beta_{i}\right)+v_{i+1} \sin \left(\alpha_{i}+\beta_{i}\right)\right)-\omega_{i} \\
& =\frac{1}{r_{i}}\left(v_{i} \sin \alpha_{i}+v_{i+1} \sin \left(\alpha_{i}+\beta_{i}\right)\right)-\omega_{i} .
\end{aligned}
$$

Therefore, in the relative coordinates (3.2) one obtains $n$ vehicle subsystems with kinematic equations of motion

$$
\begin{align*}
\dot{r}_{i} & =-v_{i} \cos \alpha_{i}-v_{i+1} \cos \left(\alpha_{i}+\beta_{i}\right)  \tag{3.3a}\\
\dot{\alpha}_{i} & =\frac{1}{r_{i}}\left(v_{i} \sin \alpha_{i}+v_{i+1} \sin \left(\alpha_{i}+\beta_{i}\right)\right)-\omega_{i}  \tag{3.3b}\\
\dot{\beta}_{i} & =\omega_{i}-\omega_{i+1}, \tag{3.3c}
\end{align*}
$$

for $i=1,2, \ldots, n$, where the indices $i+1$ are evaluated modulo $n$, and with states $r_{i} \in \mathbb{R}_{+}-\{0\}$ and $\alpha_{i}, \beta_{i} \in \mathbb{R}$, and inputs $v_{i}, \omega_{i} \in \mathbb{R}$.

System (3.3) describes the relationship between vehicle $i$ and the one that it is pursuing, $i+1$, in relative coordinates. Observe that the transformation from $q_{i}, i=1,2, \ldots, n$, into $\xi_{i}:=\left(r_{i}, \alpha_{i}, \beta_{i}\right)$ is not proper, which is not surprising since any reference to a global coordinate frame has been removed. Analytically, an immediate example of why this coordinates transformation is not proper lies in the fact that, from (3.2c), the relative coordinates are constrained by $\sum_{i=1}^{n} \beta_{i}=-n \pi$. A more detailed discussion about coordinate constraints is offered in Section 3.1.3.

### 3.1.2 Fixed-speed Pursuit

Once more, in analogy with the linear control law (2.2), an intuitive fixed-speed pursuit law for (2.12) is to assign vehicle $i$ 's angular speed $\omega_{i}$ in proportion to its heading error $\alpha_{i}$. In other words, consider $n$ unicycles, each with control inputs

$$
\begin{equation*}
v_{i}=v_{R} \text { and } \omega_{i}=k \alpha_{i} \tag{3.4}
\end{equation*}
$$

where $k, v_{R}>0$ are constants. Substituting these controls into (3.3) gives a system of $n$ cyclically interconnected and identical subsystems

$$
\begin{align*}
\dot{r}_{i} & =-v_{R}\left(\cos \alpha_{i}+\cos \left(\alpha_{i}+\beta_{i}\right)\right)  \tag{3.5a}\\
\dot{\alpha}_{i} & =\frac{v_{R}}{r_{i}}\left(\sin \alpha_{i}+\sin \left(\alpha_{i}+\beta_{i}\right)\right)-k \alpha_{i}  \tag{3.5b}\\
\dot{\beta}_{i} & =k\left(\alpha_{i}-\alpha_{i+1}\right) \tag{3.5c}
\end{align*}
$$

which is examined in the remaining sections of this chapter.

### 3.1.3 Pursuit Graphs and Coordinate Constraints

At each instant, regardless of the control law, the multivehicle system's geometric configuration in $\mathbb{R}^{2}$ can be described by a special digraph called a pursuit graph.

Definition 3.1: A pursuit graph $\Gamma_{t}$ at time $t \geq 0$ is a pair $\left(\mathcal{V}_{t}, \mathcal{E}_{t}\right)$ such that
(i) $\mathcal{V}_{t}$ is a finite set of vertices, $\left|\mathcal{V}_{t}\right|=n$, where each vertex $z_{i}(t)=\left(x_{i}(t), y_{i}(t)\right) \in$ $\mathbb{R}^{2}, i \in\{1, \ldots, n\}$, represents the position of vehicle $i$ in the plane; and,
(ii) $\mathcal{E}_{t}$ is a finite set of directed edges, $\left|\mathcal{E}_{t}\right|=n$, where each edge $e_{i}(t): \mathcal{V}_{t} \times \mathcal{V}_{t} \rightarrow$ $\mathbb{R}^{2}, i \in\{1, \ldots, n\}$, is the vector connecting $z_{i}(t)$ to its prey, $z_{i+1}(t)$.

Note that $r_{i}(t)=\left\|e_{i}(t)\right\|_{2}$, where $\|\cdot\|_{2}$ denotes the standard Euclidean norm. Furthermore, note that pursuit graphs store within their vertices each vehicle's position in the plane. In other words, $e_{i}(t)=z_{i+1}(t)-z_{i}(t)$ such that $\sum_{i=1}^{n} e_{i}(t) \equiv 0$. Let $\xi:=\left(\xi_{1}, \xi_{2}, \ldots, \xi_{n}\right)$, where $\xi_{i}=\left(r_{i}, \alpha_{i}, \beta_{i}\right)$. Then, by choosing a coordinate frame attached to, say, vehicle 1 and oriented with vehicle 1's heading, this con-
dition corresponds to constraints on the system described by the equations

$$
\begin{aligned}
& g_{1}(\xi)=r_{1} \sin \alpha_{1}+ r_{2} \sin \left(\alpha_{2}+\pi-\beta_{1}\right)+r_{3} \sin \left(\alpha_{3}+2 \pi-\beta_{1}-\beta_{2}\right)+\cdots \\
& \cdots+r_{n} \sin \left(\alpha_{n}+(n-1) \pi-\beta_{1}-\beta_{2}-\cdots-\beta_{n-1}\right)=0 \\
& g_{2}(\xi)=r_{1} \cos \alpha_{1}+ r_{2} \cos \left(\alpha_{2}+\pi-\beta_{1}\right)+r_{3} \cos \left(\alpha_{3}+2 \pi-\beta_{1}-\beta_{2}\right)+\cdots \\
& \cdots+r_{n} \cos \left(\alpha_{n}+(n-1) \pi-\beta_{1}-\beta_{2}-\cdots-\beta_{n-1}\right)=0
\end{aligned}
$$

For vehicles numbered 1 and 2, Figure 3.2 helps to illustrate how these constraint equations arise. Moreover, one has the constraint

$$
\sum_{i=1}^{n} \dot{\beta}_{i}(t)=0 \Longrightarrow \sum_{i=1}^{n} \beta_{i}(t) \equiv c
$$

for all $t \geq 0$, where the constant $c=-n \pi$ by the definition (3.2) for $\beta_{i}$. This yields a final constraint equation

$$
g_{3}(\xi)=\sum_{i=1}^{n} \beta_{i}+n \pi=0 \bmod 2 \pi
$$



Figure 3.2: Depiction of coordinates for vehicles 1 and 2
These constraints are essential to the equilibrium and stability analyses that follow in Sections 3.2 and 3.3, respectively.

### 3.1.4 Sample Simulations

Preliminary computer simulations suggest the possibility of achieving circular trajectories in the plane. Figure 3.3a shows simulation results for a system of $n=5$ vehicles, initially positioned at random, subject to the control law (3.4) with $k=1$. Notice how the vehicles converge to equally spaced motion around a circle of fixed radius with a pursuit graph that is similar to a regular pentagon. Figure 3.3 b illustrates what happens if the gain $k$ is increased from 1 to 2 , suggesting that the circle's radius may be inversely proportional to the gain $k>0$.

### 3.2 Generalized Equilibria

In this section, the term equilibria is used to describe the set of configurations for which the vehicles have fixed relative pose. Thus, to an observer aboard any one of the vehicles, all the other vehicles would appear stationary. These equilibria are the equilibrium points of the $n$ interconnected subsystems (3.5).

In order to characterize the possible equilibrium formations for the multivehicle system (3.5), an adequate description of the system's pursuit graph at equilibrium is required. The following definition for a regular polygon with coplanar vertices has been adapted from Coxeter (1948) to allow for vertices that are not necessarily distinct and for the directed edges of a pursuit graph.

Definition 3.2 (after Coxeter, 1948, p. 93): Let $n$ and $d<n$ be positive integers so that $p:=n / d>1$ is a rational number. Let $R$ be the positive rotation in the plane, about the origin, through angle $2 \pi / p$ and let $z_{1} \neq 0$ be a point in the plane. Then, the points $z_{i+1}=R z_{i}, i=1, \ldots, n-1$, and edges $e_{i}=z_{i+1}-z_{i}, i=1, \ldots, n$, define a generalized regular polygon, which is denoted $\{p\}$.

By this definition, $\{p\}$ can be interpreted as a directed graph with vertices $z_{i}$ (not necessarily distinct) connected by edges $e_{i}$ as determined by the ordering of points. The planar figure $\{p\}$ is called positively oriented if $d \leq n / 2$ and, conversely, negatively oriented if $d>n / 2$.

Since $p$ is rational, the period ${ }^{1}$ of $R$ is finite and, when $n$ and $d$ are coprime, this definition is equivalent to the well known definition of a regular polygon as a

[^3]

Figure 3.3: Five unit-speed vehicles subject to control law (3.4)
polygon that is both equilateral and equiangular. Moreover, when $d=1,\{p=n\}$ is an ordinary regular polygon (i.e., its edges do not cross one another). However, when $d>1$ is coprime to $n,\{p\}$ is a star polygon since its sides intersect at certain extraneous points, which are not included among the vertices (Coxeter, 1948, pp. 93-94). If $n$ and $d$ have a common factor $m>1$, then $\{p\}$ has $\tilde{n}=n / m$ distinct vertices and $\tilde{n}$ edges traversed $m$ times. The trivial case when $d=n$ has not been included because it corresponds to the geometrically uninteresting situation when the vehicles are all coincident.

Figure 3.4 illustrates some example possibilities for $\{p\}$ when $n=9$. In the first instance, $\{9 / 1\}$ is an ordinary polygon. In the second instance, $\{9 / 2\}$ is a star polygon since 9 and 2 are coprime. In the third instance, the edges of $\{9 / 3\}$ traverse a $\{3 / 1\}$ polygon 3 times, since $m=3$ is a common factor of 9 and 3 .


Figure 3.4: Example generalized regular polygons

Lemma 3.1 (Coxeter, 1948, p. 94): The internal angle $\psi$ at each vertex of a generalized regular polygon $\{n / d\}$ is

$$
\psi=\pi\left(1-\frac{2 d}{n}\right)
$$

Note that the sign of $\psi$ determines whether $\{p\}$ is positively or negatively oriented. The following important result reveals the possible equilibrium formations for $n$ kinematic unicycles in cyclic pursuit, subject to (3.4).

Theorem 3.1: The $3 n$-dimensional system (3.5) has $2(n-1)$ equilibrium points, described as follows: the $r_{i}$ are all equal, $r_{i}=\bar{r}$; likewise, $\alpha_{i}=\bar{\alpha}$ and $\beta_{i}=\bar{\beta}$ for
all $i \in\{1,2, \ldots, n\}$. The $2(n-1)$ values of $\bar{r}, \bar{\alpha}$, and $\bar{\beta}$ are given by

$$
\begin{aligned}
\bar{\alpha} & = \pm \frac{\pi d}{n}, d=1,2, \ldots, n-1 \\
\bar{\beta} & =\pi-2 \bar{\alpha} \\
\bar{r} & =\frac{2 v_{R}}{k \bar{\alpha}} \sin \bar{\alpha} .
\end{aligned}
$$

Finally, at each equilibrium point, the related pursuit graph is a generalized regular polygon $\{n / d\}$, with $d \in\{1,2, \ldots, n-1\}$.

Proof: For $\dot{\beta}_{i}=0,(3.5 \mathrm{c})$ yields $\alpha_{i}=\alpha_{i+1}$. Let $\bar{\alpha} \equiv \alpha_{i}$ at equilibrium. From the equilibrium condition $\dot{r}_{i}=0$ of (3.5a), $\cos \bar{\alpha}=-\cos \left(\bar{\alpha}+\beta_{i}\right)$, which implies that either $\beta_{i}=\pi$ or $\beta_{i}=\pi-2 \bar{\alpha}$. However, at equilibrium (3.5b) yields $\bar{\alpha}=0$ when $\beta_{i}=\pi$. Therefore, it is not simultaneously possible that $\beta_{i}=\pi$ and $\beta_{i+1}=\pi-2 \bar{\alpha}$ for some $\bar{\alpha} \neq 0$ at equilibrium, implying that $\beta_{i}=\beta_{i+1}$ for all $i$. Let $\bar{\beta} \equiv \beta_{i}$ at equilibrium. Again, from the condition $\dot{\alpha_{i}}=0$ of (3.5b),

$$
\begin{equation*}
r_{i}=\frac{v_{R}}{k \bar{\alpha}}(\sin \bar{\alpha}+\sin (\bar{\alpha}+\bar{\beta})) \text { for all } i \in\{1,2, \ldots, n\} . \tag{3.6}
\end{equation*}
$$

Therefore, $r_{i}=r_{i+1}$. Let $\bar{r} \equiv r_{i}$ at equilibrium.
For vehicles in cyclic pursuit, the system's pursuit graph $\Gamma_{t}=\left(\mathcal{V}_{t}, \mathcal{E}_{t}\right)$ has $\sum_{i=1}^{n} e(t) \equiv 0$. In particular, the constraint

$$
\begin{align*}
g_{2}(\bar{\xi})= & \bar{r}(\cos \bar{\alpha}+\cos (\bar{\alpha}+\pi-\bar{\beta})+\cdots  \tag{3.7}\\
& \cdots+\cos (\bar{\alpha}+(n-1)(\pi-\bar{\beta})))=0
\end{align*}
$$

of Section 3.1.3 must hold. However, when $\bar{\beta}=\pi$, (3.5b) implies that $\bar{\alpha}=0$, which subsequently implies that the left-hand side of (3.7) equals $\bar{r} n \neq 0$. Thus, $\bar{\beta}=\pi($ with $\bar{\alpha}=0)$ is not feasible for vehicles in cyclic pursuit.

Suppose $\bar{\alpha}>0$. Since $r_{i}=r_{i+1}$, the system's pursuit graph $\Gamma_{t}$ is equilateral (i.e., $\left\|e_{i}\right\|_{2}=\left\|e_{i+1}\right\|_{2}$ ). Let $\psi_{i}$ be the internal angle at each vertex of the pursuit graph. The pursuit graph is equiangular (i.e., $\psi_{i}=\psi_{i+1}$ ) since it can be checked using the geometry of Figure 3.2 that the internal angle at each vertex is given by $\bar{\psi} \equiv \psi_{i}=\alpha_{i-1}+\beta_{i-1}-\alpha_{i}=\bar{\beta}$ at equilibrium. Therefore, by Definition 3.2 , the pursuit graph must correspond to a generalized regular polygon $\{p\}$. By Lemma 3.1, the internal angle $\bar{\psi}=\bar{\beta}$ at each vertex of the polygon $\{p\}$ gives $\bar{\beta}=\pi(1-2 d / n)$, which together with $\bar{\beta}=\pi-2 \bar{\alpha}$ implies that $\bar{\alpha}=\pi d / n$, where
$d \in\{1,2, \ldots, n-1\}$. Repetition of the above argument for the case when $\bar{\alpha}<0$ yields the remaining $n-1$ equilibrium points. By Definition 3.2, these $2(n-1)$ equilibrium points must satisfy the coordinate constraints of Section 3.1.3.

To clarify why there are $2(n-1)$ equilibria and only $n-1$ pursuit graphs, note that $\bar{\alpha}>0$ and $\bar{\alpha}<0$ correspond to counterclockwise and clockwise rotation of the system's pursuit graph at equilibrium, respectively. Also, notice that there are only $n-1$ distinct values of $\bar{r}$, since $\sin \bar{\alpha} / \bar{\alpha}$ is an even function.

The case when $n$ and $d$ of Theorem 3.1 are not coprime is physically undesirable (e.g., as in the polygon $\{9 / 3\}$ of Figure 3.4) since it requires that multiple vehicles occupy the same point in space. From geometry, it is clear that, for each possible $\{n / d\}$ formation, the equilibrium angle $\bar{\alpha}= \pm \pi d / n$ corresponds to a relative heading angle for each vehicle that points it in a direction that is tangent to the circle circumscribed by the vertices of the corresponding equilibrium polygon.

Corollary 3.1: At equilibrium, the vehicles traverse a circle of radius

$$
\rho=\frac{v_{R} n}{k \pi d} .
$$

Proof: This result can be shown by employing Lemma 3.1, and the fact that, by elementary geometry (Coxeter, 1948, pp. 3, 94), $\bar{r}=2 \rho \cos (\bar{\psi} / 2)$. By solving for the radius $\rho$, one obtains the stated result.

Observe that the possible equilibrium formations depend only on one's choice of gain $k$ and fixed reference speed $v_{R}$; in fact, only on the ratio $v_{R}: k$. Therefore, in what follows it is assumed that $v_{R}=1$, without loss of generality. Consequently, following Corollary 3.1, the radius about which the vehicles travel is determined by the designable parameter $k>0$.

### 3.3 Local Stability Analysis

In general, for a given number of vehicles $n \geq 2$, which $\{n / d\}$ equilibrium polygons are asymptotically stable, and for what values of $k$ ? The contributions of this section include a local stability answer to this question, arrived at through linearization about a general $\{n / d\}$ formation. The solution follows a procedure that
is similar to the alternate proof of Theorem 2.5, given in Section 2.2.6, concerning linear agents, although the details are significantly more involved.

To facilitate notation, define $\tilde{\xi}_{i}:=\xi_{i}-(\bar{r}, \bar{\alpha}, \bar{\beta})$ and let $q:=p^{-1}=d / n$ so that $0<q<1$ and is rational. Write the kinematics of each vehicle subsystem (3.5) more compactly as $\dot{\xi}_{i}=f\left(\xi_{i}, \xi_{i+1}\right)$. Linearizing each $\xi_{i}$ model about an equilibrium point $(\bar{r}, \bar{\alpha}, \bar{\beta})$ gives $n$ identical subsystems of the form $\dot{\tilde{\xi}}_{i}=A \tilde{\xi}_{i}+B \tilde{\xi}_{i+1}$ where

$$
\begin{aligned}
A & =\left.\frac{\partial f\left(\xi_{i}, \xi_{i+1}\right)}{\partial \xi_{i}}\right|_{(\bar{r}, \bar{\alpha}, \bar{\beta})} \\
& =\left[\begin{array}{ccc}
0 & 2 \sin (q \pi) & \sin (q \pi) \\
-\frac{1}{2}(k q \pi)^{2} \csc (q \pi) & -k & -\frac{1}{2} k q \pi \cot (q \pi) \\
0 & k & 0
\end{array}\right] \\
B & =\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -k & 0
\end{array}\right] .
\end{aligned}
$$

If one views the aggregate multivehicle system as

$$
\begin{equation*}
\dot{\xi}=\hat{f}(\xi) \tag{3.8}
\end{equation*}
$$

then its linearization about $\bar{\xi}$ has the form $\dot{\tilde{\xi}}=\hat{A} \tilde{\xi}$, where

$$
\begin{equation*}
\hat{A}=\operatorname{circ}\left(A, B, 0_{3 \times 3}, \ldots, 0_{3 \times 3}\right) . \tag{3.9}
\end{equation*}
$$

### 3.3.1 Submanifolds of $\mathbb{R}^{n}$

Prior to continuing, some fundamental concepts from differential geometry are briefly reviewed. Upon a first reading, this section may be skipped and referred back to as needed. Firstly, a smooth $n$-dimensional manifold has a precise mathematical definition, which necessitates something of a build-up. For the purposes of this thesis, one may adopt a somewhat abridged, perhaps informal, view of the subject. For a more rigorous treatment, the reader is referred to Abraham and Marsden (1977) or the appendix of Isidori (1995).

The differential of a real-valued function $g_{i}(x)$, with $x \in \mathbb{R}^{n}$, is often written
using the condensed notation

$$
d g_{i}(x)=\frac{\partial g_{i}(x)}{\partial x}=\left[\begin{array}{llll}
\frac{\partial g_{i}(x)}{\partial x_{1}} & \frac{\partial g_{i}(x)}{\partial x_{2}} & \cdots & \frac{\partial g_{i}(x)}{\partial x_{n}}
\end{array}\right] .
$$

If the vector function $g(x)$ has dimension $n-m$, then the Jacobian matrix of $g(x)$ evaluated at a point $\bar{x} \in \mathbb{R}^{n}$ is given by

$$
\left.\frac{\partial g(x)}{\partial x}\right|_{\bar{x}}=\left[\begin{array}{cccc}
\partial g_{1}(x) / \partial x_{1} & \partial g_{1}(x) / \partial x_{2} & \cdots & \partial g_{1}(x) / \partial x_{n} \\
\partial g_{2}(x) / \partial x_{1} & \partial g_{2}(x) / \partial x_{2} & \cdots & \partial g_{2}(x) / \partial x_{n} \\
\vdots & \vdots & & \vdots \\
\partial g_{n-m}(x) / \partial x_{1} & \partial g_{n-m}(x) / \partial x_{2} & \cdots & \partial g_{n-m}(x) / \partial x_{n}
\end{array}\right]_{\bar{x}}
$$

The focus here is on smooth $m$-dimensional submanifolds of $\mathbb{R}^{n}$, where $0<$ $m<n$. For the current purposes, one can merely interpret these mathematical objects to be hypersurfaces in $\mathbb{R}^{n}$, or equivalently, as solutions of a vector equation

$$
g(x)=0,
$$

where the map $g: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n-m}$ is smooth (i.e., $C^{\infty}$ ) and the Jacobian matrix of $g(x)$ has rank $n-m$ for all $x \in \mathbb{R}^{n}$. In other words,

$$
\begin{equation*}
\mathcal{M}=\left\{x \in \mathbb{R}^{n}: g(x)=0\right\} \subset \mathbb{R}^{n} \tag{3.10}
\end{equation*}
$$

is a smooth $m$-dimensional submanifold of $\mathbb{R}^{n}$.
If $\mathcal{M}$ is a smooth submanifold of $\mathbb{R}^{n}$, then the tangent space to $\mathcal{M}$ at a point $\bar{x} \in \mathcal{M}$ is the set of all tangent vectors at $\bar{x}$, which is a linear subspace of dimension $m$ and is denoted $T_{\bar{x}} \mathcal{M}$. Although it has not explicitly been defined what is meant by a tangent vector, the notion of a tangent space is clear if it is simply identified as the $m$-dimensional hyperplane in $\mathbb{R}^{n}$ that is tangent to $\mathcal{M}$ at the point $\bar{x} \in \mathcal{M}$.

Consider an autonomous nonlinear system

$$
\begin{equation*}
\dot{x}=f(x), \tag{3.11}
\end{equation*}
$$

where $f: \mathcal{U} \rightarrow \mathbb{R}^{n}$ is a smooth vector field defined on an open subset $\mathcal{U}$ of $\mathbb{R}^{n}$.
Definition 3.3 (Abraham and Marsden, 1977, Definition 3.4.13): A submanifold $\mathcal{M}$ of $\mathbb{R}^{n}$ is said to be invariant under (3.11) if for all $x \in \mathcal{M}, f(x) \in T_{x} \mathcal{M}$.

This definition is equivalent to the condition (cf. Abraham and Marsden, 1977, Proposition 3.4.14) that

$$
x(0) \in \mathcal{M} \Longrightarrow x(t) \in \mathcal{M} \text { for every } t \in\left[0, t_{1}\right)
$$

where $x(t)$ is the solution to (3.11) starting at $x(0)$ and $\left[0, t_{1}\right)$ is any time interval over which the solution $x(t)$ is uniquely defined.

Lemma 3.2: A submanifold (3.10) of $\mathbb{R}^{n}$ is invariant under (3.11) if and only if

$$
\frac{\partial g(x)}{\partial x} f(x)=0 \text { for every } x \in \mathcal{M}
$$

Proof: At every point $x \in \mathcal{M}$ one can define a $(n-m)$-dimensional plane that is orthogonal to the tangent plane $T_{x} \mathcal{M}$ (Isidori, 1995, Appendix A.5) by

$$
\left(T_{x} \mathcal{M}\right)^{\perp}=\operatorname{span}\left\{d g_{1}(x), d g_{2}(x), \ldots, d g_{n-m}(x)\right\}
$$

Therefore, the inner product

$$
\left\langle d g_{i}^{\top}(x), f(x)\right\rangle=0 \text { for every } i \in\{1,2, \ldots, n-m\}, x \in \mathcal{M}
$$

is equivalent to $f(x) \in T_{x} \mathcal{M}$ for every $x \in \mathcal{M}$.

In the linear case, $\mathcal{M}=\left\{x \in \mathbb{R}^{n}: C x=0\right\}$ is a subspace of $\mathbb{R}^{n}$ and the autonomous system becomes $\dot{x}=A x$. Lemma 3.2 is therefore equivalent to saying that $\mathcal{M}$ is invariant under $A$ if and only if $C A x=0$ for every $x \in \mathcal{M}$. By following the proof of Lemma 3.2, at every point $x \in \mathcal{M}=\operatorname{Ker}(C)$ one can define a subspace $\mathcal{M}^{\perp}=\operatorname{Img}\left(C^{\top}\right)$, which is orthogonal to $\mathcal{M}$. Therefore, $A$-invariance of $\mathcal{M}$ is equivalent to $A x \perp \mathcal{M}^{\perp}$ for all $x \in \mathcal{M} \Longleftrightarrow A x \perp \operatorname{Img}\left(C^{\top}\right) \Longleftrightarrow C A x=0$.

Lemma 3.3: Assume $\bar{x} \in \mathcal{M}$ is the origin in $\mathbb{R}^{n}$ and an equilibrium point of (3.11). Let $A$ be the $n \times n$ Jacobian of $f(x)$ evaluated at $\bar{x} \in \mathcal{M}$. If $\mathcal{M}$, defined by (3.10), is invariant under (3.11), then $T_{\bar{x}} \mathcal{M}$ is an $A$-invariant subspace of $\mathbb{R}^{n}$.

Proof: Since the submanifold $\mathcal{M}$ is invariant under $f$, by Lemma 3.2

$$
\frac{\partial g(x)}{\partial x} f(x)=0 \text { for every } x \in \mathcal{M}
$$

Therefore, by differentiating, one obtains

$$
f^{\top}(x) \frac{\partial}{\partial x}\left(\frac{\partial g_{i}^{\top}(x)}{\partial x}\right)+\frac{\partial g_{i}(x)}{\partial x} \frac{\partial f(x)}{\partial x}=0
$$

for $i=1,2, \ldots, n-m$. By assumption, $f(0)=0$, which implies that

$$
\left.\frac{\partial g(x)}{\partial x}\right|_{0} A=0
$$

In other words, at the origin, the columns of $A$ must lie in the tangent space $T_{0} \mathcal{M}$, so $T_{0} \mathcal{M}$ must be an $A$-invariant subspace of $\mathbb{R}^{n}$.

### 3.3.2 Coordinate Constraints

As in the linear agents problem, for every initial condition, the system (3.8) is constrained to evolve on a $\hat{f}$-invariant submanifold $\mathcal{M}$ of $\mathbb{R}^{3 n}$. To see why this is the case, recall that under cyclic pursuit the system's pursuit graph $\Gamma_{t}$ at each instant satisfies $\sum_{i=1}^{n} e_{i}(t) \equiv 0$, resulting in the constraints of Section 3.1.3. These constraints are essential with regards to understanding how the spectrum of $\hat{A}$ relates to the stability of a given $\{n / d\}$ equilibrium polygon.

Let $g(\xi)=\left(g_{1}(\xi), g_{2}(\xi), g_{3}(\xi)\right)$. Then

$$
\begin{equation*}
\mathcal{M}=\left\{\xi \in \mathbb{R}^{3 n}: g(\xi)=0\right\} \subset \mathbb{R}^{3 n} \tag{3.12}
\end{equation*}
$$

defines a submanifold $\mathcal{M}$ of $\mathbb{R}^{3 n}$.
Lemma 3.4: The submanifold $\mathcal{M}$ is invariant under $\hat{f}$.
Proof: By Lemma 3.2, $\mathcal{M}$ is invariant under $\hat{f}$ if and only if

$$
\frac{\partial g(\xi)}{\partial \xi} \hat{f}(\xi)=0 \text { for every } \xi \in \mathcal{M}
$$

It is shown in Appendix A. 2 that this identity holds for all $\xi \in \mathcal{M}$.

Corollary 3.2: Since the submanifold $\mathcal{M}$ is invariant under $\hat{f}$, the tangent space $T_{\bar{\xi}} \mathcal{M}$ at every equilibrium point $\bar{\xi} \in \mathcal{M}$ is invariant under $\hat{A}$.

Proof: The proof follows as a direct consequence of Lemmas 3.3 and 3.4.

Therefore, by Corollary 3.2 , there exists a change of basis for $\mathbb{R}^{3 n}$ that transforms $\hat{A}$ into the upper-triangular form

$$
\left[\begin{array}{cc}
\hat{A}_{T_{\bar{\xi}} \mathcal{M}} & *  \tag{3.13}\\
0_{3 \times(3 n-3)} & \hat{A}_{T_{\bar{\xi}} \mathcal{M}}^{\star}
\end{array}\right]=: \tilde{A} .
$$

Next, in Lemma 3.5, the eigenvalues of $\hat{A}_{T_{\overline{\mathcal{M}}}}^{\star}$ are computed. A natural way to compute $\hat{A}_{T_{\bar{\xi}} \mathcal{M}}^{\star}$ would be to transform $\hat{A}$ into the aforementioned upper triangular form by converting to the canonical basis described in Section 2.2.4 (also exemplified in Section 2.2.6). Let $\varphi=P \xi$ denote this local change of coordinates, about $\bar{\xi} \in \mathcal{M}$. In this canonical basis, the last three coordinates must be identically zero for points in $T_{\bar{\xi}} \mathcal{M}$. Therefore, an appropriate choice for $P$ is

$$
P=\left.\frac{\partial \Phi(\xi)}{\partial \xi}\right|_{\bar{\xi}}
$$

where $\varphi=\Phi(\xi)$ is the change of coordinates

$$
\begin{align*}
& \varphi_{1}=r_{1}, \varphi_{2}=\alpha_{1}, \ldots, \varphi_{3 n-3}=\beta_{n-1}  \tag{3.14}\\
& \varphi_{3 n-2}=g_{1}(\xi), \varphi_{3 n-1}=g_{2}(\xi), \varphi_{3 n}=g_{3}(\xi)
\end{align*}
$$

By definition of the constraints $g_{i}(\xi)$, the last three coordinates $\varphi_{3 n-2}, \varphi_{3 n-1}, \varphi_{3 n}$ are identically zero on $\mathcal{M}$. A verification that $\Phi$ is proper (local) change of coordinates is provided in Appendix A.1.1.

Therefore, the upper triangular form (3.13) can be achieved by computing $P \hat{A} P^{-1}$. However, this is not a trivial task for general $n>1$ and it is only $\hat{A}_{T_{\bar{\xi}} \mathcal{M}}^{\star}$ that is of interest here. A more tractable approach is used in the proof of Lemma 3.5, which is to first perform the coordinates transformation $\Phi$ on $\hat{f}$ and then linearize to find $\hat{A}_{T_{\bar{\xi}} \mathcal{M}}^{\star}$. The relationship between these different approaches is illustrated by the commutative diagram below, where $\tilde{A}$ is defined in (3.13).


To demonstrate the validity of this commutative diagram, one can compute the nonlinear dynamics in the new coordinates,

$$
\dot{\varphi}=\left.\frac{\partial \Phi(\xi)}{\partial \xi} f(\xi)\right|_{\xi=\Phi^{-1}(\varphi)}=: \tilde{f}(\varphi)
$$

Thus, it must be shown that the Jacobian of $\tilde{f}$ about $\bar{\varphi}=\Phi(\bar{\xi})$ equals $P \hat{A} P^{-1}$. By taking the Jacobian of $\tilde{f}$ one obtains

$$
\left.\frac{\partial \tilde{f}(\varphi)}{\partial \varphi}\right|_{\bar{\varphi}}=\left.\frac{\partial}{\partial \varphi}\left(\left.\frac{\partial \Phi(\xi)}{\partial \xi} f(\xi)\right|_{\xi=\Phi^{-1}(\varphi)}\right)\right|_{\bar{\varphi}}
$$

By using the product and chain rules, and by the fact that $\hat{f}(\bar{\xi})=0$,

$$
\begin{aligned}
\left.\frac{\partial \tilde{f}(\varphi)}{\partial \varphi}\right|_{\bar{\varphi}} & =\left.\left.\frac{\partial \Phi(\xi)}{\partial \xi}\right|_{\bar{\xi}} \cdot \frac{\partial \hat{f}\left(\Phi^{-1}(\varphi)\right)}{\partial \varphi}\right|_{\bar{\varphi}}+\left.\left.\frac{\partial}{\partial \varphi}\left(\left.\frac{\partial \Phi(\xi)}{\partial \xi}\right|_{\xi=\Phi^{-1}(\varphi)}\right)\right|_{\bar{\varphi}} \cdot \hat{f}(\xi)\right|_{\bar{\xi}} \\
& =\left.\left.\left.\frac{\partial \Phi(\xi)}{\partial \xi}\right|_{\bar{\xi}} \cdot \frac{\partial \hat{f}(\xi)}{\partial \xi}\right|_{\bar{\xi}} \cdot \frac{\partial \Phi^{-1}(\varphi)}{\partial \varphi}\right|_{\bar{\varphi}} \\
& =P \hat{A} P^{-1}
\end{aligned}
$$

The above arguments are useful in proving the following important lemma.
Lemma 3.5: In the quotient space $\mathbb{R}^{3 n} / T_{\bar{\xi}} \mathcal{M}$, the induced linear transformation $\hat{A}_{T_{\bar{\xi}} \mathcal{M}}^{\star}: \mathbb{R}^{3 n} / T_{\bar{\xi}} \mathcal{M} \rightarrow \mathbb{R}^{3 n} / T_{\bar{\xi}} \mathcal{M}$ has (solely imaginary axis) eigenvalues

$$
\lambda_{1}=0 \text { and } \lambda_{2,3}= \pm j k \frac{\pi d}{n}
$$

Proof: Let $\varphi=\Phi(\xi)$ be the change of coordinates (3.14). Partition these new coordinates into $\varphi=\left(\varphi_{\mathrm{I}}, \varphi_{\mathrm{II}}\right)$ where $\varphi_{\mathrm{I}}=\left(\varphi_{1}, \varphi_{2}, \ldots, \varphi_{3 n-3}\right)$ and $\varphi_{\mathrm{II}}=$ $\left(\varphi_{3 n-2}, \varphi_{3 n-1}, \varphi_{3 n}\right)$. Notice that the set of coordinates in $\varphi_{\text {II }}$ are precisely the functions that define $\mathcal{M}$. Thus, in the new coordinates

$$
\begin{aligned}
\dot{\varphi}_{\mathrm{I}} & =\left.\left[\begin{array}{ll}
I_{3 n-3} & 0_{(3 n-3) \times 3}
\end{array}\right] \hat{f}(\xi)\right|_{\xi=\Phi^{-1}(\varphi)} \\
\dot{\varphi}_{\mathrm{II}} & =\left.\frac{\partial g(\xi)}{\partial \xi} \hat{f}(\xi)\right|_{\xi=\Phi^{-1}(\varphi)}
\end{aligned}
$$

Moreover, the equilibrium $\bar{\varphi}=\Phi(\bar{\xi})$ is equal to $\bar{\xi}$, except that the last three com-
ponents are instead zero. By computing the linearization about this equilibrium,

$$
\begin{aligned}
\dot{\varphi}_{\mathrm{I}} & =\left[\begin{array}{ll}
I_{3 n-3} & 0_{(3 n-3) \times 3}
\end{array}\right] \hat{A} \varphi \\
\dot{\varphi}_{\mathrm{II}} & =\left.\frac{\partial}{\partial \varphi}\left(\left.\frac{\partial g(\xi)}{\partial \xi} \hat{f}(\xi)\right|_{\xi=\Phi^{-1}(\varphi)}\right)\right|_{\bar{\varphi}} \varphi \\
& \left.\stackrel{(\mathrm{A})}{=} \frac{\partial}{\partial \varphi}\left[\begin{array}{c}
-k \alpha_{1} g_{2}(\xi)-\sin \left(g_{3}(\xi)\right) \\
k \alpha_{1} g_{1}(\xi)+\cos \left(g_{3}(\xi)\right)-1 \\
0
\end{array}\right]_{\xi=\Phi^{-1}(\varphi)}\right|_{\bar{\varphi}} \varphi \\
& =\left.\frac{\partial}{\partial \varphi}\left[\begin{array}{c}
-k \varphi_{2} \varphi_{3 n-1}-\sin \varphi_{3 n} \\
k \varphi_{2} \varphi_{3 n-2}+\cos \varphi_{3 n}-1
\end{array}\right]\right|_{\bar{\varphi}} \varphi \\
& =\left[\begin{array}{lll|lll}
0 & \cdots & 0 & 0 & -k \bar{\alpha} & -1 \\
0 & \cdots & 0 & k \bar{\alpha} & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0
\end{array}\right] \varphi=\left[\begin{array}{lll}
0_{3 \times(3 n-3)} & \hat{A}_{T_{\bar{\xi}} \mathcal{M}}^{\star}
\end{array}\right] \varphi
\end{aligned}
$$

where the lengthy derivation of equivalence (A) can be found in Appendix A.2. The $3 \times 3$ block $\hat{A}_{T_{\bar{\xi}} \mathcal{M}}^{\star}$ has eigenvalues $\lambda_{1,2,3}=\{0, \pm j k \bar{\alpha}\}$, with $\bar{\alpha}= \pm \pi d / n$ from Theorem 3.1, concluding the proof.

Therefore, just as in the linear agents problem, these three imaginary axis eigenvalues of $\hat{A}$ can be ignored when determining the stability of a given $\{n / d\}$ formation and stability can be assessed based on its remaining $3 n-3$ eigenvalues.

### 3.3.3 Spectral Analysis

Recall that the $3 n \times 3 n$ matrix $\hat{A}$, the linearization of $\hat{f}$, has the block circulant form (3.9). The present section exploits this fact and the background material of Section 2.2.3 to further isolate the eigenvalues of $\hat{A}$.

Lemma 3.6: The eigenvalues of $\hat{A}$ are the collection of all eigenvalues of

$$
\begin{aligned}
& A+B \\
& A+\omega B \\
& A+\omega^{2} B \\
& \quad \vdots \\
& A+\omega^{n-1} B
\end{aligned}
$$

where $\omega^{i-1}:=e^{j 2(i-1) \pi / n} \in \mathbb{C}$ is the $i$-th of $n$ roots of unity.
Proof: By Theorem 2.9, since $\hat{A}$ is block circulant it can be diagonalized using the Fourier matrix $F_{n}$; specifically

$$
\operatorname{diag}\left(D_{1}, D_{2}, \ldots, D_{n}\right)=\left(F_{n} \otimes I_{3}\right) \hat{A}\left(F_{n} \otimes I_{3}\right)^{*}
$$

where the $n$ diagonal blocks $D_{i}$ of dimension $3 \times 3$ are given by

$$
\left[\begin{array}{c}
D_{1} \\
D_{2} \\
\vdots \\
D_{n}
\end{array}\right]=\left(\sqrt{n} F_{n}^{*} \otimes I_{3}\right)\left[\begin{array}{c}
M_{1} \\
M_{2} \\
\vdots \\
M_{n}
\end{array}\right]
$$

with $M_{1}=A, M_{2}=B$, and $M_{i}=0_{3 \times 3}$ for $i=\{3,4, \ldots, n\}$. By expanding the above for each block $D_{i}$ one obtains

$$
\begin{aligned}
D_{1} & =A+B \\
D_{2} & =A+\omega B \\
D_{3} & =A+\omega^{2} B \\
& \vdots \\
D_{n} & =A+\omega^{n-1} B
\end{aligned}
$$

as the diagonal blocks for diagonalized $\hat{A}$. This implies that the eigenvalues of $\hat{A}$ must be the collection of all eigenvalues of $A+\omega^{i-1} B, i=1,2, \ldots, n$.

Therefore, each diagonal block in the diagonalization has the same form $D_{i}=$ $A+\omega^{i-1} B, i \in\{1,2, \ldots, n\}$, given by

$$
D_{i}=\left[\begin{array}{ccc}
0 & 2 \sin (q \pi) & \sin (q \pi) \\
-\frac{1}{2}(k q \pi)^{2} \csc (q \pi) & -k & -\frac{1}{2} k q \pi \cot (q \pi) \\
0 & k\left(1-\omega^{i-1}\right) & 0
\end{array}\right]
$$

From Lemma 3.6, one can observe two facts. The first is that the eigenvalues of $D_{1}=A+B$ are among the eigenvalues of $\hat{A}$ for every $n$. The characteristic polynomial of $D_{1}$ is

$$
\begin{aligned}
p_{D_{1}}(\lambda) & =\lambda^{3}+k \lambda^{2}+(k q \pi)^{2} \lambda \\
& =\lambda\left(\lambda^{2}+k \lambda+(k q \pi)^{2}\right),
\end{aligned}
$$

so the eigenvalues of $D_{1}$ are always

$$
\begin{align*}
\lambda_{1} & =0 \\
\lambda_{2,3} & =-\frac{k}{2} \pm j \frac{k}{2} \sqrt{4(q \pi)^{2}-1} \tag{3.15}
\end{align*}
$$

As predicted by Lemma 3.5, one zero eigenvalue has been unveiled, while the remaining eigenvalues have $\operatorname{Re}\left(\lambda_{2,3}\right)<0$ for every $0<q<1$ and $k>0$.

The second fact is that, when the number of vehicles $n$ is even, the eigenvalues of the matrix $D_{i^{\star}}=A-B$, with $i^{\star}:=1+n / 2$, are among the eigenvalues of $\hat{A}$. The characteristic polynomial of $D_{i^{\star}}$ is

$$
p_{D_{i^{\star}}}(\lambda)=\lambda^{3}+k \lambda^{2}+k^{2}\left((q \pi)^{2}+q \pi \cot (q \pi)\right) \lambda+k^{3}(q \pi)^{2}
$$

for which one can construct the Routh array

$$
\begin{array}{c|cc}
\lambda^{3} & 1 & k^{2}\left((q \pi)^{2}+q \pi \cot (q \pi)\right)  \tag{3.16}\\
\lambda^{2} & k & k^{3}(q \pi)^{2} \\
\lambda^{1} & k^{2} q \pi \cot (q \pi) & 0 \\
\lambda^{0} & k^{3}(q \pi)^{2} & 0
\end{array}
$$

By the Routh-Hurwitz criterion, for stability one would need that $\cot (q \pi)>0$ (due to the $\lambda^{1}$ element of the first column), or equivalently $0<q<1 / 2$. Moreover, in
the special case when $q=1 / 2$ the characteristic polynomial factors as

$$
\begin{equation*}
p_{D_{i^{\star}}}(\lambda)=\left(\lambda+j \frac{k \pi}{2}\right)\left(\lambda-j \frac{k \pi}{2}\right)(\lambda+k), \tag{3.17}
\end{equation*}
$$

which yields two imaginary axis eigenvalues of the form predicted by Lemma 3.5, and one stable eigenvalue. To illustrate, consider the simplest case, when $n=2$.

Proposition 3.1: The $\{2 / 1\}$ equilibrium polygon is locally asymptotically stable.
Proof: When $n=2$ the matrix $\hat{A}$ has the form

$$
\hat{A}=\left[\begin{array}{ll}
A & B \\
B & A
\end{array}\right]
$$

By Lemma 3.6, the eigenvalues (3.15) of $D_{1}=A+B$ must be among those of $\hat{A}$. Moreover, $i^{\star}=2$ and so the eigenvalues of $D_{2}=A-B$, which are the roots of (3.17), must be the remaining eigenvalues of $\hat{A}$. By disregarding the imaginary axis eigenvalues according to Lemma 3.5, one can directly conclude that the $\{2 / 1\}$ polygon formation is locally asymptotically stable.

Return now to the general case, when $n \geq 2$.
Lemma 3.7: The stability of $\hat{A}$ is independent of $k>0$.
Proof: Suppose $\hat{A}$ has been block diagonalized into $n$ diagonal blocks $D_{i}=A+$ $\omega^{i-1} B$ according to Lemma 3.6. The claim of Lemma 3.7 becomes obvious when each block $D_{i}$ is factored as $D_{i}=k T \tilde{D}_{i} T^{-1}$, where $T=\operatorname{diag}((1 / k) \sin (q \pi), 1,1)$ (recall $0<q<1$ ) and

$$
\tilde{D}_{i}=\left[\begin{array}{ccc}
0 & 2 & 1 \\
-\frac{1}{2}(q \pi)^{2} & -1 & -\frac{1}{2} q \pi \cot (q \pi) \\
0 & 1-\omega^{i-1} & 0
\end{array}\right]
$$

so that $\sigma\left(D_{i}\right)=k \sigma\left(\tilde{D}_{i}\right)$, where $\sigma(\cdot)$ denotes the spectrum of a matrix. Since $k>0$, the stability of the matrix $\tilde{D}_{i}$ implies the stability of $D_{i}$.

Therefore, whether a specific $\{n / d\}$ polygon formation is locally asymptotically stable or not is independent of the chosen gain $k>0$, and one can proceed
by studying only the transformed blocks $\tilde{D}_{i}$. In other words, for a given $n$, only the polygon density $d$ influences the stability of $\hat{A}$.

### 3.3.4 Stable Pursuit Formations

Recapitulating, about a given $\{n / d\}$ equilibrium polygon, the linearized multivehicle system has the form $\dot{\xi}=\hat{A} \xi$, where $\hat{A}$ is a block circulant matrix. It has been shown (in Lemma 3.5) that $\hat{A}$ has exactly three imaginary axis eigenvalues that do not influence the stability of a given $\{n / d\}$ formation. By capitalizing on its block circulant structure, $\hat{A}$ was block diagonalized into $n 3 \times 3$ blocks, $D_{i}$. It was then shown (in Lemma 3.7) that the stability of each matrix $D_{i}$, and hence the stability of $\hat{A}$, is independent of $k>0$, leaving that stability is dependent only on the density $d$ for a given $n$ via the $3 \times 3$ transformed matrices $\tilde{D}_{i}$.

Thus, modulo the imaginary axis eigenvalues of Lemma 3.5, when are these matrices $\tilde{D}_{i}, i=1,2, \ldots, n$, asymptotically stable? The answer to this question will expose which formations of Theorem 3.1 are locally asymptotically stable.

Unfortunately, the blocks $\tilde{D}_{i}$ are, in general, complex matrices. To be explicit about this fact, one can write the $n$ roots of unity $\omega^{i-1}=w_{i}+j z_{i} \in \mathbb{C}$, where

$$
w_{i}=\cos \left(2 \pi \frac{i-1}{n}\right) \text { and } z_{i}=\sin \left(2 \pi \frac{i-1}{n}\right) .
$$

In this general case, the characteristic polynomial of $\tilde{D}_{i}$ is

$$
\begin{equation*}
p_{\tilde{D}_{i}}(\lambda)=\lambda^{3}+\lambda^{2}+\left(a_{2}+j b_{2}\right) \lambda+\left(a_{3}+j b_{3}\right) \tag{3.18}
\end{equation*}
$$

with coefficients

$$
\begin{align*}
a_{2} & =(q \pi)^{2}+\frac{1}{2} q \pi\left(1-w_{i}\right) \cot (q \pi) \\
b_{2} & =-\frac{1}{2} q \pi z_{i} \cot (q \pi) \\
a_{3} & =\frac{1}{2}\left(1-w_{i}\right)(q \pi)^{2}  \tag{3.19}\\
b_{3} & =-\frac{1}{2} z_{i}(q \pi)^{2} .
\end{align*}
$$

For $c \in \mathbb{C}$, let $\bar{c}$ denote the complex conjugate of $c$.
Theorem 3.2 (after Barnett, 1983, Theorem 3.16): Consider a complex poly-
nomial of the third degree

$$
p(\lambda)=\lambda^{3}+c_{1} \lambda^{2}+c_{2} \lambda+c_{3}
$$

where $c_{1}, c_{2}, c_{3} \in \mathbb{C}$. Define the Hermitian matrix

$$
H=\left[\begin{array}{ccc}
c_{1}+\bar{c}_{1} & c_{2}-\bar{c}_{2} & c_{3}+\bar{c}_{3}  \tag{3.20}\\
-c_{2}+\bar{c}_{2} & \bar{c}_{2}+c_{2}-\bar{c}_{3}-c_{3} & c_{3}-\bar{c}_{3} \\
c_{3}+\bar{c}_{3} & -c_{3}+\bar{c}_{3} & c_{2} \bar{c}_{3}+\bar{c}_{2} c_{3}
\end{array}\right]
$$

The polynomial $p(\lambda)$ is asymptotically stable if and only if $H$ is positive definite.

This theorem is equivalent to a variation of the Routh-Hurwitz criterion for complex polynomials (Barnett, 1983, p. 179). A Hermitian matrix $H$ is positive definite if and only if its leading principal minors, denoted $h_{1}, h_{2}$, and $h_{3}$, are positive. Apply Theorem 3.2 to the characteristic polynomial (3.18) of $\tilde{D}_{i}$. By computing the leading principal minors of the corresponding $H$, one obtains

$$
\begin{aligned}
& h_{1}=2 \\
& h_{2}=4\left(a_{2}-a_{3}-b_{2}^{2}\right) \\
& h_{3}=8\left(a_{2}^{2} a_{3}+a_{2} b_{2} b_{3}-2 a_{2} a_{3}^{2}-3 a_{3} b_{2} b_{3}-b_{3}^{2}-a_{3} b_{2}^{2} a_{2}-b_{2}^{3} b_{3}+a_{3}^{3}\right)
\end{aligned}
$$

Clearly $h_{1}>0$. Stability of a given $\tilde{D}_{i}$ matrix, therefore, depends on the signs of $h_{2}$ and $h_{3}$. For stability, substituting (3.19) and using $z_{i}^{2}=1-w_{i}^{2}$, one would need for the second leading principal minor that

$$
\begin{align*}
h_{2}\left(q, w_{i}\right)= & 2\left(1+w_{i}\right) q \pi+2\left(1-w_{i}\right) \cot (q \pi)  \tag{3.21}\\
& -\left(1-w_{i}^{2}\right) q \pi \cot ^{2}(q \pi)>0
\end{align*}
$$

and for the third leading principal minor that

$$
\begin{align*}
& h_{3}\left(q, w_{i}\right)=\left(1+w_{i}-w_{i}^{2}-w_{i}^{3}\right)\left((q \pi)^{2}+q \pi \cot (q \pi)\right) \\
& \quad-\left(1-w_{i}-w_{i}^{2}+w_{i}^{3}\right)\left((q \pi)^{2} \cot ^{2}(q \pi)+q \pi \cot ^{3}(q \pi)\right) \\
& \quad+2\left(1-2 w_{i}+w_{i}^{2}\right) \cot ^{2}(q \pi)-2\left(1-w_{i}^{2}\right)>0 \tag{3.22}
\end{align*}
$$

The decision to write $h_{2}$ and $h_{3}$ as functions of the real component $w_{i}$ rather than the imaginary component $z_{i}$ was arbitrary. In what follows, these functions $h_{2}$
and $h_{3}$ will be used to determine which $\{n / d\}$ equilibrium polygons are stable and which are not. For a given $n$, define the set

$$
\mathcal{W}_{n}=\left\{w_{i}=\operatorname{Re}\left(\omega^{i-1}\right): i=1,2, \ldots, n\right\} .
$$

Lemma 3.8: Every $\{n / d\}$ equilibrium polygon with $n / 2<d<n$ is unstable.
Proof: When $n$ is even, it has already been shown that the eigenvalues of $A-B$ (a real matrix) are among the eigenvalues of $\hat{A}$ (i.e., if $i^{\star}:=1+n / 2, w_{i^{\star}}=-1$ is always a root of unity). By the Routh array (3.16) on page $55, D_{i^{\star}}=A-B$ is unstable when $1 / 2<d / n<1$, or equivalently $n / 2<d<n$.

When $n$ is odd, look at $h_{2}$ and consider the set of points that are not stable

$$
\mathcal{H}_{2}=\left\{(\mu, w): h_{2}(\mu, w) \leq 0, \mu \in(0,1), w \in(-1,1)\right\},
$$

which is illustrated by region $\mathcal{U}$ of Figure 3.5a. Note that $\mu$ and $w$ are taken on a continuum, whereas the arguments of $h_{2}$ in (3.21), $q$ and $w_{i} \in \mathcal{W}_{n}$, take on rational and discrete values, respectively.

Let int $\mathcal{H}_{2}$ denote the interior of $\mathcal{H}_{2}$. It is a fact that the pair $(\mu, \cos (2 \pi \mu)) \in$ $\operatorname{int} \mathcal{H}_{2}$ for every $\mu \in(1 / 2,2 / 3]$, as illustrated by the dotted line in Figure 3.5a. This fact, which is most easily checked numerically, will be useful in what follows.

Let $d^{\star}:=(n+1) / 2$, the smallest integer satisfying the condition $n / 2<d<n$ of the lemma. Let $i^{\star}:=d^{\star}+1$, which gives

$$
w_{i^{\star}}=\operatorname{Re}\left(\omega^{d^{\star}}\right)=\cos \left(\frac{2 \pi d^{\star}}{n}\right) \in \mathcal{W}_{n} .
$$

Note that $1 / 2<d^{\star} / n \leq 2 / 3$ for every $n \geq 3$, which (by the previously stated fact) implies that for every $n \geq 3$ there exists a $w_{i^{\star}} \in \mathcal{W}_{n}$ such that $\left(d^{\star} / n, w_{i^{\star}}\right) \in \operatorname{int} \mathcal{H}_{2}$, i.e., every $\left\{n / d^{\star}\right\}$ polygon is unstable.

It is left to show that the remaining densities $d^{\star}<d<n$ satisfying the condition $n / 2<d<n$ of the lemma are also unstable. Since $d^{\star} / n<d / n<1$, the point $\left(d / n, w_{i^{\star}}\right) \in \operatorname{int} \mathcal{H}_{2}$ also lives in the region $\mathcal{U}$. This is because it lies directly to the right of the unstable point $\left(d^{\star} / n, w_{i^{\star}}\right) \in \operatorname{int} \mathcal{H}_{2}$ in Figure 3.5a.

Before stating the principal result of this chapter, consider the set of points


Figure 3.5: Parameter $w$ as a function of $\mu$ for $h_{2}$ and $h_{3}$
that are not stable

$$
\begin{equation*}
\mathcal{H}_{3}=\left\{(\mu, w): h_{3}(\mu, w) \leq 0, \mu \in(0,1 / 2], w \in(-1,1)\right\}, \tag{3.23}
\end{equation*}
$$

which is illustrated by the region $\mathcal{U}$ of Figure 3.5b. Define the following functions

$$
\begin{aligned}
& \bar{w}(\mu):=\frac{2 \tan (\mu \pi)}{\mu \pi(1+\mu \pi \tan (\mu \pi))}-1 \\
& \underline{w}(\mu):=\cos (2 \pi \mu),
\end{aligned}
$$

which describe the upper and lower boundaries of the region $\mathcal{U}$ in Figure 3.5b. These functions were obtained by solving, with the aid of computer algebra software, the equation $h_{3}(\mu, w)=0$ on the relevant domain $\mu=(0,1 / 2]$ and $w \in(-1,1)$ and numerically checking that the region $\mathcal{U}$ indeed corresponds to the set $\mathcal{H}_{3}$. As a result, the definition (3.23) is equivalent to

$$
\mathcal{H}_{3}=\{(\mu, w): \mu \in(0,1 / 2], w \in[\underline{w}(\mu), \bar{w}(\mu)]\} .
$$

Theorem 3.3 (Main Stability Result): A given $\{n / d\}$ equilibrium polygon is locally asymptotically stable if and only if $0<d \leq n / 2$ and

$$
\begin{equation*}
\underline{w}\left(\frac{d-1}{n}\right)>\bar{w}\left(\frac{d}{n}\right) . \tag{3.24}
\end{equation*}
$$

Proof: According to the proof of Lemma 3.8, $h_{2}<0$ for every $\{n / d\}$ polygon with $n / 2<d<n$. Thus, a necessary condition for stability is that $0<d \leq n / 2$. Notice that $h_{2}>0$ for every $0<d<n / 2$ (see Figure 3.5a). Proceed by assuming that this condition holds for the given $\{n / d\}$ polygon. The special case when $d=n / 2$ will be considered separately. Moreover, observe that every matrix $D_{i}$ has a complex conjugate matrix $D_{n-i+2}$, hence the spectrum of $D_{i}$ and that of its conjugate are also complex conjugates.

Define $i^{\star}:=d+1$ so that $w_{i^{\star}}=\cos (2 \pi d / n) \equiv \underline{w}(d / n)$. Thus, the point $\left(d / n, w_{i^{*}}\right)$ lies exactly on the lower boundary of $\mathcal{H}_{3}$ in Figure 3.5 b. Together, the matrix $D_{i^{\star}}$ and its conjugate $D_{n-i^{\star}+2}$ have two imaginary axis eigenvalues (one each) of the form $\lambda= \pm j k \pi d / n$, while the remaining eigenvalues have $\operatorname{Re}(\lambda) \neq 0$. These facts were verified with the assistance of computer algebra software. The eigenvalues with $\operatorname{Re}(\lambda) \neq 0$ cannot be unstable, otherwise the point $\left(d / n, w_{i^{\star}}\right)$
would not lie on the boundary of $\mathcal{H}_{3}$. According to Lemma 3.5, the two imaginary axis eigenvalues can be disregarded, together with the zero eigenvalue of $D_{1}$, since they have no connection to the stability of the given $\{n / d\}$ polygon. Since the point $\left(d / n, w_{i^{\star}}\right)$ lies on the lower boundary of $\mathcal{H}_{3}$, all points $\left(d / n, w_{i}\right), w_{i} \in \mathcal{W}_{n}$ with $w_{i}<w_{i^{\star}}$ lie outside the unstable set $\mathcal{H}_{3}$. Therefore, look at the points $\left(d / n, w_{i}\right), w_{i} \in \mathcal{W}_{n}$ with $w_{i}>w_{i^{\star}}$.

Define the index $i^{\prime}:=i^{\star}-1=d$, corresponding to $w_{i^{\prime}} \in \mathcal{W}_{n}, w_{i^{\prime}}>w_{i^{\star}}$ that is closest to $w_{i^{\star}}$. This new value is given by $w_{i^{\prime}}=\cos (2 \pi(d-1) / n) \equiv \underline{w}((d-1) / n)$. If $w_{i^{\prime}}>\bar{w}(d / n)$, then the point $\left(d / n, w_{i}\right) \notin \mathcal{H}_{3}$ for all $w_{i} \in \mathcal{W}_{n}, w_{i}>w_{i^{\star}}$. Therefore, by Theorem 3.2, stability is equivalent to $\underline{w}((d-1) / n)>\bar{w}(d / n)$.

In the special case when $d=n / 2$, the matrix $D_{i^{\star}}$ is real and has eigenvalues according to the roots of (3.17), as shown on page 56. Therefore, one is stable and the remaining two imaginary axis eigenvalues should be ignored according to Lemma 3.5. The rest of the proof follows as for $0<d<n / 2$.

The following sequence of corollaries employs this main stability result to explicitly disclose which $\{n / d\}$ equilibrium polygons are stable and which are not.

Corollary 3.3: Every $\{n / 1\}$ polygon is locally asymptotically stable.
Proof: Let $d=1$ and $n \geq 2$. Then, from Figure 3.5b, $\underline{w}(0)=1$ and $\bar{w}(d / n)<1$ for every $n \in\{2,3, \ldots\}$ so that the conditions of Theorem 3.3 are satisfied.

Recall that the variable $\mu$ is intended to represent $d / n$. Thus, with reference to Figure 3.5b, for some fixed density $d$ the condition (3.24) is equivalent to

$$
\begin{equation*}
\underline{w}(\mu-\mu / d)>\bar{w}(\mu) . \tag{3.25}
\end{equation*}
$$

The graphs of $\underline{w}(\mu-\mu / d)$ versus $\mu$ for $d \in\{2,3,4,5,6\}$ are illustrated by the dotted curves in Figure 3.6. These curves are shown superimposed on Figure 3.5b and the three small circles in Figure 3.6 indicate intersection with the boundary $\bar{w}(\mu)$. Notice that only when $d \in\{3,4,5\}$ does $\underline{w}(\mu-\mu / d)$ intersect the curve $\bar{w}(\mu)$ on the real interval $\mu \in(0,1 / 2]$.

Corollary 3.4: Every $\{n / 2\}$ polygon with $n \geq 4$ is locally asymptotically stable.
Proof: When $d=2$, the necessary condition $0<d \leq n / 2$ of Theorem 3.3 dictates that $n \geq 4$. Moreover, from Figure 3.6, the inequality (3.24) is satisfied


Figure 3.6: The left-hand side of (3.25) versus $\mu$ for $d \in\{2,3,4,5,6\}$ (dotted) superimposed on Figure 3.5b; circles indicate intersection
for every $n \in\{4,5, \ldots\}$.

Corollary 3.5: Every $\{n / d\}$ polygon with $d \geq 6$ is unstable.
Proof: When $d \geq 6$, from Figure 3.6, (3.24) is never satisfied; i.e.,

$$
\underline{w}\left(\frac{d-1}{n}\right)<\bar{w}\left(\frac{d}{n}\right)
$$

for every $6 \leq d \leq n / 2$.

Using Theorem 3.3 and Figure 3.6, the stability of the remaining polygons $\{n / d\}, d \in\{3,4,5\}$ can now be determined by replacing (3.25) with an equality.

Corollary 3.6: For some $\{n / d\}$ with $d \in\{3,4,5\}$, let $\bar{\mu}$ be the unique solution to

$$
\underline{w}\left(\mu-\frac{\mu}{d}\right)=\bar{w}(\mu) .
$$

Then $\{n / d\}$ is locally asymptotically stable if and only if $d<\bar{\mu} n$.
Proof: For $d \in\{3,4,5\}$ the graphs of $\underline{w}(\mu-\mu / d)$ and $\bar{w}(\mu)$ intersect exactly once in the domain $\mu \in(0,1 / 2]$ (see the circles in Figure 3.6). Let $\bar{\mu}$ be this point of intersection and note that $d / n<\bar{\mu}$ results in stability, while $d / n>\bar{\mu}$ gives instability as per the condition (3.25) and Theorem 3.3.

These points of intersection $\bar{\mu}$ (solved for numerically) are listed in Table 3.1 and are shown as circles in Figure 3.6. Employing these values, it can be shown that polygon $\{10 / 3\}$ is stable, while $\{9 / 3\}$ is not. Similarly, $\{21 / 4\}$ is stable, while $\{20 / 4\}$ is not, and finally $\{54 / 5\}$ is stable, while $\{53 / 5\}$ is not. Table 3.2 lists all possible equilibrium formations and gives their stability.

Table 3.1: Table of $\bar{\mu}$ values for polygon densities $d \in\{3,4,5\}$

| $d$ | $\bar{\mu}$ (approx.) |
| :---: | :---: |
| 3 | 0.3318678173 |
| 4 | 0.1999447110 |
| 5 | 0.0942114573 |

Table 3.2: Equilibrium formations with stable formations shaded

| $d=1$ | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2 / 1\}$ | $\{3 / 2\}$ | $\{4 / 3\}$ | $\{5 / 4\}$ | $\{6 / 5\}$ | $\{7 / 6\}$ |
| $\{3 / 1\}$ | $\{4 / 2\}$ | $\{5 / 3\}$ | $\{6 / 4\}$ | $\{7 / 5\}$ | $\{8 / 6\}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\{7 / 1\}$ | $\{8 / 2\}$ | $\{9 / 3\}$ | $\{10 / 4\}$ | $\{11 / 5\}$ | $\{12 / 6\}$ |
| $\{8 / 1\}$ | $\{9 / 2\}$ | $\{10 / 3\}$ | $\{11 / 4\}$ | $\{12 / 5\}$ | $\{13 / 6\}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\{17 / 1\}$ | $\{18 / 2\}$ | $\{19 / 3\}$ | $\{20 / 4\}$ | $\{21 / 5\}$ | $\{22 / 6\}$ |
| $\{18 / 1\}$ | $\{19 / 2\}$ | $\{20 / 3\}$ | $\{21 / 4\}$ | $\{22 / 5\}$ | $\{23 / 6\}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\{49 / 1\}$ | $\{50 / 2\}$ | $\{51 / 3\}$ | $\{52 / 4\}$ | $\{53 / 5\}$ | $\{54 / 6\}$ |
| $\{50 / 1\}$ | $\{51 / 2\}$ | $\{52 / 3\}$ | $\{53 / 4\}$ | $\{54 / 5\}$ | $\{55 / 6\}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

### 3.3.5 Additional Sample Simulations

Figures 3.7 a and 3.7 b show computer simulation results for $n=7$ vehicles, where in each case the forward speed $v_{R}=1$ and gain $k=1$. However, due to differing initial conditions, the vehicles of Figure 3.7a form a $\{7 / 1\}$ polygon at equilibrium, whereas the vehicles of Figure 3.7b converge to a $\{7 / 2\}$-polygon formation.

### 3.4 Stationary Polygons

Notice that in the simulation examples of Figures 3.3 and 3.7 the resulting formations appear stationary in the plane (i.e., the circles do not drift over time). Although it has been shown that the equilibrium formations for fixed-speed unicycles in cyclic pursuit are generalized regular polygons and that only some of these formations are locally asymptotically stable, it has not yet been established that these equilibria are stationary. This matter is resolved in this section.

To this end, it is useful to view each vehicle's position in the plane as a point in the complex plane; specifically, $z_{i}=x_{i}+j y_{i} \in \mathbb{C}$. Thus, the centroid of the vehicles is given by $z_{c}=(1 / n) \sum_{i=1}^{n} z_{i}$ and the velocity of the centroid is


Figure 3.7: Fixed-speed pursuit generating $\{7 / 1\}$ and $\{7 / 2\}$ formations
$\dot{z}_{c}=(1 / n) \sum_{i=1}^{n} \dot{z}_{i}$. By substituting the unicycle model (2.12), one obtains

$$
\dot{z}_{c}=\frac{1}{n} \sum_{i=1}^{n} v_{i} e^{j \theta_{i}} .
$$

The control law (3.4) has the property that $v_{i}=v_{R}, i=1,2, \ldots, n$. Therefore,

$$
\dot{z}_{c}=\frac{v_{R}}{n} \sum_{i=1}^{n} e^{j \theta_{i}} .
$$

Moreover, it is clear that the centroid of the vehicles equals the centroid of the generalized regular polygon $\{n / d\}$ at equilibrium. The definition of the relative coordinate $\beta_{i}$ is $\beta_{i}=\theta_{i}-\theta_{i+1}-\pi$, which has a constant value $\bar{\beta}$ for all $i=$ $1,2, \ldots, n$ at equilibrium. In other words, $\theta_{i+1}=\theta_{i}-\bar{\beta}-\pi$ at equilibrium. But, consequent to Theorem 3.1 it holds that $\bar{\beta}= \pm \pi(1-2 d / n)$, which implies that $\theta_{i+1}=\theta_{i} \pm 2 \pi d / n \bmod 2 \pi$. Consequently, every heading angle $\theta_{i}$ can be written as a function of the first; namely

$$
\begin{aligned}
\theta_{2} & =\theta_{1} \pm 2 \pi d / n \\
\theta_{3} & =\theta_{2} \pm 2 \pi d / n=\theta_{1} \pm 4 \pi d / n \\
& \vdots \\
\theta_{n} & =\theta_{n-1} \pm 2 \pi d / n=\theta_{1} \pm(n-1) 2 \pi d / n .
\end{aligned}
$$

Using this, the velocity of the centroid can be written as

$$
\begin{aligned}
\dot{z}_{c} & =\frac{v_{R}}{n} \sum_{i=1}^{n} e^{\theta_{1} \pm j 2 \pi(i-1) d / n} \\
& =\frac{v_{R}}{n} e^{\theta_{1}} \sum_{i=1}^{n} e^{ \pm j 2 \pi(i-1) d / n}
\end{aligned}
$$

If $d$ and $n$ are coprime, then $\sum_{i=1}^{n} e^{ \pm j 2 \pi(i-1) d / n} \equiv 0$ because these are the $n$ roots of unity. If $d$ and $n$ have a common factor $m>1$ then $\sum_{i=1}^{n} e^{ \pm j 2 \pi(i-1) d / n} \equiv 0$ because these are the $n / m$ roots of unity (traversed $m$ times). Together, these facts imply that $\dot{z}_{c} \equiv 0$. This, in turn, implies that every $\{n / d\}$ generalized regular polygon formation is stationary at equilibrium, despite the motion of the vehicles.

### 3.5 On the $n$-vehicle Weave

The focus of this chapter has been on equilibrium formations of the multivehicle system (3.5). These correspond to configurations for which the vehicles have fixed relative pose. However, it turns out that these regular formations are not the only stable behaviours of (3.5). Repeated simulations indicate that when the vehicles do not converge to a generalized regular polygon formation, they instead fall into a different kind of order: one in which the relative coordinates (3.2) follow periodic trajectories. What is more, in this new mode of organization each vehicle's motion is identical to that of the next, only $1 / n$-th of a cycle out of step in time. This purely nonlinear phenomenon shall be referred to as the $n$-vehicle weave.

It is not within the scope of this thesis to provide a complete analysis of the weave. Instead, based on simulation results, some qualitative observations are made that might serve as a starting point for analysis.

### 3.5.1 Simulating the Weave

Figure 3.8a depicts a system of $n=6$ unicycles subject to the fixed-speed pursuit law (3.4) with $v_{R}=1$ and $k=1$. Rather than converging to one of the equilibrium formations of Theorem 3.1, the vehicles "weave" in and out, while the formation as a whole moves along a linear trajectory (in the direction indicated by the dashed arrow in Figure 3.8a). In this case, it can be verified that the vehicles' centroid always lies on the centre line defined by the arrow in the figure. Simulations indicate that this average formation trajectory travels at a constant speed. However, its steady-state heading depends on the initial conditions. Figure 3.8b shows the steady-state trajectories of the vehicles in Figure 3.8a, but with the motion of their centroid subtracted from the motion of each vehicle.

Furthermore, the weaving pattern converges to a steady-state amplitude, or width $w_{k}$, as indicated in Figure 3.8a. The steady-state width of the weave presented in Figure 3.8a is approximately $w_{1}=4.3668$ units in the figure (each tick mark indicates a unit). If one increases the gain $k$ from 1 to 2 , the width drops to $w_{2}=2.1832 \approx w_{1} / 2$. Likewise, if the gain is decreased to $k=0.5$, then the width increases to $w_{0.5}=8.7334 \approx 2 w_{1}$. Thus, one might infer from simulation that the width $w_{k}$ of the weave is inversely proportional to the gain $k$. Moreover, simulations indicate that the width $w_{k}$ is independent of the initial conditions that generate the weave. Notice how this behaviour is similar to the inverse relationship

(a)

(b) Centroid motion removed (steady-state)

Figure 3.8: Weaving trajectories for $n=6$ vehicles
between $k$ and the equilibrium radius $\rho$ in Corollary 3.1.
In Figure 3.8, observe how the trajectories repeatedly intersect along the line followed by the centroid of the formation. Repeated simulations indicate that fixed pairs of vehicles regularly collide at points along this line. Specifically, in Figure 3.8, the vehicle pairs $(i, i+3), i=1,2,3$, always collide. No other collisions occur during the weave. Another potentially useful observation is that, during the weave, the configuration of vehicles (including their orientations) appears to exhibit dihedral symmetry. The dihedral symmetry group of order 1 , denoted $D_{1}$, consists of the (identity) rotation through angle $2 \pi$ and exactly one reflection symmetry about a unique line in the plane (see Chapter 6). Such a line of symmetry clearly exists in Figure 3.8, and is given by the dashed arrow representing the overall motion of the formation.

However, the dihedral symmetry described above exists only when the number of vehicles is even. What happens when $n$ is odd? Figure 3.9 shows a system of $n=5$ vehicles subject to the fixed-speed pursuit law (3.4) with $v_{R}=1$ and $k=1$. The inverse relationship between the width $w_{k}$ and gain $k$, observed in the case of $n=6$ vehicles, still holds. However, the trajectory of the vehicles' centroid, in this case, is not confined to a line. Instead, it follows a periodic trajectory (not shown) whose average, in time, represents the general motion of the formation.


Figure 3.9: Weaving trajectories for $n=5$ vehicles
Another observation in the case when $n$ is odd is that no collisions occur during the weave. The five vehicles in Figure 3.9 may arrive "close" to one-another at
times, but they never occupy the same point at any instant (unlike the six vehicles in Figure 3.8). Moreover, if the weave is widened (by decreasing $k$ ) the minimum distance between vehicles, denoted $d_{k}$, increases linearly. Figure 3.10 shows the relative distances from vehicle 1 to vehicles $i=2,3,4,5$ corresponding to the simulation depicted in Figure 3.9. The closest any other vehicle comes to vehicle 1 is approximately $d_{1}=0.3841$ units when $k=1$. Repeating the simulation with $k=2$ yields $d_{2}=0.1921 \approx d_{1} / 2$; likewise, $k=0.5$ yields $d_{0.5}=0.7682 \approx 2 d_{1}$. Thus, for some real vehicle size, one could in practise compute the maximum $k$ (alternatively, minimum $w_{k}$ ) allowable in order that collisions do not occur.


Figure 3.10: Relative distances from vehicle 1 to vehicle $i=2,3,4,5$

Finally, so as to gain some further understanding of the underlying order that is the weave, consider Figure 3.11a, which depicts the coordinates $\alpha_{i}, i=1,2,3,4,5$, of (3.5) as a function of time. Obviously the solutions are periodic; let the period be $T$ seconds. Furthermore, the behaviour of each vehicle is identical to that of the next, only shifted $T / 5$ seconds in time. More generally,

$$
\begin{equation*}
\alpha_{i}(t)=\alpha_{1}(t+(n-i+1) T / n), \tag{3.26}
\end{equation*}
$$

for $i=1,2, \ldots, n$. The same phenomenon occurs in the coordinates $r_{i}$ and $\beta_{i}$. In other words, in analyzing the periodic solution $\xi(t) \in \mathbb{R}^{3 n}$ that is the weave, it may be possible to reduce the problem to one in $\mathbb{R}^{3}$ by exploiting (3.26). Figure 3.11 b shows this periodic orbit in the $\left(r_{i}, \alpha_{i}, \beta_{i}\right)$-space when $n=5$. Although the data used to generate Figure 3.11b corresponds to vehicle $i=1$, the orbit is identical in the case of $i=2,3,4,5$ (merely shifted $1 / 5$-th of a cycle in time).

### 3.5.2 An Invariant Subspace

In this section, the potential for the existence of steady-state solutions other than the equilibrium solutions of Theorem 3.1 is conjectured by defining an invariant subspace $\mathcal{W} \subset \mathbb{R}^{3 n}$ in which no fixed points exist. Let $m:=\lceil n / 2\rceil$ and define

$$
\mathcal{W}:=\left\{\xi \in \mathbb{R}^{3 n}: r_{i}=r_{i+m}, \alpha_{i}=-\alpha_{i+m}, \beta_{i}=-\beta_{i+m} \text { for } i=1,2, \ldots, m\right\}
$$

For example, the initial conditions in Figures 3.8 and 3.9 each belong to this subspace. For the $n=5$ vehicles in Figure 3.9, $m=\lceil 5 / 2\rceil=3$. Specifically, $r_{i}(0)=1, i=1,2, \ldots, 6, \alpha_{1}(0)=-\alpha_{4}(0)=-\pi / 2, \alpha_{2}(0)=-\alpha_{5}(0)=-\pi / 2$, and $\alpha_{3}(0)=-\alpha_{1}=\pi / 2$. Similar relationships hold for the $\beta_{i}(0)$ coordinates.

For convenience, define the constraint functions

$$
\begin{aligned}
g_{1}(\xi) & =r_{1}-r_{1+m} \\
g_{2}(\xi) & =\alpha_{1}+\alpha_{1+m} \\
g_{3}(\xi) & =\beta_{1}+\beta_{1+m} \\
g_{4}(\xi) & =r_{2}-r_{2+m} \\
& \vdots \\
g_{3 m-1}(\xi) & =\alpha_{m}+\alpha_{2 m} \\
g_{3 m}(\xi) & =\beta_{m}+\beta_{2 m} .
\end{aligned}
$$

Therefore, $\mathcal{W}=\left\{\xi \in \mathbb{R}^{3 n}: g_{i}(\xi)=0, i=1,2, \ldots, 3 m\right\}$.
Proposition 3.2: The subspace $\mathcal{W}$ is invariant under the dynamics (3.8).
Proof: By Lemma 3.2, the subspace $\mathcal{W}$ is invariant under (3.8) if and only if

$$
\frac{\partial g(\xi)}{\partial \xi} \hat{f}(\xi)=0
$$


(a) Heading errors $\alpha_{i}, i=1,2,3,4,5$

(b) The weave in coordinates $\left(r_{1}, \alpha_{1}, \beta_{1}\right)$

Figure 3.11: Periodic solutions of (3.5) corresponding to the $n=5$ weave
for all $\xi \in \mathcal{W}$. Therefore, for every $i \in\{1,2, \ldots, 3 m\}$, it needs to be shown that $\partial g_{i}(\xi) / \partial \xi \cdot \hat{f}(\xi)=0$ when $\xi \in \mathcal{W}$. Hence, compute

$$
\begin{aligned}
& \left.\frac{\partial}{\partial \xi}\left(r_{i}-r_{i+m}\right) \cdot \hat{f}(\xi)\right|_{\xi \in \mathcal{W}} \\
& \quad=-v_{R}\left(\cos \alpha_{i}+\cos \left(\alpha_{i}+\beta_{i}\right)\right)+v_{R}\left(\cos \alpha_{i+m}+\cos \left(\alpha_{i+m}+\beta_{i+m}\right)\right) \\
& \quad=-v_{R}\left(\cos \alpha_{i}+\cos \left(\alpha_{i}+\beta_{i}\right)\right)+v_{R}\left(\cos \alpha_{i}+\cos \left(\alpha_{i}+\beta_{i}\right)\right) \\
& \quad=0
\end{aligned}
$$

One also has that

$$
\begin{aligned}
& \left.\frac{\partial}{\partial \xi}\left(\alpha_{i}+\alpha_{i+m}\right) \cdot \hat{f}(\xi)\right|_{\xi \in \mathcal{W}} \\
& \quad=\frac{v_{R}}{r_{i}}\left(\sin \alpha_{i}+\sin \left(\alpha_{i}+\beta_{i}\right)\right)-k \alpha_{i}+\frac{v_{R}}{r_{i+m}}\left(\sin \alpha_{i+m}+\sin \left(\alpha_{i+m}+\beta_{i+m}\right)\right) \\
& \quad-k \alpha_{i+m} \\
& \quad=0
\end{aligned}
$$

And finally,

$$
\begin{aligned}
& \left.\frac{\partial}{\partial \xi}\left(\beta_{i}+\beta_{i+m}\right) \cdot \hat{f}(\xi)\right|_{\xi \in \mathcal{W}} \\
& \quad=k\left(\alpha_{i}-\alpha_{i+1}\right)+k\left(\alpha_{i+m}-\alpha_{i+m+1}\right) \\
& \quad=k\left(\alpha_{i}+\alpha_{i+m}\right)-k\left(\alpha_{i+1}+\alpha_{i+1+m}\right) \\
& \quad=0
\end{aligned}
$$

which concludes the verification.

What is exceptional about $\mathcal{W}$ is that, not only is it invariant under the system's dynamics, it also contains no equilibrium points, following Theorem 3.1 (save the origin, which is the degenerate case). Therefore, any "stable" solution $\xi(t)$ must either flow towards a limit-cycle or be chaotic in nature. On the other hand, simulations that begin with initial conditions $\xi(0) \in \mathcal{W}$ consistently converge to the weave (e.g., as in Figures 3.8 and 3.9), suggesting the former.

## Chapter 4

## Varying-speed Pursuit Formations

Equilibrium pursuit formations for fixed-speed vehicles were studied in Chapter 3. However, in Section 2.3 a coordination strategy was described in which each vehicle $i$ 's forward speed $v_{i}$ is assigned in proportion to the distance error $r_{i}$. This is in addition to assigning the angular speed $\omega_{i}$ in proportion to the heading error $\alpha_{i}$. Consequently, the aim of this chapter is to extend the fixed-speed pursuit scenario to the case of varying-speed vehicles.

### 4.1 Nonlinear Equations of Pursuit

The transformation from interconnected unicycles to relative coordinates $\xi_{i}=$ $\left(r_{i}, \alpha_{i}, \beta_{i}\right)$, described by (3.2)-(3.3) on page 38 and detailed in Section 3.1.1, is useful in this chapter as it was in the last. However, the varying-speed pursuit scenario described above corresponds to $n$ unicycles, each with control inputs

$$
\begin{equation*}
v_{i}=k_{r} r_{i} \text { and } \omega_{i}=k_{\alpha} \alpha_{i}, \tag{4.1}
\end{equation*}
$$

where $k_{r}, k_{\alpha}>0$ are constant gains. Substituting these controls into (3.3) gives a system of $n$ cyclically interconnected and identical subsystems

$$
\begin{align*}
\dot{r}_{i} & =-k_{r}\left(r_{i} \cos \alpha_{i}+r_{i+1} \cos \left(\alpha_{i}+\beta_{i}\right)\right)  \tag{4.2a}\\
\dot{\alpha}_{i} & =k_{r}\left(\sin \alpha_{i}+\frac{r_{i+1}}{r_{i}} \sin \left(\alpha_{i}+\beta_{i}\right)\right)-k_{\alpha} \alpha_{i}  \tag{4.2b}\\
\dot{\beta}_{i} & =k_{\alpha}\left(\alpha_{i}-\alpha_{i+1}\right), \tag{4.2c}
\end{align*}
$$

for $i=1,2, \ldots, n$, and where the indices $i+1$ are evaluated modulo $n$. This system (4.2) of $n$ cyclically interconnected vehicles is examined in the remaining
sections of this chapter.

### 4.1.1 Sample Simulations

Preliminary computer simulations indicate the possibility of achieving circular pursuit trajectories similar to those observed in Chapter 3 for fixed-speed vehicles. Figure 4.1 shows the trajectories for a system of $n=6$ unicycles, initially positioned at random, where the gain $k_{\alpha}=1$ and gain $k_{r}=(\pi / 12) \csc (\pi / 6)=k^{\star}$. This specific choice of gains corresponds to (4.3), to be discussed in the next section. In this case, the unicycles converge to equally spaced motion around a circle of fixed radius with a pursuit graph similar to a regular hexagon. However, in contrast to the outcomes observed in Chapter 3, Figures 4.2 a and 4.2 b show unicycles converging to a point and diverging, respectively, while simultaneously approaching equally spaced motion that resembles a regular hexagon.


Figure 4.1: Six vehicles subject to control law (4.1) with $k_{\alpha}=1$ and $k_{r}=k^{\star}$


Figure 4.2: Six vehicles subject to control law (4.1) with $k_{r} \neq k^{\star}$

### 4.2 Generalized Equilibria

This section mirrors the analysis of Section 3.2 by revealing the set of possible equilibrium formations for multivehicle systems subject to (4.1).

Theorem 4.1: The $3 n$-dimensional system (4.2) has $2(n-1)$ equilibrium points, described as follows: the $r_{i}$ are all equal, $r_{i}=\bar{r}>0$; likewise, $\alpha_{i}=\bar{\alpha}$ and $\beta_{i}=\bar{\beta}$ for all $i \in\{1,2, \ldots, n\}$. The equilibrium values of $\bar{\alpha}$ and $\bar{\beta}$ are given by

$$
\begin{aligned}
\bar{\alpha} & = \pm \frac{\pi d}{n}, d=1,2, \ldots, n-1 \\
\bar{\beta} & =\pi-2 \bar{\alpha} .
\end{aligned}
$$

At each equilibrium point, the related pursuit graph is a generalized regular polygon $\{n / d\}$, with $d \in\{1,2, \ldots, n-1\}$. Finally, such an $\{n / d\}$-polygon equilibrium formation for (4.2) exists if and only if

$$
\begin{equation*}
k_{r} / k_{\alpha}=\frac{\pi d}{2 n} \csc \left(\frac{\pi d}{n}\right)=: k^{\star} . \tag{4.3}
\end{equation*}
$$

Proof: When $\dot{\beta}_{i}=0$, (4.2) yields $\alpha_{i}=\alpha_{i+1}$. Let $\bar{\alpha} \equiv \alpha_{i}$ at equilibrium. Moreover, when $\dot{r}_{i}=0$,

$$
\begin{equation*}
\frac{-\cos \bar{\alpha}}{\cos \left(\bar{\alpha}+\beta_{i}\right)}=\frac{r_{i+1}}{r_{i}}>0 \tag{4.4}
\end{equation*}
$$

Finally, when $\dot{\alpha}_{i}=0$,

$$
\begin{equation*}
k_{\alpha} \bar{\alpha}=k_{r}\left(\sin \bar{\alpha}-\cos \bar{\alpha} \tan \left(\bar{\alpha}+\beta_{i}\right)\right) \tag{4.5}
\end{equation*}
$$

by substituting (4.4). The left-hand side of (4.5) is constant, thus $\beta_{i}$ should satisfy $\beta_{i}=\beta_{i+1}+\pi a$, with $a \in \mathbb{Z}$. But since, by assumption, the right-hand side of (4.4) is strictly positive, its left-hand side cannot change sign, which implies $a$ is even. Consequently, by (4.4), $r_{i+1} / r_{i}=c, i=1,2, \ldots, n$, where $c$ is some real and positive constant. Therefore, compute

$$
r_{2}=c r_{1} \Rightarrow r_{3}=c^{2} r_{1} \Rightarrow \cdots \Rightarrow r_{n}=c^{n-1} r_{1} \Rightarrow r_{n+1}=c^{n} r_{1} .
$$

But, since $r_{n+1}=r_{1}, r_{1}=c^{n} r_{1}$, which implies that $c^{n}=1$. Because $c$ is real and positive, $c=1$, implying that $r_{i}=r_{i+1}$ for all $i$. Let $\bar{r} \equiv r_{i}$ at equilibrium.

Moreover, following the proof of Theorem 3.1 on page 44, (4.4) implies that either $\beta_{i}=\pi$ or $\beta_{i}=\pi-2 \bar{\alpha}$. At equilibrium (4.2b) yields $\bar{\alpha}=0$ when $\beta_{i}=\pi$, implying that $\beta_{i}=\beta_{i+1}$ for all $i$. However, the constraint (3.7), which also holds for varying speed vehicles, implies (see the proof of Theorem 3.1) that $\beta=\pi$ (with $\bar{\alpha}=0$ ) is not feasible for vehicles in cyclic pursuit.

The remainder of the proof that equilibrium formations are generalized regular polygons follows from the proof of Theorem 3.1 on page 44.

Finally, at equilibrium, (4.2b) is equivalent to

$$
\begin{aligned}
k_{r} / k_{\alpha} & =\bar{\alpha}(\sin \bar{\alpha}+\sin (\bar{\alpha}+\bar{\beta}))^{-1} \\
& = \pm \frac{\pi d}{n}\left(\sin \left( \pm \frac{\pi d}{n}\right)+\sin \left( \pm \frac{\pi d}{n}\right)\right)^{-1} \\
& =\frac{\pi d}{2 n} \csc \left(\frac{\pi d}{n}\right)=: k^{\star}(n) .
\end{aligned}
$$

In other words, the ratio $k^{\star}$ must be defined in order that an equilibrium (with equilibrium distance $\bar{r}>0$ ) exists, concluding the proof.

Therefore, without loss of generality, one can choose $k_{\alpha}=1$ and $k_{r}=k^{\star}$ to ensure the existence of generalized regular polygon equilibria. For example, an equilibrium formation $\{6 / 1\}$ has $k^{\star}=(\pi / 12) \csc (\pi / 6)$, which is precisely the gain used to generate the simulation results of Figure 4.1.

### 4.3 Global Stability Analysis for Two Vehicles

In general, a stability analysis of the multivehicle system (4.2) is not a simple task. As has just been proved, equilibria in relative coordinates exist only for a certain gain ratio $k^{\star}$. This critical gain, in turn, depends on the number of vehicles, $n$. This section investigates the special case when $n=2$, since the analysis is simplified in that $r_{1}=r_{2}, \alpha_{2}=\alpha_{1}+\beta_{1}$, and $\alpha_{1}=\alpha_{2}+\beta_{2}$ (see Figure 4.3).

Consequently, choosing $k_{\alpha}=1$ and $k_{r}=k>0$, the system (4.2) reduces to

$$
\begin{align*}
& \dot{r}_{1}=-k r_{1}\left(\cos \alpha_{1}+\cos \left(\alpha_{1}+\beta_{1}\right)\right)  \tag{4.6a}\\
& \dot{\alpha}_{1}=k\left(\sin \alpha_{1}+\sin \left(\alpha_{1}+\beta_{1}\right)\right)-\alpha_{1}  \tag{4.6b}\\
& \dot{\beta}_{1}=-\beta_{1}  \tag{4.6c}\\
& \dot{r}_{2}=-k r_{2}\left(\cos \alpha_{2}+\cos \left(\alpha_{2}+\beta_{2}\right)\right) \\
& \dot{\alpha}_{2}=k\left(\sin \alpha_{2}+\sin \left(\alpha_{2}+\beta_{2}\right)\right)-\alpha_{2} \\
& \dot{\beta}_{2}=-\beta_{2},
\end{align*}
$$

where $k=k_{r}$. Since the vehicle equations are decoupled, the indices are, hereafter, dropped to simplify notation. One may then proceed by analyzing (4.6).


Figure 4.3: Coordinates for $n=2$ vehicles
The behaviour of this two-vehicle system depends on the selected gain $k$. However, note that when $\beta(0)=-2 \alpha(0)$, subsystems (4.6b) and (4.6c) respectively reduce to $\dot{\alpha}=-\alpha$ and $\dot{\beta}=-\beta$ for all $t \geq 0$, independent of any particular $k$.

Theorem 4.2: Consider $n=2$ unicycles subject to (4.1), each with kinematics (4.6). Let $\mathcal{W}=\{\xi=(\alpha, \beta): \beta=-2 \alpha\}$ and $k^{\star}=\pi / 4$ after (4.3). Then,
(i) if $0<k<k^{\star}$ or if $\xi(0) \in \mathcal{W}$ and $0<k<5 \pi / 4$, the vehicles converge to a common point;
(ii) if $k^{\star}<k<5 \pi / 4$ and $\xi(0) \notin \mathcal{W}$, the vehicles diverge, or;
(iii) if $k=k^{\star}$ and $\xi(0) \notin \mathcal{W}$, the vehicles converge to equally spaced motion around a circle.

When $k \geq 5 \pi / 4$, the analysis is further complicated, as will become clear in the proof. In proving Theorem 4.2, the following theorem is useful.

Theorem 4.3 (cf. Isidori, 1999, Theorem 10.3.1): Consider a composite system

$$
\begin{aligned}
\dot{\alpha} & =f_{\alpha}(\alpha, \beta) \\
\dot{\beta} & =f_{\beta}(\beta) .
\end{aligned}
$$

Suppose the origin of $\dot{\alpha}=f_{\alpha}(\alpha, 0)$ is locally asymptotically stable. Let $\mathcal{S}$ be a set with the property that for any $\tilde{\alpha}(0) \in \mathcal{S}$, the solution $\tilde{\alpha}(t)$ of $\dot{\tilde{\alpha}}=f_{\alpha}(\tilde{\alpha}, 0)$ with initial condition $\tilde{\alpha}(0)$ is defined for all $t \geq 0$ and such that

$$
\lim _{t \rightarrow \infty} \tilde{\alpha}(t)=0
$$

Pick any $\beta(0)$ and let $\beta(t)$ be the solution of $\dot{\beta}=f_{\beta}(\beta)$ with initial condition $\beta(0)$. Suppose $\beta(t)$ is defined for all $t \geq 0$ and such that

$$
\lim _{t \rightarrow \infty} \beta(t)=0
$$

Pick any $\alpha(0) \in \mathcal{S}$ and let $\alpha(t)$ be the solution of $\dot{\alpha}=f_{\alpha}(\alpha, \beta(t))$ with initial condition $\alpha(0)$. Suppose $\alpha(t)$ is defined for all $t \geq 0$, is bounded, and is such that $\alpha(t) \in \mathcal{S}$ for all $t \geq 0$. Then, it is also true that

$$
\lim _{t \rightarrow \infty} \alpha(t)=0
$$

Proof of Theorem 4.2: Since (4.6b) and (4.6c) do not depend on $r$, they can be viewed as an autonomous system in $\xi=(\alpha, \beta)$. Let $(\bar{\alpha}, \bar{\beta}=0)$ denote an equilibrium point of (4.6b,c). From (4.6b), $\bar{\alpha}$ must satisfy

$$
\begin{equation*}
2 k \sin \bar{\alpha}-\bar{\alpha}=0 \tag{4.7}
\end{equation*}
$$

at equilibrium. If $k \leq 1 / 2,(4.6 \mathrm{~b}, \mathrm{c})$ has only one equilibrium point, namely $(0,0)$, since $\bar{\alpha}=0$ is the only solution to (4.7). However, when the gain $k$ is increased to $1 / 2<k<5 \pi / 4$, a bifurcation occurs so that the system acquires two equilibrium points (locations dependent on $k$ ) in addition to the origin.

In general, the following four cases exist.

Case I: $0<k \leq 1 / 2$
In this case, as already noted, $(0,0)$ is the sole equilibrium point. System $(4.6 \mathrm{~b}, \mathrm{c})$ can be viewed as a pair of cascade connected subsystems (cf. Theorem 4.3)

$$
\begin{aligned}
\dot{\alpha} & =f_{\alpha}(\alpha, \beta) \\
\dot{\beta} & =f_{\beta}(\beta)
\end{aligned}
$$

where $\beta$ is an input to (4.6b). First, it will be shown that the origin of

$$
\begin{equation*}
\dot{\alpha}=f_{\alpha}(\alpha, 0) \tag{4.8}
\end{equation*}
$$

is globally asymptotically stable (GAS).
Let $V: \mathbb{R} \rightarrow \mathbb{R}$ be the continuously differentiable function

$$
V(\alpha)=\frac{1}{2} \alpha^{2}
$$

which has the derivative along (4.8) given by

$$
\dot{V}(\alpha)=-\alpha(\alpha-2 k \sin \alpha)
$$

But $\alpha>0 \Rightarrow \alpha>2 k \sin \alpha \Rightarrow \dot{V}<0$ and $\alpha<0 \Rightarrow \alpha<2 k \sin \alpha \Rightarrow \dot{V}<0$. Since $V(0)=0, V(\alpha)>0$ in $\mathbb{R}-\{0\}, V(\alpha)$ is radially unbounded, and $\dot{V}(\alpha)<0$ in $\mathbb{R}-\{0\}$, the origin of (4.8) must be GAS by the Barbashin-Krasovskii theorem (Khalil, 2002, Theorem 4.2). Choose $\mathcal{S}=\mathbb{R}$.

It is clear that the origin of $\dot{\beta}=-\beta$ is GAS.
Next, it is proved that trajectories of $\dot{\alpha}=f_{\alpha}(\alpha, \beta(t))$ are bounded for all $t \geq 0$ and for every $\alpha(0) \in \mathcal{S}$ by showing that trajectories of the full system (4.6b,c) are bounded for all trajectories starting at $\xi(0) \in \mathbb{R}^{2}$. Consider the positive definite function $V_{\Omega}: \mathbb{R}^{2} \rightarrow \mathbb{R}$

$$
V_{\Omega}(\xi)=\frac{1}{2}\left(\alpha^{2}+\frac{\beta^{2}}{2}\right)
$$

which has a derivative along the solutions of $(4.6 \mathrm{~b}, \mathrm{c})$ given by

$$
\begin{aligned}
\dot{V}_{\Omega} & =\alpha g(\xi)-\alpha^{2}-\frac{\beta^{2}}{2} \\
& \leq-\frac{1}{2}\left(\alpha^{2}+\beta^{2}\right)+\frac{1}{2} g^{2}(\xi) \\
& \leq-\frac{1}{2}\|\xi\|_{2}^{2}+k^{2} \\
& <0
\end{aligned}
$$

for all $\|\xi\|_{2}>\sqrt{2} k$, where $g(\xi)=k[\sin \alpha+\sin (\alpha+\beta)]$. Let $\Omega=\left\{\xi \in \mathbb{R}^{2}: V_{\Omega} \leq c\right\}$ with $c>k^{2}$, which corresponds to a ball of radius $\rho>\sqrt{2 c}$ so that $\Omega$ defines a compact, positively invariant set with respect to $(4.6 \mathrm{~b}, \mathrm{c})$. Since one can take $\rho \rightarrow \infty$, it follows that solutions to $\dot{\alpha}=f_{\alpha}(\alpha, \beta(t))$ are bounded for all $t \geq 0$ and for all $\alpha(0) \in \mathcal{S}$. Having satisfied the conditions of Theorem 4.3, it holds that

$$
\lim _{t \rightarrow \infty} \alpha(t)=0
$$

for all $\alpha(0) \in \mathbb{R}$, which implies that the origin of the full system (4.6b,c) is GAS when $0<k \leq 1 / 2$. Interestingly, when $k=1 / 2$, the linearization of $(4.6 \mathrm{~b}, \mathrm{c})$ cannot determine the origin's stability. In a neighbourhood of the origin, $(\cos \alpha+$ $\cos (\alpha+\beta))>0$, which by (4.6a) implies that, after some finite time $t^{\star}>0, r \rightarrow 0$ as $t \rightarrow \infty$ (i.e., the vehicles converge to a common point).

Case II: $1 / 2<k<\pi / 4$
In the cases that remain, the origin of $(4.6 \mathrm{~b}, \mathrm{c})$ is a saddle point and two equilibrium solutions to (4.7) exist, namely $\pm|\bar{\alpha}|$ (see Figure 4.4). It can be checked that $\mathcal{W}=\{\xi: \beta=-2 \alpha\}$ is invariant, making it a stable manifold of the origin. Thus, following the conclusion of Case I, for every $\xi(0) \in \mathcal{W}, r \rightarrow 0$ as $t \rightarrow \infty$ for all $k$.

Consider a proper change of coordinates from $(\alpha, \beta)$ to new coordinates $(\chi, \beta)$, where $\chi=2 \alpha+\beta$ and $\dot{\chi}=f_{\chi}(\chi, \beta)$ with

$$
f_{\chi}(\chi, \beta)=2 k \sin \left(\frac{\chi}{2}\right) \cos \left(\frac{\beta}{2}\right)-\frac{\chi-\beta}{2}
$$

Let $\mathcal{S}=\{\tilde{\chi}: \tilde{\chi}>0\} \subset \mathbb{R}$. Define $V: \mathbb{R} \rightarrow \mathbb{R}$ by $V(\tilde{\chi})=1 / 2(\tilde{\chi}-2|\bar{\alpha}|)^{2}$, which


Figure 4.4: Phase portrait for $(4.6 \mathrm{~b}, \mathrm{c})$ with $k=k^{\star}$
has a derivative along the solutions of $\dot{\tilde{\chi}}=f(\tilde{\chi}, 0)$ given by

$$
\dot{V}(\tilde{\chi})=\underbrace{(\tilde{\chi}-2|\bar{\alpha}|)}_{(*)} \underbrace{\left(2 k \sin \left(\frac{\tilde{\chi}}{2}\right)-\frac{\tilde{\chi}}{2}\right)}_{(* *)} .
$$

It is easy to see that the term denoted $(*)<0$ for all $\tilde{\chi}<2|\bar{\alpha}|$ and $(*)>0$ for all $\tilde{\chi}>2|\bar{\alpha}|$, where $\tilde{\chi} \in \mathcal{S}$. Moreover, for $\tilde{\chi} \in \mathcal{S}$

$$
\begin{aligned}
(* *)<0 & \Longleftrightarrow 2 k \sin \left(\frac{\tilde{\chi}}{2}\right)<\frac{\tilde{\chi}}{2} \\
& \Longleftrightarrow \Longleftrightarrow \frac{\sin (\tilde{\chi} / 2)}{\tilde{\chi} / 2}<\frac{\sin |\bar{\alpha}|}{|\bar{\alpha}|} \\
& \Longleftrightarrow{ }^{(\mathrm{b})} \\
& \tilde{\chi}>2|\bar{\alpha}| .
\end{aligned}
$$

The equivalence (a) comes from (4.7) and the equivalence (b) follows from the fact that $|\bar{\alpha}|<\pi$ for $k<5 \pi / 4$ and the function $\sin x / x$ is strictly positive and monotone decreasing on $[0, \pi)$. It follows that $(* *)>0$ for $\tilde{\chi}<2|\bar{\alpha}|$ when $\tilde{\chi} \in \mathcal{S}$. Since $V(0)=0, V(\tilde{\chi})>0$ in $\mathcal{S}-\{2|\bar{\alpha}|\}$, and $\dot{V}(\tilde{\chi})<0$ in $\mathcal{S}-\{2|\bar{\alpha}|\}$, the equilibrium point $\tilde{\chi}=2|\bar{\alpha}|$ of $\dot{\tilde{\chi}}=f_{\chi}(\tilde{\chi}, 0)$ is asymptotically stable (AS) by Lyapunov's stability theorem (Khalil, 2002, Theorem 4.1). Moreover, it can be checked (using the same argument given for $(* *)$ above) that the set $\mathcal{S}$ is invariant with respect to $\dot{\tilde{\chi}}=f_{\chi}(\tilde{\chi}, 0)$, which implies that

$$
\lim _{t \rightarrow \infty} \tilde{\chi}(t)=2|\bar{\alpha}|
$$

for every trajectory starting in $\mathcal{S}$.
Again, it is clear that the origin of $\dot{\beta}=-\beta$ is GAS, and that trajectories of the full system are bounded (see Case I).

It remains to show that trajectories of $\dot{\chi}=f_{\chi}(\chi, \beta(t))$ that start in $\mathcal{S}$, remain in $\mathcal{S}$ for all $t \geq 0$. Suppose the converse is true and that for some $\chi(0) \in \mathcal{S}$ it happens that $\chi\left(t_{1}\right)=0$ at some time $t_{1}>0$. Then $\xi\left(t_{1}\right) \in \mathcal{W}$. Since it has already been established that $\mathcal{W}$ is itself an invariant set, it must have been that $\chi(t)=0$ for all future and past times. But this is a contradiction. Hence, trajectories of $\dot{\chi}=f_{\chi}(\chi, \beta(t))$ starting in $\mathcal{S}$ must remain in $\mathcal{S}$ for all $t \geq 0$.

Therefore, the conditions of Theorem 4.3 have been satisfied and so

$$
\lim _{t \rightarrow \infty} \chi(t)=2|\bar{\alpha}|
$$

for every $\alpha(0) \in \mathcal{S}$, where $\chi(t)$ is the solution of $\dot{\chi}=f_{\chi}(\chi, \beta(t))$. In the original $(\alpha, \beta)$ coordinates, $\chi>0$ corresponds to the condition that $\beta>-2 \alpha$. Thus, together with the GAS of $\beta=0$, this implies that all solutions starting in the set $\mathcal{S}_{+}=\{\xi: \beta>-2 \alpha\}$ converge to the equilibrium point $(|\bar{\alpha}|, 0)$. An identical argument can be used to show that all solutions starting in the set $\mathcal{S}_{-}=\{\xi$ : $\beta<-2 \alpha\}$ converge to the equilibrium point $(-|\bar{\alpha}|, 0)$. For $1 / 2<k<\pi / 4$ this corresponds to $\bar{\alpha} \in(-\pi / 2,0) \cup(0, \pi / 2)$, wherein (4.6a) yields $r \rightarrow 0$ as $t \rightarrow \infty$.

Case III: $k=\pi / 4$
In this case, after Theorem 4.1, the nonzero equilibria correspond to a $\{2 / 1\}$ polygon and are $( \pm \pi / 2,0)$ since $k^{\star}=\pi / 4$ according to (4.3). Indeed, these equilibria are AS following the technique of Case II. However, as $t \rightarrow \infty, \dot{r} \rightarrow 0$, thus $r \rightarrow \bar{r}$, where $\bar{r}>0$ is some diameter of encirclement.

Still, as noted in Case II, if $\xi(0) \in \mathcal{W}, r \rightarrow 0$ as $t \rightarrow \infty$ for all $k$.

Case IV: $\pi / 4<k<5 \pi / 4$
When $k \geq 5 \pi / 4$, (4.7) has more than three equilibria, further complicating the analysis. Therefore, gains equal to or exceeding $5 \pi / 4$ are disallowed.

Again, following the technique of Case II, for every $\xi(0) \notin \mathcal{W}$, the two equilibria $\bar{\alpha} \in(-\pi,-\pi / 2) \cup(\pi / 2, \pi)$ are AS, which by (4.6a) yields $r \rightarrow \infty$ as $t \rightarrow \infty$ (i.e., the vehicles diverge).

Whether the vehicles circle each other in the counterclockwise or clockwise direction depends on whether they start in the region of attraction of the positive $\left(\mathcal{S}_{+}\right)$or negative $\left(\mathcal{S}_{-}\right)$equilibrium point respectively.

Also, the set of initial conditions $\xi(0) \in \mathcal{W}$, for which changes in $k$ have no effect, corresponds to vehicles that start with $\alpha_{1}(0)=\alpha_{2}(0)+\beta_{2}(0)=-\alpha_{2}(0)$ (see Figure 4.5a). Figure 4.5 b shows the special case when $\alpha_{1}(0)=\alpha_{2}(0)=0$. Figure 4.5 c illustrates the case when $\alpha_{1}(0)=\pi$ and $\alpha_{2}(0)=-\pi$. Note that the same geometric arrangement can be described by $\alpha_{1}(0)=\alpha_{2}(0)=\pi$. However, in this case the vehicle's behaviour depends on $k$.


Figure 4.5: Possible configurations for $\xi(0) \in \mathcal{W}$

### 4.4 Geometry of Pursuit

In the general case, when $n \geq 2$, the number of $\{n / d\}$-polygon formations, increases with $n$, making a global analysis difficult. On the other hand, by employing the techniques developed in Chapter 3, it is possible to study the local stability of these formations through linearization. What is more, it is of interest to understand how the gains $k_{r}$ and $k_{\alpha}$ influence the system's steady-state behaviour.

Several of the ideas (and the notational conventions) introduced in Section 3.3 are also useful in the varying-speed case. For example, one can view the aggregate multivehicle system (4.2) as an the autonomous nonlinear system

$$
\begin{equation*}
\dot{\xi}=\hat{f}(\xi), \tag{4.9}
\end{equation*}
$$

where $\xi=\left(\xi_{1}, \xi_{2}, \ldots, \xi_{n}\right)$. Again, let $\hat{A}$ denote the Jacobian of $\hat{f}$. Prior to linearizing (4.9) about a given equilibrium formation, it is helpful to make two fundamental geometric observations about the possible trajectories of (4.9).

### 4.4.1 Pursuit Constraints

Firstly, the constraints $g(\xi)=0$ defined in Section 3.1.3 on pages $40-42$ apply equally to varying-speed unicycles. Recall that these three real-valued constraints define a submanifold $\mathcal{M}=\left\{\xi \in \mathbb{R}^{3 n}: g(\xi)=0\right\} \subset \mathbb{R}^{3 n}$.

Lemma 4.1: The submanifold $\mathcal{M}$ is invariant under the flow of (4.9).
Proof: By Lemma 3.2 on page $49, \mathcal{M}$ is invariant under $\hat{f}$ if and only if

$$
\frac{\partial g(\xi)}{\partial \xi} \hat{f}(\xi)=0 \text { for every } \xi \in \mathcal{M}
$$

It is shown in Appendix A. 3 that this identity holds for all $\xi \in \mathcal{M}$.

Corollary 4.1: Given $\bar{\xi} \in \mathcal{M}$, the tangent space $T_{\bar{\xi}} \mathcal{M}$ is an invariant subspace of the linearization at $\bar{\xi}$ of (4.9).

Proof: The proof is a direct consequence of Lemmas 3.3 (page 49) and 4.1.

Therefore, by Corollary 4.1, there exists a change of basis for $\mathbb{R}^{3 n}$ that transforms the linearized system matrix $\hat{A}$ into the upper-triangular form

$$
\left[\begin{array}{cc}
\hat{A}_{T_{\bar{\xi}} \mathcal{M}} & * \\
0_{3 \times(3 n-3)} & \hat{A}_{T_{\bar{\xi}} \mathcal{M}}^{\star}
\end{array}\right]
$$

By employing the approach discussed in Section 3.3.2, the following lemma reveals three imaginary axis eigenvalues that do not influence the formation stability.

Lemma 4.2: In the quotient space $\mathbb{R}^{3 n} / T_{\bar{\xi}} \mathcal{M}$, the induced linear transformation $\hat{A}_{T_{\bar{\xi}} \mathcal{M}}^{\star}: \mathbb{R}^{3 n} / T_{\bar{\xi}} \mathcal{M} \rightarrow \mathbb{R}^{3 n} / T_{\bar{\xi}} \mathcal{M}$ has (solely imaginary axis) eigenvalues

$$
\lambda_{1}=0 \text { and } \lambda_{2,3}= \pm j k \frac{\pi d}{n}
$$

Proof: The proof is almost identical to the proof of Lemma 3.5 on page 52; only the dynamics have changed. Let $\varphi=\Phi(\xi)$ be the change of coordinates

$$
\begin{aligned}
& \varphi_{1}=r_{1}, \varphi_{2}=\alpha_{1}, \ldots, \varphi_{3 n-3}=\beta_{n-1} \\
& \varphi_{3 n-2}=g_{1}(\xi), \varphi_{3 n-1}=g_{2}(\xi), \varphi_{3 n}=g_{3}(\xi)
\end{aligned}
$$

Partition these new coordinates into $\varphi=\left(\varphi_{\mathrm{I}}, \varphi_{\mathrm{II}}\right)$ where $\varphi_{\mathrm{I}}=\left(\varphi_{1}, \varphi_{2}, \ldots, \varphi_{3 n-3}\right)$ and $\varphi_{\mathrm{II}}=\left(\varphi_{3 n-2}, \varphi_{3 n-1}, \varphi_{3 n}\right)$. Notice that the set of coordinates in $\varphi_{\text {II }}$ are precisely the functions that define $\mathcal{M}$. Thus, in the new coordinates

$$
\begin{aligned}
\dot{\varphi}_{\mathrm{I}} & =\left.\left[\begin{array}{ll}
I_{3 n-3} & 0_{(3 n-3) \times 3}
\end{array}\right] \hat{f}(\xi)\right|_{\xi=\Phi^{-1}(\varphi)} \\
\dot{\varphi}_{\mathrm{II}} & =\left.\frac{\partial g(\xi)}{\partial \xi} \hat{f}(\xi)\right|_{\xi=\Phi^{-1}(\varphi)}
\end{aligned}
$$

Moreover, the equilibrium $\bar{\varphi}=\Phi(\bar{\xi})$ is equal to $\bar{\xi}$, except that the last 3 components are instead zero. By computing the linearization about this equilibrium,

$$
\begin{aligned}
\dot{\varphi}_{\mathrm{I}} & =\left[\begin{array}{ll}
I_{3 n-3} & 0_{(3 n-3) \times 3}
\end{array}\right] \hat{A} \varphi \\
\dot{\varphi}_{\mathrm{II}} & =\left.\frac{\partial}{\partial \varphi}\left(\left.\frac{\partial g(\xi)}{\partial \xi} \hat{f}(\xi)\right|_{\xi=\Phi^{-1}(\varphi)}\right)\right|_{\bar{\varphi}} \varphi \\
& \left.\stackrel{(\mathrm{B})}{=} \frac{\partial}{\partial \varphi}\left[\begin{array}{c}
-\alpha_{1} g_{2}(\xi)-k r_{1} \sin \left(g_{3}(\xi)\right) \\
\alpha_{1} g_{1}(\xi)+k r_{1} \cos \left(g_{3}(\xi)\right)-k^{\star} r_{1} \\
0
\end{array}\right]_{\xi=\Phi^{-1}(\varphi)}\right|_{\bar{\varphi}} \\
& \varphi \\
& =\left.\frac{\partial}{\partial \varphi}\left[\begin{array}{c}
-\varphi_{2} \varphi_{3 n-1}-k \varphi_{1} \sin \varphi_{3 n} \\
\varphi_{2} \varphi_{3 n-2}+k \varphi_{1} \cos \varphi_{3 n}-k^{\star} \varphi_{1}
\end{array}\right]\right|_{\bar{\varphi}} \varphi \\
& =\left[\begin{array}{lll|lll}
0 & \cdots & 0 & -\bar{\alpha} & -k \bar{r} \\
0 & \cdots & 0 & \bar{\alpha} & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0
\end{array}\right] \varphi=\left[\begin{array}{lll}
0_{3 \times(3 n-3)} & \hat{A}_{T_{\bar{\xi}} \mathcal{M}}^{\star}
\end{array}\right] \varphi
\end{aligned}
$$

where the lengthy derivation of equivalence (B) can be found in Appendix A.3. The $3 \times 3$ block $\hat{A}_{T_{\bar{\xi}} \mathcal{M}}^{\star}$ has eigenvalues $\lambda_{1,2,3}=\{0, \pm j k \bar{\alpha}\}$, with $\bar{\alpha}= \pm \pi d / n$ from Theorem 4.1, concluding the proof.

Therefore, just as in Chapter 3, when determining the stability of a given $\{n / d\}$ formation, three imaginary axis eigenvalues of $\hat{A}$ may be ignored and the formation's stability can be assessed based on the remaining $3 n-3$ eigenvalues. Reiterating, this is because the multivehicle system is constrained to evolve, at $\bar{\xi} \in \mathcal{M}$, along the tangent space $T_{\bar{\xi}} \mathcal{M} \subset \mathbb{R}^{3 n}$ and not in the quotient space $\mathbb{R}^{3} n / T_{\bar{\xi}} \mathcal{M}$ corresponding to the above mentioned imaginary axis eigenvalues.

### 4.4.2 Formation Subspace

The second geometric observation about the trajectories of (4.9) is that for every polygon density $d \in\{1,2, \ldots, n-1\}$ there exists a set of points in $\mathcal{M} \subset \mathbb{R}^{3 n}$, denoted $\mathcal{F}_{d} \subset \mathcal{M}$, where the pursuit graph $\Gamma_{t}$ corresponding to (4.9) is a generalized regular polygon of the form $\{n / d\}$. Let $\mathcal{F}_{d}$ be called a formation subspace.

To see this, define a 1-dimensional affine subspace of $\mathbb{R}^{3 n}$

$$
\begin{gathered}
\mathcal{F}_{d}=\left\{\xi \in \mathbb{R}^{3 n}: r_{i}=r_{i+1} \text { for } i=2,3, \ldots, n, \alpha_{i}=\bar{\alpha}=\pi d / n\right. \\
\text { and } \left.\beta_{i}=\bar{\beta}=\pi-2 \bar{\alpha} \text { for } i=1,2, \ldots, n\right\} .
\end{gathered}
$$

Alternatively, $\mathcal{F}_{d}$ can be defined by $3 n-1$ constraint functions

$$
\begin{aligned}
& h_{1}(\xi)=\alpha_{1}-\bar{\alpha}, h_{2}(\xi)=\beta_{1}-\bar{\beta}, h_{3}(\xi)=r_{2}-r_{3}, h_{4}(\xi)=\alpha_{2}-\bar{\alpha} \\
& h_{5}(\xi)=\beta_{2}-\bar{\beta}, h_{6}(\xi)=r_{3}-r_{4}, \ldots, h_{3 n-1}=\beta_{n}-\bar{\beta}
\end{aligned}
$$

Thus, each formation subspace $\mathcal{F}_{d}, d \in\{1,2 \ldots, n-1\}$, corresponds to the set of configurations for which the pursuit graph $\Gamma_{t}$ forms a generalized regular polygon ${ }^{1}\{n / d\}$, parameterized by its radius. This fact will be formalized in Lemma 4.4. The $3 n-1$ constraints $h(\xi)$, given above, leave exactly one degree of freedom in $\mathcal{F}_{d}$ : namely, the polygon's radius.

Lemma 4.3: The submanifold $\mathcal{F}_{d}$ is invariant under $\hat{f}$.
Proof: By Lemma 3.2 on page $49, \mathcal{F}_{d}$ is invariant under $\hat{f}$ if and only if

$$
\frac{\partial h(\xi)}{\partial \xi} \hat{f}(\xi)=0 \text { for all } \xi \in \mathcal{F}
$$

Therefore, compute

$$
\begin{aligned}
\left.\frac{\partial\left(r_{i}-r_{i+1}\right)}{\partial \xi} \hat{f}(\xi)\right|_{\xi \in \mathcal{F}_{d}} & =k_{r}\left(\left(r_{i+1}-r_{i+2}\right) \cos (\bar{\alpha}+\bar{\beta})-\left(r_{i}-r_{i+1}\right) \cos \bar{\alpha}\right)=0 \\
\left.\frac{\partial\left(\alpha_{i}-\bar{\alpha}\right)}{\partial \xi} \hat{f}(\xi)\right|_{\xi \in \mathcal{F}_{d}} & =k_{r}(\sin \bar{\alpha}+\sin (\bar{\alpha}+\bar{\beta}))-k_{\alpha} \bar{\alpha}=0 \\
\left.\frac{\partial\left(\beta_{i}-\bar{\beta}\right)}{\partial \xi} \hat{f}(\xi)\right|_{\xi \in \mathcal{F}_{d}} & =k_{\alpha}(\bar{\alpha}-\bar{\alpha})=0
\end{aligned}
$$

for $i=1,2, \ldots, n$, which concludes the proof.

In Theorem 4.1, it was shown that at equilibrium the pursuit graph $\Gamma_{t}$ corresponding to (4.9) is a generalized regular polygon $\{n / d\}$. Next, it is established that $\Gamma_{t}$ is in fact a generalized regular polygon for every $\xi \in \mathcal{F}_{d}$.

[^4]Lemma 4.4: For $\xi \in \mathcal{F}_{d}$, the $n$-unicycle pursuit graph $\Gamma_{t}$ corresponding to (4.9) is a generalized regular polygon $\{p\}$, where $p=n / d$ and $d \in\{1,2, \ldots, n-1\}$.

Proof: Since, for $\xi \in \mathcal{F}_{d}, r_{i}=r_{i+1}$, the system's pursuit graph $\Gamma_{t}$ is equilateral (i.e., $\left\|e_{i}\right\|_{2}=\left\|e_{i+1}\right\|_{2}$ ). Let $\psi_{i}$ be the internal angle at vertex $i$ of the pursuit graph. The pursuit graph is equiangular (i.e., $\psi_{i}=\psi_{i+1}$ ) since it can be checked using the geometry of Figure 3.1 that the internal angle at each vertex is given by

$$
\bar{\psi} \equiv \psi_{i}= \begin{cases}\alpha_{i-1}+\beta_{i-1}-\alpha_{i}=\bar{\beta} & \text { for } \bar{\alpha}>0 \\ -\alpha_{i-1}-\beta_{i-1}+\alpha_{i}=-\bar{\beta} & \text { for } \bar{\alpha}<0\end{cases}
$$

for $\xi \in \mathcal{F}$. Therefore, $\Gamma_{t}$ is a $\{n / d\}$ polygon.

From this, it is also possible to conclude that the constant angle $\bar{\beta}$ is always independent of the chosen gains $k_{r}$ and $k_{\alpha}$.

Corollary 4.2: The angle $\bar{\beta}= \pm \pi(1-2 d / n)$ and is independent of $k_{r}$ and $k_{\alpha}$.
Proof: By Lemma 3.1 on page 44, the internal angles of $\{p=n / d\}$ must sum to $n \bar{\psi}=n \pi(1-2 d / n)$. From Lemma 4.4 , for $\xi \in \mathcal{F}_{d}$, the pursuit graph $\Gamma_{t}$ is a generalized regular polygon $\{p\}$. Therefore, the internal angle $\bar{\psi}= \pm \bar{\beta}$ at each vertex gives $\bar{\beta}= \pm \pi(1-2 d / n)$, independent of $k_{r}$ and $k_{\alpha}$.

Notice that, for points on $\mathcal{F}_{d}$ the controller gains $k_{r}, k_{\alpha}>0$ and the constant angles $\bar{\alpha}, \bar{\beta} \in[-\pi, \pi]$ satisfy

$$
\begin{equation*}
k_{r} / k_{\alpha}=\bar{\alpha}(\sin \bar{\alpha}+\sin (\bar{\alpha}+\bar{\beta}))^{-1} \tag{4.10}
\end{equation*}
$$

With $\bar{\beta}$ independent of the gains $k_{r}$ and $k_{\alpha}$, for a given $\{n / d\}$ formation the corresponding equilibrium value $\bar{\alpha}$ is then determined by equation (4.10). Thus, the system's steady-state behaviour depends only on the ratio $k_{r} / k_{\alpha}$.

In summary, Lemma 4.4 says that for every $n>1$ there are associated affine subspaces, denoted $\mathcal{F}_{d}$ where $d \in\{1,2, \ldots, n-1\}$, each one invariant under (4.9) and in which the pursuit graph $\Gamma_{t}$ corresponding to (4.9) is a generalized polygon of type $\{n / d\}$. The dynamics on each of these affine subspaces may be zero or varying, as will be shown next, depending on the ratio of controller gains $k_{r} / k_{\alpha}$.

### 4.5 Local Stability Analysis for $k_{r} / k_{\alpha}=k^{\star}$

The purpose of this section is to determine, for the case when $k_{r} / k_{\alpha}=k^{\star}$, precisely which $\{n / d\}$ equilibrium formations are locally asymptotically stable. In this case, according to Theorem 4.1, every point $\xi \in \mathcal{F}_{d}$ is an equilibrium point of (4.9). Moreover, the equilibrium values for $\bar{\alpha}$ and $\bar{\beta}$ are those given by Theorem 4.1.

### 4.5.1 Block Circulant Linearization

As in Chapter 3, linearizing (4.2) about an equilibrium point $\bar{\xi}_{i}=(\bar{r}, \bar{\alpha}, \bar{\beta})$ gives $n$ identical linear subsystems, each of the form $\dot{\tilde{\xi}}_{i}=A \tilde{\xi}_{i}+B \tilde{\xi}_{i+1}$, where $\tilde{\xi}_{i}=\xi_{i}-\bar{\xi}_{i}$ and the matrices $A$ and $B$ are given by

$$
\begin{aligned}
A & =\left.\frac{\partial f\left(\xi_{i}, \xi_{i+1}\right)}{\partial \xi_{i}}\right|_{(\bar{r}, \bar{\alpha}, \bar{\beta})} \\
& =\left[\begin{array}{ccc}
-\frac{1}{2} q \pi \cot (q \pi) & q \pi \bar{r} & \frac{1}{2} q \pi \bar{r} \\
-\frac{1}{2 \bar{r}} q \pi & -1 & -\frac{1}{2} q \pi \cot (q \pi) \\
0 & 1 & 0
\end{array}\right] \\
B & =\left[\begin{array}{ccc}
\frac{1}{2} q \pi \cot (q \pi) & 0 & 0 \\
\frac{1}{2 \bar{r}} q \pi & 0 & 0 \\
0 & -1 & 0
\end{array}\right] .
\end{aligned}
$$

Therefore, the full system Jacobian of $\hat{f}$ has the block circulant form

$$
\hat{A}=\operatorname{circ}\left(A, B, 0_{3 \times 3}, \ldots, 0_{3 \times 3}\right)
$$

### 4.5.2 Spectral Analysis

For a given $\{n / d\}$-polygon formation, let $\mathcal{F}_{d}^{0}$ denote the invariant subspace formed by the affine subspace $\mathcal{F}_{d}$ expressed in $\tilde{\xi}$ coordinates (i.e., shifted so that the origin is an equilibrium point $\bar{\xi} \in \mathcal{F}_{d}$ ).

Lemma 4.5: The restriction of $\hat{A}$ to $\mathcal{F}_{d}^{0}$ equals zero.
In other words, there is a zero eigenvalue in $\hat{A}$ corresponding to motion along $\mathcal{F}_{d} \subset \mathcal{M}$. This result is rather obvious, since every point in the affine subspace $\mathcal{F}_{d}$ is an equilibrium point when $k_{r} / k_{\alpha}=k^{\star}$ (cf. Theorem 4.1). Therefore, combining the results of Lemmas 4.2 and 4.5, this leaves $3 n-4$ eigenvalues of $\hat{A}$,
which together determine the local stability of a given $\{n / d\}$ equilibrium formation. Thus, if a given polygon formation subspace $\mathcal{F}_{d}$ is locally asymptotically stable, then the formation's radius at equilibrium depends on the initial vehicle configurations. Figure 4.6 conceptually illustrates this situation for this case when the formation in question is locally asymptotically stable.


Figure 4.6: Trajectories in $\mathcal{M}$ locally approaching $\mathcal{F}_{d}$ with $k=k^{\star}$
Naturally, the block circulant structure of $\hat{A}$ can be exploited to further isolate its eigenvalues. As in Chapter 3, this can be accomplished through block diagonalization of the system matrix $\hat{A}$. Recall that $\omega^{i-1}:=e^{j 2 \pi(i-1) / n} \in \mathbb{C}$ denotes the $i$-th of $n$ roots of unity, where $j:=\sqrt{-1}$. Again, let $q:=p^{-1}=d / n$.

Lemma 4.6: The matrix $\hat{A}$ can be block diagonalized into $\operatorname{diag}\left(D_{1}, D_{2}, \ldots, D_{n}\right)$, where each $3 \times 3$ block is given by $D_{i}=A+\omega^{i-1} B, i=1,2, \ldots, n$.

The proof of Lemma 4.6 follows from Davis (1994, Theorem 5.6.4) and is essentially the same as Lemma 3.6 on page 54. Consequently, each diagonal block has the same form

$$
D_{i}=\left[\begin{array}{ccc}
\frac{\pi}{2} q \cot (q \pi)\left(\omega^{i-1}-1\right) & q \pi \bar{r} & \frac{\pi}{2} q \bar{r} \\
\frac{\pi}{2 \bar{r}} q\left(\omega_{i-1}-1\right) & -1 & -\frac{\pi}{2} q \cot (q \pi) \\
0 & 1-\omega^{i-1} & 0
\end{array}\right] .
$$

Lemma 4.7: The stability of $\hat{A}$ is independent of $\bar{r}>0$.

Proof: Every matrix $D_{i}$ can be factored as $D_{i}=T \tilde{D}_{i} T^{-1}$, where

$$
T=\left[\begin{array}{lll}
\bar{r} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

(recall $0<q<1$ ) and

$$
\tilde{D}_{i}=\left[\begin{array}{ccc}
\frac{1}{2} q \pi \cot (q \pi)\left(\omega^{i-1}-1\right) & q \pi & \frac{1}{2} q \pi \\
\frac{1}{2} q \pi\left(\omega^{i-1}-1\right) & -1 & -\frac{1}{2} q \pi \cot (q \pi) \\
0 & 1-\omega^{i-1} & 0
\end{array}\right]
$$

This implies that, $\sigma\left(D_{i}\right)=\sigma\left(\tilde{D}_{i}\right)$ and, thus, that $\sigma\left(D_{i}\right)$ is independent of $\bar{r}>0$.

### 4.5.3 Stable Pursuit Formations

Observe that the eigenvalues of $D_{1}=A+B$ are among the eigenvalues of $\hat{A}$ for every $n$. The characteristic polynomial of $D_{1}$ is $p_{D_{1}}(\lambda)=\lambda^{2}(\lambda+1)$, so the eigenvalues of $D_{1}$ are $\lambda_{1,2}=0$ and $\lambda_{3}=-1$. Therefore, as predicted by Lemmas 4.2 and 4.5 , two zero eigenvalues have been discovered, while the remaining eigenvalue of $D_{1}$ has $\operatorname{Re}\left(\lambda_{3}\right)=-1<0$ for every $0<q<1$.

Proposition 4.1: The $\{2 / 1\}$ formation is locally asymptotically stable.

Proof: When $n=2, q=1 / 2$ and $D_{2}=A-B$ is the only block to analyze, apart from $D_{1}=A+B$. The eigenvalues of $D_{2}$ are $\lambda_{4,5}= \pm j \frac{\pi}{2}$ and $\lambda_{6}=-1$ (computations not shown). By disregarding the imaginary axis eigenvalues according to Lemma 4.2, one may conclude that the $\{2 / 1\}$ polygon is stable.

Of course, this result is already known from Section 4.3. However, the short proof serves to emphasize the local analysis technique. As in the previous chapter, the blocks $\tilde{D}_{i}$ are in general complex matrices. In this general case, the characteristic polynomial of $\tilde{D}_{i}$ is

$$
p_{\tilde{D}_{i}}(\lambda)=\lambda^{3}+c_{1} \lambda^{2}+c_{2} \lambda+c_{3},
$$

where the complex coefficients $c_{1}, c_{2}, c_{3} \in \mathbb{C}$ are

$$
\begin{aligned}
& c_{1}=1+\frac{1}{2} q \pi\left(1-\omega^{i-1}\right) \cot (q \pi) \\
& c_{2}=\frac{1}{2} q \pi\left(1-\omega^{i-1}\right)(q \pi+2 \cot (q \pi)) \\
& c_{3}=\frac{1}{4}(q \pi)^{2}\left(1-\omega^{i-1}\right)^{2}\left(1+\cot ^{2}(q \pi)\right)
\end{aligned}
$$

Therefore, to determine the stability of $\tilde{D}_{i}$ one merely needs to check that the leading principal minors of the Hermitian matrix (3.20) on page 58. The computed expressions for the minors, denoted $h_{1}, h_{2}$, and $h_{3}$, are rather lengthy and, therefore, are not explicitly reproduced here. Instead, these minors are represented graphically, as in Section 3.3.4. Let $\omega^{i-1}=w_{i}+j z_{i}$ and let $-1<w<1$ and $0<\mu<1$ represent $w_{i}$ and $q$, respectively, on a continuum.

For a given $n$, define the set

$$
\mathcal{W}_{n}=\left\{w_{i}=\operatorname{Re}\left(\omega^{i-1}\right): i=1,2, \ldots, n\right\} .
$$

With reference to the leading principal minor $h_{2}$, consider the set of points that are not stable $\mathcal{H}_{2}=\left\{(\mu, w): h_{2} \leq 0, \mu \in(0,1), w \in(-1,1)\right\}$, illustrated by the region marked $\mathcal{U}$ in Figure 4.7a. Let int $\mathcal{H}_{2}$ denote the interior of $\mathcal{H}_{2}$.

Lemma 4.8: Every $\{n / d\}$ formation with $n / 2<d<n$ is unstable.

Proof: It will be shown that for all $\{n / d\}$, with $n / 2<d<n$, there exists an index $i^{\star} \in\{1,2, \ldots, n\}$ with associated $w_{i^{\star}} \in \mathcal{W}_{n}$ such that $\left(q, w_{i^{\star}}\right) \in \operatorname{int} \mathcal{H}_{2}$, implying that the block $D_{i^{\star}}$ is unstable. It is a fact (most easily checked numerically) that the pair $(\mu, \cos (2 \pi \mu)) \in \mathcal{H}_{2}$ for every $\mu \in(1 / 2,1)$, as illustrated by the dotted line in Figure 4.7a. Take $i^{\star}:=d+1$. Observe that $w_{i^{\star}}=\cos (2 \pi q) \in \mathcal{W}_{n}$. However, by the above stated fact, this implies that $\left(q, w_{i^{\star}}\right) \in \operatorname{int} \mathcal{H}_{2}$. Hence, $D_{i^{\star}}$ is unstable, implying that the formation $\{n / d\}$ is unstable.

Now, consider the set of points that are not stable, with reference to the leading principal minor $h_{3}, \mathcal{H}_{3}=\left\{(\mu, w): h_{2} \leq 0, \mu \in(0,1 / 2), w \in(-1,1)\right\}$, illustrated by the region marked $\mathcal{U}$ in Figure 4.7 b . It is a fact that the lower boundary of $\mathcal{U}$ in Figure 4.7 b is given by the function $\underline{w}(\mu)=\cos (2 \pi \mu)$, which was obtained by solving (with the help of computer algebra software) the equation $h_{3}(\mu, w)=0$ on the relevant domain $\mu=(0,1 / 2]$ and $w \in(-1,1)$. As a result, the above definition


Figure 4.7: Parameter $w$ as a function of $\mu$ for $h_{2}$ and $h_{3}$
for the set $\mathcal{H}_{3}$ is equivalent to $\mathcal{H}_{3}=\{(\mu, w): \mu \in(0,1 / 2], w \in[\underline{w}(\mu), 1)\}$. Let $\partial \mathcal{H}_{3}$ denote the boundary of $\mathcal{H}_{3}$.

Theorem 4.4 (Main Stability Result): For $n \geq 2$, the only locally asymptotically stable equilibrium polygons are those of the form $\{n / 1\}$.

Proof: By Lemma 4.8, one needs only to study those polygons with $1 \leq d \leq$ $n / 2$. It will first be shown that for all $\{n / d\}$, with $1 \leq d \leq n / 2$, there exists an index $i^{\prime} \in\{1,2, \ldots, n\}$ with associated $w_{i^{\prime}} \in \mathcal{W}_{n}$ such that $\left(q, w_{i^{\prime}}\right) \in \operatorname{int} \mathcal{H}_{3}$, implying the block $D_{i^{\prime}}$ is unstable.

Let $i^{\prime}=d$ and observe that $w_{i^{\prime}}=\cos (2 \pi(d-1) / n) \in \mathcal{W}_{n}$, which implies that $\underline{w}(q)<w_{i^{\prime}}<1$. In other words, $w_{i^{\prime}} \in \operatorname{int} \mathcal{H}_{3}$, implying that every polygon with $1<d \leq n / 2$ is also unstable. Thus, the only remaining polygons are those with $d=1$. To show that these remaining $\{n / 1\}$ polygons are stable for all $n$, it will be proved that for all indices $i \in\{1,2, \ldots, n\}$ the blocks $D_{i}$ are indeed stable, modulo the imaginary axis eigenvalues that can be ignored according to Lemmas 4.2 and 4.5. First, note that $h_{1}=2+\mu \pi(1-w) \cos (\pi \mu)>0$ for every $(\mu, w)$ on the relevant domain. Moreover, Figure 4.7a shows the parameter $w$ as a function of $\mu$ for $h_{2}$, in addition to the boundary of $\mathcal{H}_{3}$. From the figure, it is clear that every point not in the set $\mathcal{H}_{3}$ also has $h_{2}>0$ on the relevant domain. Thus, to show stability on this domain it is enough to show that $h_{3}>0$.

Let $d=1$. First, check that the blocks $D_{i}$ with $i \in\{3,4, \ldots, n-1\}$ are stable (i.e, yield $h_{3}>0$ ), which is equivalent to verifying that $w_{i}<\underline{w}(\mu)$ for all $i \in\{3,4, \ldots, n-1\}$ (see Figure 4.7b). But this is always true since $w_{i}=$ $\cos (2 \pi(i-1) / n)<\cos (2 \pi / n)$ for all $i \in\{3,4, \ldots, n-1\}$. Now, choose $i=1$. In this case, $D_{1}=A+B$, which is known to have eigenvalues $\lambda_{1,2}=0$ and $\lambda_{3}=-1$. Thus, according to Lemmas 4.2 and 4.5, these two zero eigenvalues can be ignored since they do not influence the stability of a given $\{n / d\}$ polygon.

For $d=1, i^{\star}=d+1=2$ and $w_{2}=\cos (2 \pi / n)$. Thus the point $\left(q, w_{2}\right) \in \partial \mathcal{H}_{3}$ (i.e., it lies exactly on the boundary of $\mathcal{H}_{3}$ in Figure 4.7b). Together, the matrix $D_{2}$ and its complex conjugate $D_{n}$ have two imaginary axis eigenvalues (one each) of the form $\lambda= \pm j \pi / n$, while the remaining eigenvalues have $\operatorname{Re}(\lambda) \neq 0$. These facts were verified with the assistance of computer algebra software. According to Lemma 4.2, these two imaginary axis eigenvalues can be ignored. Of course, the eigenvalues with $\operatorname{Re}(\lambda) \neq 0$ cannot be unstable, otherwise the point $\left(q, w_{2}\right)$ would
lie in $\operatorname{int} \mathcal{H}_{3}$, as opposed to in $\partial \mathcal{H}_{3}$, concluding the proof.

In summary, for unicycles in cyclic pursuit under the control law (4.1), with $k_{\alpha}=1$ and $k_{r}=k^{\star}$, equilibrium formations of the type $\{n / 1\}$ with $n \geq 2$ are locally asymptotically stable, while the remaining formations with $2 \leq d<n$ are not. Moreover, the equilibrium distance between unicycles $\bar{r}>0$ depends on the initial vehicle configurations. Lastly, the findings of this section are consistent with the observed simulation results of Figure 4.1.

### 4.6 Local Stability Analysis for $k_{r} / k_{\alpha} \neq k^{\star}$

In this section, the ratio of controller gains $k_{r} / k_{\alpha}$ is allowed to take on values other than $k^{\star}$, as in Figure 4.2. Once more, presume that $k_{\alpha}=1$ and $k_{r}=k$ without loss of generality (see Section 4.2). In order to utilize the local stability result of Theorem 4.4, consider the case when $k=k^{\star} \pm \epsilon$, where $\epsilon>0$. Thus, $k$ remains in some $\epsilon$-neighbourhood of the critical gain $k^{\star}$. The objective is to (locally) explain the simulation results of Figures 4.2a and 4.2b, where the unicycles converge and diverge, respectively, but appear to do so in formation. Define $\varphi=\Phi(\xi)$ by

$$
\begin{align*}
& \varphi_{1}=r_{1}, \varphi_{2}=\alpha_{1}-\bar{\alpha}, \varphi_{3}=\beta_{1}-\bar{\beta}, \varphi_{4}=r_{2} / r_{3}-1, \ldots, \varphi_{3 n-2}=r_{n} / r_{1}-1 \\
& \varphi_{3 n-1}=\alpha_{n}-\bar{\alpha}, \varphi_{3 n}=\beta_{n}-\bar{\beta} \tag{4.11}
\end{align*}
$$

so that the last $3 n-1$ coordinates are again zero on the affine subspace $\mathcal{F}_{d} \subset \mathcal{M}$, defined in Section 4.4.2. A verification that the above change of coordinates is proper is provided in Appendix A.1.2. In contrast to the previous section, no point with $r_{i}>0$ in $\mathcal{F}_{d}$ is an equilibrium point because $k \neq k^{\star}$. Thus, in the new coordinates one obtains the dynamics

$$
\begin{aligned}
\dot{\varphi}_{1}= & -k\left(r_{1} \cos \alpha_{1}+r_{2} \cos \left(\alpha_{1}+\beta_{1}\right)\right)_{\xi=\Phi^{-1}(\varphi)} \\
= & -k \varphi_{1}\left(\cos \left(\varphi_{2}+\bar{\alpha}\right)+\left(\varphi_{4}+1\right)\left(\varphi_{7}+1\right) \cdots\right. \\
& \left.\cdots\left(\varphi_{3 n-2}+1\right) \cos \left(\varphi_{2}+\bar{\alpha}+\varphi_{3}+\bar{\beta}\right)\right)
\end{aligned}
$$

while the remaining coordinates are such that, if

$$
\begin{aligned}
\varphi_{\mathrm{I}} & :=\varphi_{1} \\
\varphi_{\mathrm{II}} & :=\left[\begin{array}{lll}
\varphi_{2} \varphi_{3} & \cdots & \varphi_{3 n}
\end{array}\right]^{\top},
\end{aligned}
$$

one obtains the following upper triangular structure

$$
\begin{align*}
\dot{\varphi}_{\mathrm{I}} & =f_{\mathrm{I}}\left(\varphi_{\mathrm{I}}, \varphi_{\mathrm{II}}\right)  \tag{4.12a}\\
\dot{\varphi}_{\mathrm{II}} & =f_{\mathrm{II}}\left(\varphi_{\mathrm{II}}\right) . \tag{4.12b}
\end{align*}
$$

Notice how the set of points with $\varphi_{\mathrm{II}}=0$ exactly corresponds to a given affine subspace $\mathcal{F}_{d}$ and that $f_{\mathrm{II}}(0)=0$. Thus, if $\varphi_{\mathrm{II}}(t) \rightarrow 0$ as $t \rightarrow \infty$, the multivehicle system's pursuit graph approaches a generalized regular polygon of type $\{n / d\}$, whether the distance between unicycles converges to a constant value or not.

Lemma 4.9: For a given $\{n / d\}$ formation, the equilibrium point $\varphi_{\mathrm{II}}=0$ of (4.12b) is locally asymptotically stable for all $k$ sufficiently near $k^{\star}$ if and only if $d=1$.

The proof of Lemma 4.9 follows immediately from Theorem 4.4. Firstly, recall that $\mathcal{F}_{d} \subset \mathcal{M}$, implying that the Jacobian of $f_{\text {II }}$ at $\varphi_{\text {II }}=0$ must possess the three imaginary axis eigenvalues (which are independent of $k$ ) revealed in the proof of Lemma 4.2. Now, it is well known that the eigenvalues of a matrix are continuous functions of its elements. Since the elements of $A_{\text {II }}$ are also continuous functions of the parameter $k=k^{\star} \pm \epsilon$, any stable eigenvalues of $A_{\text {II }}$ will remain in the left-half complex plane for sufficiently small $\epsilon$. Likewise, any unstable eigenvalues will also remain in the right-half complex plane, implying by Theorem 4.4 that the only locally asymptotically stable formations are those of the type $\{n / 1\}$. In other words, there exists a sufficiently small neighbourhood of $\mathcal{F}_{1}$ wherein $\alpha_{i} \rightarrow \bar{\alpha}$, $\beta_{i} \rightarrow \bar{\beta}$, and the ratio of distances $r_{i} / r_{i+1} \rightarrow 1$. Equivalently, the unicycles converge to a generalized regular polygon formation of type $\{n / 1\}$, as per Lemma 4.4. How this polygon formation's radius changes with time, as a function of the chosen controller gain $k>0$, is examined next in Theorems 4.5 and 4.6.

The right-hand side of equation (4.10) defines a function $k(\bar{\alpha})$. Differentiating this with respect to $\bar{\alpha}$ (recall that $\bar{\beta}$ is constant according to Corollary 4.2) gives

$$
\frac{\partial k}{\partial \bar{\alpha}}=(\sin \bar{\alpha}+\sin (\bar{\alpha}+\bar{\beta}))^{-2}(\sin \bar{\alpha}+\sin (\bar{\alpha}+\bar{\beta})-\bar{\alpha}(\cos \bar{\alpha}+\cos (\bar{\alpha}+\bar{\beta})))
$$

which equals $(1 / 2) \csc (\pi / n)$ for $\bar{\alpha}=\pi / n$ and $k=k^{\star}$. Since $\csc (\pi \mu)>0$ for $\mu \in(0,1)$ and by the continuity of $k(\bar{\alpha})$, the slope of the graph of (4.10) is positive for $k$ in an $\epsilon$-neighbourhood of $k^{\star}$.

Let $\bar{\alpha}$ be the solution to (4.10) when $k=k^{\star} \pm \epsilon$. Thus, when $k=k^{\star}-\epsilon$ (respectively, $k=k^{\star}+\epsilon$ ) and for sufficiently small $\epsilon>0$ it holds that $0<\bar{\alpha}=$ $\pi / n-\delta(\epsilon)<\pi$ (respectively, $0<\bar{\alpha}=\pi / n+\delta(\epsilon)<\pi$ ), where $\delta(\epsilon)=\bar{\alpha}-\pi / n>0$.

Theorem 4.5 (Converging Vehicles): If $k=k^{\star}-\epsilon$, for small enough $\epsilon>0$ and $\xi(0)$ in a sufficiently small neighbourhood of $\mathcal{F}_{1}$, the $n$-unicycle pursuit graph corresponding to (4.9) converges to a generalized regular polygon of type $\{n / 1\}$ while $r_{i}(t) \rightarrow 0$ as $t \rightarrow \infty, i=1,2, \ldots, n$.

Proof: The proof follows by applying Theorem 4.3 on page 81 to the composite system (4.12). By Lemma 4.9, for a given $\{n / d\}$ polygon formation, the origin of (4.12b) is locally asymptotically stable if and only if $d=1$. Let $\varphi_{\mathrm{II}}(t)$ denote the solution of $(4.12 \mathrm{~b})$ starting at $\varphi_{\mathrm{II}}(0)$ and let $\mathcal{R}_{A} \subset \mathbb{R}^{3 n-1}$ be a sufficiently small neighbourhood of the origin such $\lim _{t \rightarrow \infty} \varphi_{\mathrm{II}}(t)=0$ for every $\varphi_{\mathrm{II}}(0) \in \mathcal{R}_{A}$.

Next, it is shown that the origin of

$$
\begin{equation*}
\dot{\varphi}_{\mathrm{I}}=f_{\mathrm{I}}\left(\varphi_{\mathrm{I}}, 0\right) \tag{4.13}
\end{equation*}
$$

is globally asymptotically stable (GAS). Let $V_{I}: \mathbb{R}_{+} \rightarrow \mathbb{R}$ be the continuously differentiable function $V_{\mathrm{I}}\left(\varphi_{\mathrm{I}}\right)=\varphi_{\mathrm{I}}^{2} / 2$, which has the derivative along (4.13)

$$
\begin{aligned}
\dot{V}_{\mathrm{I}}\left(\varphi_{\mathrm{I}}\right) & =-k \varphi_{\mathrm{I}}^{2}(\cos \bar{\alpha}+\cos (\bar{\alpha}+\bar{\beta})) \\
& =-k \varphi_{\mathrm{I}}^{2}(\cos (\pi / n-\delta(\epsilon))-\cos (\pi / n+\delta(\epsilon)))
\end{aligned}
$$

Since $0<\pi / n<\pi$ and $\delta(\epsilon)>0$, it holds that $\dot{V}_{\mathrm{I}}\left(\varphi_{\mathrm{I}}\right)<0$ on $\mathbb{R}_{+}-\{0\}$. This, together with the facts $V_{\mathrm{I}}(0)=0, V_{\mathrm{I}}\left(\varphi_{\mathrm{I}}\right)>0$ in $\mathbb{R}_{+}-\{0\}$, and $V_{\mathrm{I}}\left(\varphi_{\mathrm{I}}\right)$ is radially unbounded, implies that the origin of (4.13) is GAS by the Barbashin-Krasovskii theorem (Khalil, 2002, Theorem 4.2). Choose $\mathcal{S}:=\mathbb{R}_{+}$.

Finally, it must be shown that the trajectories of

$$
\begin{equation*}
\dot{\varphi}_{\mathrm{I}}=f_{\mathrm{I}}\left(\varphi_{\mathrm{I}}, \varphi_{\mathrm{II}}(t)\right) \tag{4.14}
\end{equation*}
$$

are bounded for all $t \geq 0, \varphi_{\mathrm{I}}(0) \in \mathcal{S}$, and sufficiently small $\varphi_{\mathrm{II}}(0)$. This is done by defining the product set

$$
\Omega=\left\{V_{\mathrm{I}}\left(\varphi_{\mathrm{I}}\right) \leq c_{1}\right\} \times\left\{V_{\mathrm{II}}\left(\varphi_{\mathrm{II}}\right) \leq c_{2}\right\},
$$

where $c_{1}, c_{2}>0$. The solution $\varphi(t)$ starting at $\varphi(0) \in \Omega$ is bounded for all $t \geq 0$ if $\Omega$ is a compact and positively invariant set. Since the origin of (4.12b) is asymptotically stable, by a converse Lyapunov theorem (Khalil, 2002, Theorem 4.17) there exists a positive definite smooth function $V_{\text {II }}\left(\varphi_{\text {II }}\right)$ and a continuous positive definite function $W\left(\varphi_{\mathrm{II}}\right)$, both defined for all $\varphi_{\mathrm{II}} \in \mathcal{R}_{A}$, such that $V_{\mathrm{II}}\left(\varphi_{\mathrm{II}}\right) \rightarrow \infty$ as $\varphi_{\text {II }} \rightarrow \partial \mathcal{R}_{A}$ and

$$
\begin{equation*}
\frac{\partial V_{\mathrm{II}}}{\partial \varphi_{\mathrm{II}}} f_{\mathrm{II}}\left(\varphi_{\mathrm{II}}\right) \leq-W\left(\varphi_{\mathrm{II}}\right) \tag{4.15}
\end{equation*}
$$

for all $\varphi_{\text {II }} \in \mathcal{R}_{A}$. Therefore $V_{\text {II }}$ is negative on the boundary $\left\{V_{\text {II }}\left(\varphi_{\text {II }}\right)=c_{2}\right\}$ for sufficiently small $c_{2}$. Now, consider the derivative of $V_{\mathrm{I}}$, yielding

$$
\begin{aligned}
\dot{V}_{\mathrm{I}}(\varphi) \leq & -k \varphi_{1}^{2}\left(\cos \left(\varphi_{2}+\bar{\alpha}\right)+\left(\varphi_{4}+1\right)\left(\varphi_{7}+1\right) \cdots\right. \\
& \left.\cdots\left(\varphi_{3 n-2}+1\right) \cos \left(\varphi_{2}+\bar{\alpha}+\varphi_{3}+\bar{\beta}\right)\right)
\end{aligned}
$$

Let $\gamma:=\left(\varphi_{4}+1\right)\left(\varphi_{7}+1\right) \cdots\left(\varphi_{3 n-2}+1\right)$. Since $0<\pi / n<\pi$ and because $\varphi_{2}, \varphi_{3} \rightarrow 0$ and $\gamma \rightarrow 1$ as $\varphi_{\text {II }} \rightarrow 0$, there exists a neighbourhood $\mathcal{R}_{N} \subset \mathcal{R}_{A}$ of $\varphi_{\text {II }}=0$ wherein

$$
\begin{aligned}
& \cos \left(\varphi_{2}+\bar{\alpha}\right)+\gamma \cos \left(\varphi_{2}+\bar{\alpha}+\varphi_{3}+\bar{\beta}\right) \\
& \quad=\cos \left(\pi / n-\delta(\epsilon)+\varphi_{2}\right)-\gamma \cos \left(\pi / n+\delta(\epsilon)-\varphi_{2}-\varphi_{3}\right)>0
\end{aligned}
$$

Thus, $\dot{V}_{\mathrm{I}}$ is negative on the boundary $\left\{V_{\mathrm{I}}\left(\varphi_{\mathrm{I}}\right)=c_{1}, V_{\mathrm{II}}\left(\varphi_{\mathrm{II}} \leq c_{2}\right\}\right.$ for any $c_{1}>$ 0 , provided $c_{2}>0$ is chosen small enough. Hence, for any given $c_{1}>0$ and sufficiently small $c_{2}>0, \Omega$ is a compact and positively invariant set. Given initial conditions $\varphi_{\mathrm{I}}(0) \in \mathcal{S}$ and $\varphi_{\mathrm{II}}(0) \in \mathcal{R}_{N}$, one can always choose $c_{1}, c_{2}>0$ such that $\varphi(0) \in \Omega$, from which it follows that the trajectories of (4.14) are bounded for all $t \geq 0$ and all $\varphi_{\mathrm{I}}(0) \in \mathcal{S}$.

By having satisfied the conditions of Theorem 4.3, it holds that

$$
\lim _{t \rightarrow \infty} \varphi_{\mathrm{I}}(t)=0
$$

for every $\varphi_{\mathrm{I}}(0) \in \mathbb{R}_{+}$and $\varphi_{\mathrm{II}}(0) \in \mathcal{R}_{N}$, where $\mathcal{R}_{N}$ is a sufficiently small neighbourhood of $\varphi_{\mathrm{II}}=0$. Equivalently, $r_{i}(t) \rightarrow 0$ as $t \rightarrow \infty$ for $i=1,2, \ldots, n$ (i.e., the vehicles converge to a point), concluding the proof.

This situation is conceptually depicted in Figure 4.8.


Figure 4.8: Trajectories in $\mathcal{M}$ locally approaching $\mathcal{F}_{1}$ with $k<k^{\star}$

Theorem 4.6 (Diverging Vehicles): If $k=k^{\star}+\epsilon$, for small enough $\epsilon>0$ and $\xi(0)$ in a sufficiently small neighbourhood of $\mathcal{F}_{1}$, the $n$-unicycle pursuit graph corresponding to (4.9) converges to a generalized regular polygon of type $\{n / 1\}$ while $r_{i}(t) \rightarrow \infty$ as $t \rightarrow \infty, i=1,2, \ldots, n$.

Proof (sketch): The proof is the same the proof of Theorem 4.5, but with $\varphi_{1}$ in the original coordinates transformation (4.11) replaced by $\varphi_{1}=1 / r_{1}$. A simple computation shows that this yields nearly the same dynamics

$$
\begin{aligned}
\dot{\varphi}_{1}= & k \varphi_{1}\left(\cos \left(\varphi_{2}+\bar{\alpha}\right)+\left(\varphi_{4}+1\right)\left(\varphi_{7}+1\right) \cdots\right. \\
& \left.\cdots\left(\varphi_{3 n-2}+1\right) \cos \left(\varphi_{2}+\bar{\alpha}+\varphi_{3}+\bar{\beta}\right)\right) .
\end{aligned}
$$

By repeating the arguments given in the proof of Theorem 4.5, one finds that $r_{i}(t) \rightarrow \infty$ as $t \rightarrow \infty$ for $i=1,2, \ldots, n$ (i.e., the vehicles diverge).

In both cases, $k=k^{\star} \pm \epsilon$, repeated computer simulations indicate that the region of convergence for $\mathcal{F}_{1}$, with respect to variations in the parameter $k$ about $k^{\star}$, is relatively large. Example simulations for $k \neq k^{*}$ are provided in Figures 4.2a
and 4.2 b , illustrating how the unicycles converge to a $\{6 / 1\}$ polygon formation while, at the same time, either converging or diverging, respectively.

### 4.7 Stationary Polygons

In Section 3.4 it was confirmed that the stable generalized regular polygon formations for fixed-speed unicycles in cyclic pursuit are stationary. This section demonstrates that an identical result holds for varying-speed vehicles. Once more, it is useful to view each vehicle's position as a point in the complex plane. The centroid of the vehicles is given by $z_{c}=(1 / n) \sum_{i=1}^{n} z_{i} \in \mathbb{C}$. By differentiating the centroid's location and substituting the unicycle model (2.12), one obtains

$$
\dot{z}_{c}=\frac{1}{n} \sum_{i=1}^{n} v_{i} e^{j \theta_{i}} .
$$

At equilibrium, after Theorem 4.1, the control law (4.1) has the property that $v_{i}=v_{i+1}=k^{\star} \bar{r}$, where $k^{\star}, \bar{r}>0$ are constants. Following Theorem 4.4, it can be shown that the centroid remains stationary (i.e., $\dot{z}_{c}=0$ at equilibrium) by setting $d=1$ and proceeding in exactly the same manner as in Section 3.4.

Similarly, for the case when $k=k^{\star} \pm \epsilon$, Theorems 4.5 and 4.6 say that $v_{i}(t) \rightarrow$ $v_{i+1}(t)$ as $t \rightarrow \infty, i=1,2, \ldots, n$, implying by the same arguments given above that $\dot{z}_{c} \rightarrow 0$ as $t \rightarrow \infty$ (i.e, the centroid approaches a fixed point).

### 4.8 On the $n$-vehicle Weave

The so-called $n$-vehicle weave has already been introduced for the case of fixedspeed vehicles in Section 3.5. Similar periodic solutions exist for the case of varying-speed vehicles subject to the control law (4.1), so the discussion in Section 3.5 will not be reproduced here. Instead, this brief section focuses solely on the qualitative differences between fixed- and varying-speed vehicles. As was remarked in Section 3.5, it is not within the scope of this thesis to provide a complete analysis of the weave. Only observations, based on simulation, are offered.

In Section 3.5 it was remarked how, for fixed-speed vehicles, the radius of any given equilibrium formation and the width of the weave are both inversely proportional to the selected controller gain $k$. An analogous result holds in the
varying-speed case. In this case, let $k_{\alpha}=1$ and define $k:=k_{r}$. Only when $k=k^{\star}$ (see Theorem 4.1) do the vehicles converge to an equilibrium formation of fixed radius. Loosely speaking, if $k<k^{\star}$, then the vehicles converge to a point. Likewise, if $k>k^{\star}$, they diverge (see Figures 4.1 and 4.2).

The same is true of the weave. Figure 4.9 shows $n=6$ weaving vehicles subject to the control law (4.1). Notice how the initial conditions belong to the subspace $\mathcal{W}$ defined in Section 3.5.2. In Figure 4.9a, the vehicles converge, with $k_{r}=0.6$, while in Figure 4.9b they diverge, with $k=0.7$. Hence, one might surmise that there exists a weaving pattern of fixed width for some gain $0.6<k<0.7$. Numerical simulations suggests this value lives in a neighbourhood of $k \approx 0.65$ (see Figure 4.10), which is different from the equilibrium gain $k^{\star}=$ $(\pi / 12) \csc (\pi / 6) \approx 0.5236$ corresponding to the stable $\{5 / 1\}$-polygon formation.


Figure 4.9: Trajectories of (4.2) corresponding to the $n=6$ weave


Figure 4.10: Varying-speed $n=6$ weave; $k=0.65$

## Chapter 5

## Experiments in Multivehicle Coordination

In Chapters 3 and 4, cyclic pursuit was studied in a purely theoretical way as a means for achieving certain regular geometric formations in the plane for a system of identical kinematic unicycles. In particular, Chapter 3 revealed that, subject to the unicycle inputs (3.4) on page 40, some generalized regular polygon formations are locally asymptotically stable, while others are not. Table 3.2 on page 65 lists all possible equilibrium polygons and gives their stability. From an engineering perspective, the question that remains is whether the cyclic pursuit algorithm is sufficiently robust to unmodelled vehicle dynamics and inevitable delays due to sensing and information processing. Therefore, consequent to the theory of Chapter 3, this chapter summarizes the apparatus and results of experiments conducted towards evaluating the practicality of fixed speed pursuit as an implementable multivehicle coordination strategy.

### 5.1 Experimental Purpose

During recent years, a number of research groups have developed testbeds for experimentation in multivehicle control. In most cases, multivehicle testbeds are designed without a specific set of control experiments in mind, thus not for the sole purpose of validating a particular theory. Some examples are the MIT Multivehicle Testbed (King, Kuwata, Alighanbari, Bertucelli, and How, 2004), Caltech's MVWT-II Multivehicle Wireless Testbed (Jin et al., 2004), the Brigham Young Unmanned Air Vehicle Testbed (McLain and Beard, 2004), and the University of Illinois's HoTDeC, or Hovercraft Testbed for Decentralized Control (Vladimerou,

Stubbs, Rubel, Fulford, and Dullerud, 2004), to name only a few.
Likewise, a fleet of ten so-called Argo Rovers ${ }^{1}$ have recently been constructed at the Space Robotics Laboratory of the University of Toronto Institute for Aerospace Studies (UTIAS). The robots were designed to be capable of lengthy autonomous operation and have each been equipped with host of sensing, communication, and actuation devices (Mirza, Beach, Earon, and D'Eleuterio, 2004).

Since the theoretical results of Chapters 3 and 4 were based on ideal kinematic unicycles, one might naturally question whether the pursuit control law (3.4) has more general applicability (e.g., to real vehicles, possessing nontrivial dynamics, such as the Argo Rovers). Also, despite the growing amount of theoretical research on coordination control strategies employing local interaction-based techniques, there are relatively few instances of experimental research validating their worth. Pursuant to this, the purpose of the experiments described here is twofold:
(i) Determine if the theoretical results of Chapter 3, obtained for kinematic unicycles, can be observed in practice using the four-wheeled Argo Rovers;
(ii) Investigate the practicality of (3.4) as a multivehicle coordination strategy given real hardware restrictions (e.g., processing delays, sensor limitations).

A description of the experimental procedure, a detailed summary of the results, and a discussion of the observations follow in the body of this chapter.

### 5.2 Overview of the Rovers

One of the Argo Rovers is shown in Figure 5.1, posing in UTIAS's indoor-outdoor testing facility called MarsDome. Built using the Tamiya TXT-1 $4 \times 4$ Pick-up chassis, the rovers were designed to be fully autonomous mobile robots suitable for outdoor use in reasonable environmental conditions.

### 5.2.1 Microelectronics and Software

Each rover possesses a 700 MHz Pentium ${ }^{\circledR}$ III processor-based computer (Cell Computing ${ }^{\circledR}$ Plug-N-Run) with a 1 GB microdrive, 256 MB of RAM, 2 PCMCIA slots, 2 USB ports, and runs the Debian-Linux operating system. All Linux-based

[^5]

Figure 5.1: An Argo Rover in the MarsDome at UTIAS
software is capable of accessing the onboard sensors and actuators by way of a Siemens C164 20 MHz 16-bit microcontroller (for processing low-level hardware routines). All custom software for the rovers is developed using the $\mathrm{C} / \mathrm{C}++$ languages in Linux. Furthermore, the rovers are each fitted with a wireless Ethernet PCMCIA card used for remote software development, operation, and potentially for direct communication between the rovers.

### 5.2.2 Power Delivery System

In order to conserve payload space and to lower the rover's centre of gravity (i.e., for improved stability), each rover is powered by 1.2 V Saft Nickel-Metal Hydride (NiMH) VH F battery cells, ten of which are located (in series) inside each rubber tire. Current is subsequently delivered to the individual rover systems by way of a custom designed circular slip-ring within each wheel hub.

### 5.2.3 Motion Actuators and Encoders

Front and rear wheel steering axis angles are adjustable independently via servomotor driven mechanisms (Hitec model HS-300) at each wheel axis. Thus, each wheel axis angle is directly specifiable (in software) through a servomotor input command $u_{\phi} \in[-1,1]$. For example, if the rear axis is fixed with a zero steering angle then a servomotor input of $u_{\phi} \approx 0$ at the front wheel axis would result in straight-line driving, while $u_{\phi}>0$ and $u_{\phi}<0$ would correspond to right and left
car-like steering (in the forward direction), respectively.
Moreover, each rover is propelled at all four wheels by a geared throttle motor (Alan's Models part number 1105/7; gear ratio $6: 1$ ), allowing the rover to easily move forward or backward. The rovers are capable of traveling at speeds of not much more than $0.5 \mathrm{~m} / \mathrm{s}$ with a minimum turning radius of approximately 0.65 m . The throttle is specifiable (in software) through a motor input command $u_{f} \in[-1,1]$. The vehicle is stopped when $u_{f} \approx 0$ and moves forward and backward for $u_{f}>0$ and $u_{f}<0$, respectively.

The rovers are all mounted with US Digital Corporation rotary optical encoders (generating 512 cycles per shaft turn) in the hub of each wheel.

### 5.2.4 Camera-based Vision Systems

Each rover is equipped with two CCD array cameras (Logitech ${ }^{\circledR}$ QuickCam ${ }^{\circledR}$ Pro 3000) capable of acquiring up to $640 \times 480$ pixel resolution images at a frequency of 30 Hz . Furthermore, each camera is fixed to a stereovision head using custom supports allowing for individual pan and tilt by way of servomotor mechanisms.

### 5.3 Design and Implementation

This section describes, in detail, the hardware and software engineering designs used to fulfil the experimental purpose described in Section 5.1.

### 5.3.1 Rover Dynamics

As a design tool and, perhaps more importantly, to illustrate exactly how different the Argo Rovers are from ideal kinematic unicycles, a simple model of the rover dynamics is first developed. In doing so, it is assumed that each wheel rolls without laterally slipping, thus having similar nonholonomic characteristics to the already studied kinematic unicycles.

Owing to limited workspace in the laboratory environment, the rovers were operated with their front and rear wheel axes "locked" for tightest turning, meaning that each rover's front and rear steering angles are always equal and opposite. Let $\phi_{f}$ and $\phi_{r}$ denote these angles, respectively, as illustrated in Figure 5.2. In practice, locking of the wheel axes was accomplished in software (not physically) by assigning the appropriate servomotor inputs $u_{\phi_{f}}$ and $u_{\phi_{r}}$ so that $\phi_{f}=-\phi_{r}=: \phi$.


Figure 5.2: Top view of a rover with wheel-axes "locked"

Therefore, in this case, each rover's configuration can be described by the vector of coordinates $q=(x, y, \theta, \phi)$. If the front and rear wheel-axis pairs are each modelled as just a single wheel (see Figure 5.3), then one can use the nonholonomic constraints, which act at each wheel to prevent it from slipping laterally, to develop the kinematic rover model

$$
\left[\begin{array}{c}
\dot{x}  \tag{5.1}\\
\dot{y} \\
\dot{\theta} \\
\dot{\phi}
\end{array}\right]=\left[\begin{array}{c}
\cos \phi \cos \theta \\
\cos \phi \sin \theta \\
\frac{1}{l} \sin \phi \\
0
\end{array}\right] v_{f}+\left[\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right] \omega_{\phi},
$$

where the rover is driven by a forward velocity input $v_{f}$ acting at the front wheelaxis point $p_{f}$ (or at the axis-point $p_{r}$ - since the wheels are locked it does not matter) in the direction of the wheel and is steered by an angular steering velocity input $\omega_{\phi}$. In the vehicle dynamics literature, this is known as a body-centred-axis model (Ellis, 1969). The length $l$ is approximately 0.3 m . A detailed derivation of the model (5.1) is provided in Appendix B.


Figure 5.3: Body centred axis model with wheel-axes locked

The kinematic model (5.1) constitutes a first step in describing the rover as a mechanical system. In reality, the rovers have mass, and thus dynamics. One can extend the above kinematic model to include dynamic effects due to translation and rotation of the rover's body mass. Let $m$ denote the rover's mass (approximately 15 kg ), $I_{p}$ its body moment of inertia about the point $p$ in Figure 5.2, and $I_{s}$ the effective inertia that needs to be overcome by the steering actuator (assumed constant). Suppose one ignores friction and the (minor) inertial effects due to rotation of the wheels. Then, the dynamical equations of motion are

$$
\begin{align*}
\dot{x} & =v_{f} \cos \phi \cos \theta  \tag{5.2a}\\
\dot{y} & =v_{f} \cos \phi \sin \theta  \tag{5.2b}\\
\dot{\theta} & =v_{f} \frac{1}{l} \sin \phi  \tag{5.2c}\\
\dot{\phi} & =\omega_{\phi}  \tag{5.2d}\\
\dot{v}_{f} & =\left(m \cos ^{2} \phi+\frac{1}{l^{2}} I_{p} \sin ^{2} \phi\right)^{-1}\left(v_{f} \omega_{\phi}\left(m-\frac{1}{l^{2}} I_{p}\right) \cos \phi \sin \phi+f\right)  \tag{5.2e}\\
\dot{\omega}_{\phi} & =\tau / I_{s} \tag{5.2f}
\end{align*}
$$

where $f$ is the throttle force input, divided evenly between the front and rear wheels, acting in the direction of the wheels, and $\tau$ is the representative steering torque input. Again, a detailed derivation of the model (5.2) can be found in Appendix B. Since, on the real rovers, the steering angle $\phi$ is directly specifiable by way of the steering servomotor inputs, the steering torque $\tau$ of (5.2f) is not actually an available input. Instead, the real steering mechanism dynamics are a function of the unmodelled servomotor characteristics, making $f$ and $\phi$ the assignable inputs (through $u_{f}$ and $u_{\phi}$, respectively).

### 5.3.2 Speed Regulation

So as to mimic the control law (3.4), it was necessary to equalize the forward speeds of all the rovers. Because of inevitable differences in the physical characteristics among the rovers, it was not enough to provide the same input signal to the throttle motors on each of the rovers. Therefore, a basic speed regulator was designed for each rover using feedback from its four wheel-encoders. By extending the Argo Rover dynamic model (5.2) to account for throttle actuator dynamics, a PI compensator design was selected (see why in Appendix B.4).

## Rolling Speed Estimation

The actual rolling speed $v_{f}$ was estimated by acquiring and differentiating position data from the rotary encoders located in each of the four rover wheel hubs. Rotary encoder counts were sampled at an interval of $T=0.1 \mathrm{~s}$ (or 10 Hz ) and differentiated using the backward finite-divided-difference formula (Chapra and Canale, 1988, page 527)

$$
\dot{d}(k T) \approx \frac{1}{2 T}(3 d(k T)-4 d(k T-T)+d(k T-2 T)) .
$$

The distance traveled by each wheel at every time step $t=k T, k=0,1,2, \ldots$, was determined based on a known translation of approximately $0.53 \mathrm{~m} / 512$ encoder counts. Subsequently, the estimated speed $v_{f}(k T)$ was computed as the average

$$
v_{f}(k T)=\frac{1}{4}\left(\dot{d}_{f l}(k T)+\dot{d}_{f r}(k T)+\dot{d}_{r l}(k T)+\dot{d}_{r r}(k T)\right),
$$

where $\dot{d}_{f l}, \dot{d}_{f r}, \dot{d}_{r l}$, and $\dot{d}_{r r}$, denote the estimated front-left, front-right, rearleft, and rear-right wheel velocities, respectively. There may exist more accurate techniques for computing $\hat{v}_{f}$ (e.g., see Barfoot, 2003), however the above method was sufficient, especially given the noise present in actual speed estimates.

In fact, there was a great deal of noise present in the speed estimates, which may have been due to several factors, possibly including: (i) noise amplification due to numerical differentiation; (ii) shaking and swaying of the rover body, transmitted through the suspension; (iii) nonsymmetric placement of rechargeable batteries inside the tires; (iv) significant play in the transmission and steering mechanisms. In order to attenuate high frequency noise in the velocity estimates, the differentiated encoder data was low-pass filtered (LPF) as follows:

$$
\hat{v}_{f}(k T)=\sum_{i=0}^{4} c_{i} v_{f}(k T-i T)
$$

where $c=(0.3,0.25,0.2,0.15,0.1)$ is the vector of employed coefficients, generating a -3 dB cutoff frequency of approximately 1.35 Hz .

## Digital Implementation

A PI compensator design was implemented digitally on each rover's computer by using finite-difference approximations, again at a sampling interval of $T=0.1 \mathrm{~s}$. Let $v_{R}$ denote the reference forward velocity such that the velocity error at time $t=k T$, where $k=0,1,2, \ldots$, is given by

$$
e_{v}(k T):=v_{R}-\hat{v}_{f}(k T)
$$

Let $u_{v} \in[-1,1]$ be the servomotor input command, specifiable in software, where -1 and 1 denote full throttle reverse and forward, respectively. Thus, the discretetime PI controller is given by

$$
u_{v}(k T)=u_{P}(k T)+u_{I}(k T),
$$

where $u_{P}(k T)=k_{P} e(k T)$ and $u_{I}(k T)$ are the proportional and integral terms, respectively. A simple trapezoidal integration rule (Chapra and Canale, 1988, page 479) was used to integrate the error so that $u_{I}(k T)$ is given by

$$
\begin{equation*}
u_{I}(k T)=u_{I}(k T-T)+k_{I} \frac{T}{2}\left(e_{v}(k T)+e_{v}(k T-T)\right), \tag{5.3}
\end{equation*}
$$

with $u_{I}(0)=0$. Through online tuning experiments, proportional and integral gains of $k_{P}=1.5$ and $k_{I}=2.5$, respectively, were found to work reasonably well.

An example of the speed regulator response is provided in Figure 5.4, as estimated using encoder data. It was found that top speed for the rovers was approximately $0.5 \mathrm{~m} / \mathrm{s}$. Given sufficient time, the steady state response recorded at $v_{R}=0.1 \mathrm{~m} / \mathrm{s}$ between the times $t=4 \mathrm{~s}$ and 10 s was typical at most speeds. Due to particularities of the transmission (significant play), chassis (significant sway), and wheel designs (placement of the batteries), the small fluctuations present and noticeable in the figure were unavoidable and could be audibly discerned while the rovers were running, even during open-loop driving.

### 5.3.3 Multivehicle Pursuit Using Vision

Since the steering angle $\phi$ is directly specifiable by way of the rover's steering servomotor inputs, the steering torque input $\tau$ of (5.2) is not actually an available input. Instead, the steering mechanism dynamics are set by the fixed servomotor


Figure 5.4: Sample speed regulator response with changing references
characteristics. Thus, in order to mimic the pursuit law $\omega_{i}=k_{\alpha} \alpha_{i}$ from (3.4) on page 40 , the actual steering mechanism dynamics were simply ignored and the steering angle $\phi$ was itself computed so as to approximate $\dot{\theta}_{i}=k_{\alpha} \alpha_{i}$ on each rover. In other words, using equation (5.2c) one obtains

$$
\phi_{i}=\arcsin \left(\frac{l k_{\alpha} \alpha_{i}}{v_{f_{i}}}\right)
$$

where $\alpha_{i}$ is the usual heading error, as in Figure 3.1 on page 38, and $v_{f_{i}}$ is the $i$-th rover's speed. However, due to the noisiness of speed estimates, $v_{f_{i}}$ was replaced with the constant reference speed $v_{R}$ so that, incorporating the speed regulator system described above, the implemented rover pursuit strategy was

$$
\begin{equation*}
v_{f_{i}}=v_{R} \text { and } \phi_{i}=\arcsin \left(\frac{l k_{\alpha} \alpha_{i}}{v_{R}}\right) . \tag{5.4}
\end{equation*}
$$

By using one of the cameras, $\alpha_{i}$ was computed for each rover $i$, in real time.
Note that computation of the control law (5.4) for each rover was based on sensing and data processing carried out locally, thus in a completely decentralized fashion (i.e., no global positioning techniques were used). This differs, for example, from the overhead camera global positioning system used by Jin et al. (2004).

## Target Recognition using Colours

In order for rover $i$ to estimate the angle $\alpha_{i}$ to its target at each instant, the rovers were each fitted with a cylinder of different coloured cardboard. Therefore, ordering of the vehicles was accomplished by ordering the colours (e.g., for three rovers in cyclic pursuit, green was predesignated to pursue orange, orange to pursue red, and red to pursue green). The right camera on each rover (although it matters not which one) was used to acquire $160 \times 120$ pixel (low resolution) images every $T=0.1 \mathrm{~s}$. An example image is provided in Figure 5.5a. Note that the cameras were not always in perfect (manual) focus, however this did not appear to adversely influence the effectiveness of the target recognition technique.

Colour detection was done by scanning the pixels in an acquired image and comparing each pixel's hue colour value with a preset target hue ${ }^{2}$ value. For

[^6]example, the green cylinder in Figure 5.5a was found to have a nominal hue value of approximately 115 on a $0-239$ hue value scale. Hue values, rather than raw red-green-blue (RGB) values, were used since hue is a known measure of the colour of a pixel and is thus less susceptible to changes in lighting conditions. The nominal hue values for the orange and red cylinders used were 19 and 1 , respectively.


Figure 5.5: Acquired camera image and image plane geometry

## Computation of the Heading Error

The location of pixels within a tolerance of $\pm 5$ hue value units from the nominal were subsequently recorded and their horizontal positions averaged to compute the horizontal centroid of the cylinder of desired colour in the image (see the distance $\Delta$ in Figure 5.5b, measured in pixels). Let $\beta$ denote the angle from the camera's optic axis to the point that is $\Delta$ pixels from the optic axis (note that $\Delta$ changes sign if the target switches sides of the optic axis) such that $\beta=\arctan (\Delta / f)$, where $f \approx 200$ pixels is the focal length of the camera. Let $\gamma$ denote the angle between the camera's optic axis and the rover's heading $\theta$, from (5.2). Therefore, simply ignoring any error due to the fact that the camera's optic centre did not exactly
correspond to the rover's centre point $p$ in Figure 5.2, the overall heading error was approximated at each time step $t=k T$ using the formula $\alpha(k T)=\gamma+\beta(k T)$.

## Camera Servoing

Because the horizontal field-of-view (FOV) of the onboard cameras was small (approximately 34 degrees), the rovers lost track of their targets very easily in initial experiments. On the other hand, recall that the theoretical unicycles of Chapter 3 are capable of omnidirectional target sensing.

To augment the camera's FOV its panning servomotor was employed, increasing the FOV to approximately 150 degrees. This was done by adjusting the angle $\gamma$ so as to actively centre the target cylinder in the image plane. Let $u_{c} \in[-1,1]$ denote the panning servomotor input, specifiable in software, where -1 and 1 denote full panning left and right, respectively. Similar to what was done for speed regulation, a PI compensator was used to track the angle $\beta=0$ such that

$$
u_{c}(k T)=u_{c}(0)-k_{P} \beta(k T)+u_{I}(k T),
$$

where $u_{c}(k T)$ is the camera's panning servomotor input at time $t=k T, k=$ $0,1,2, \ldots$, and $u_{I}(k T)$ is the same as in (5.3) except with $e_{v}(t)$ replaced by $-\beta(t)$. The value $u_{c}(0)$ was included to account for situations when the initial camera input was not zero. Through online experiments, proportional and integral gains of $k_{P}=0.9$ and $k_{I}=0.5$ were found to work reasonably well.

Because there are no position sensors on the camera panning servomotors, the camera angle was computed at each time step by mapping the current servomotor input $u_{c}(k T)$ to its corresponding camera angle $\gamma(k T)$. Fortunately, the relationship was found to be approximately linear, at 7 degrees per 0.1 units of change in the panning servomotor input. Thus, in radians

$$
\gamma(k T)=\frac{7 \pi}{18} \times u_{c}(k T)
$$

yielding a heading error of $\alpha(k T)=\gamma(k T)+\beta(k T)$.

## Steering Actuation for Pursuit

Let $u_{\phi} \in[-1,1]$ denote the steering angle servomotor input, where -1 and 1 represent full left and full right steering, respectively. Recall that the kinematic
unicycles of Chapter 3 are capable of steering arbitrarily tightly. At each time step $t=k T$, the desired steering angle was computed as per (5.4), yielding

$$
\phi(k T)=\arcsin \left(\frac{l k_{\alpha} \alpha(k T)}{v_{R}}\right)
$$

As for the camera panning servomotors, due to the nonexistence of position sensors on the steering angle servomotors the appropriate servomotor input was computed by mapping the current desired steering angle $\phi(k T)$ to the appropriate servomotor input $u_{\phi}(k T)$. Again, the relationship was found to be approximately linear, at 2 degrees per 0.1 units of change in the steering servomotor. Thus, in radians

$$
u_{\phi}(k T)=\frac{9}{\pi} \times \phi(k T)
$$

adequately describes this relationship.

### 5.4 Experiments and Observations

A variety of experiments were conducted using teams of two, three, and four rovers. Despite the significant physical differences between ideal kinematic unicycles and the Argo Rover systems, which is a natural conclusion of Section 5.3, the outcome was positive. Preliminary experiments were done using two rovers. In this case, the only theoretically possible formation is the $\{2 / 1\}$ polygon (i.e., two vehicles diametrically opposite each other on a circular path). To ensure the two rovers were within each other's FOV, it was necessary to rotate their stereovision heads by 90 degrees (e.g., see the rovers in Figure 5.7). This rotation was accounted for in software by adding (for rotation left) or subtracting (for rotation right) $\pi / 2$ to the angle $\gamma$ in Figure 5.5b. The inside, body-centred camera was always used for computing $\alpha$. It was found that, so long as no rover lost the other from its view, the vehicles always converged to a $\{2 / 1\}$-polygon formation.

### 5.4.1 Stability of the $\{3 / 1\}$ Formation

In theory, the possible equilibrium formations for three vehicles are the $\{3 / 1\}$ and $\{3 / 2\}$ polygons. Although both resemble equilateral triangles, it is the vehicles' ordering on the circle circumscribed by each polygon that is different. As per Definition 3.2 on page 42 , a $\{3 / 1\}$ formation corresponds to the case when the
$i$-th vehicle's target, $i+1$, lies at a heading error of $\alpha_{i}= \pm \pi / 3$. Conversely, a $\{3 / 2\}$ formation has $\alpha_{i}= \pm 2 \pi / 3$. According to Table 3.2, of the two possible equilibria only the $\{3 / 1\}$-polygon formation is locally asymptotically stable.

Figure 5.6 shows three unit-speed unicycles, subject to the control law (3.4), converging to a $\{3 / 1\}$ formation (the simulation ends after 45 s ). On the other hand, Figure 5.7 presents a sequence of captured images of three Argo Rovers in cyclic pursuit, subject to (5.4) with red (vehicle 1) in pursuit of green (vehicle 2 ), green of orange (vehicle 3), and thus, orange of red. Qualitatively speaking, Figures 5.6 and 5.7 indicate like behaviour in that both the unicycles and rovers converge to a $\{3 / 1\}$ formation.


Figure 5.6: Three unicycles subject to control law (3.4) with $k_{\alpha}=0.6$

Figure 5.8 shows the heading errors $\alpha_{i}, i=1,2,3$, as a function of time for each of the simulated unicycles in Figure 5.6 (dotted lines) and for the actual rovers of Figure 5.7. The time axis in Figure 5.8 corresponds (approximately) to the times noted for each image frame in Figure 5.7. The actual rover heading errors were recorded only every second, although they were computed every 0.1 s. Clearly, owing to their physical differences, the unicycle and rover trajectories should not be expected to match in the transient. However, Figure 5.8 shows that their steady-state behaviours both tend to equally spaced motion around a

(a) At $t=0 \mathrm{~s}$ (initial condition)

(c) At $t=4 \mathrm{~s}$

(e) At $t=8 \mathrm{~s}$
(b) At $t=2 \mathrm{~s}$

(d) At $t=6 \mathrm{~s}$

(f) At $t=10 \mathrm{~s}$

Figure 5.7: Generating a $\{3 / 1\}$ formation with $k_{\alpha}=0.2$
stationary circle of fixed radius. Convergence of the real rovers to a stable $\{3 / 1\}$ formation, with $\alpha_{i}=-\pi / 3$, is clear from Figure 5.7.

Additional experiments were performed where the rovers were first allowed to achieve a steady-state $\{3 / 1\}$ formation. Subsequently, one of the rovers was deliberately perturbed from this equilibrium by either altering its heading, halting it temporarily, or slightly changing its location. So long as the rovers were able to maintain their targets within view, the group always returned to a $\{3 / 1\}$-polygon configuration, further demonstrating its stability as a formation.


Figure 5.8: Kinematic unicycle (dotted lines, cf. Figure 5.6) together with actual rover (solid lines, cf. Figure 5.7) target heading errors

### 5.4.2 Formation Radius and the $\{4 / 1\}$ Polygon

Results equivalent to those described in Section 5.4 .1 for the $\{3 / 1\}$-polygon formation were also observed using teams of four vehicles. Figure 5.9 shows four
rovers maintaining a $\{4 / 1\}$-polygon formation. Furthermore, in Chapter 3 it was proved that the kinematic unicycles traverse a circle of radius $\rho=v_{R} n / k_{\alpha} \pi d$ at equilibrium, where $\{n / d\}$ is the formation (see Corollary 3.1 on page 3.1 ). Therefore, by increasing (resp. decreasing) the gain $k_{\alpha}$ one should have expected to observe a proportional decrease (resp. increase) in the radius traversed by the rovers, which was indeed the case. Figure 5.9 shows four rovers in cyclic pursuit, each with gain $k_{\alpha}=0.3$, after having stabilized to a $\{4 / 1\}$-polygon configuration. At approximately $t=7 \mathrm{~s}$, the gain $k_{\alpha}$ was decreased from 0.3 to 0.1 on all the rovers. The sequence of images shows how the rovers continued to maintain a $\{4 / 1\}$ formation while, at the same time, the polygon's radius effectively tripled in size. Identical results were also observed for groups of two and three rovers.

### 5.4.3 The $\{3 / 2\}$ Formation

In theory, the $\{3 / 2\}$ formation for unicycles is unstable (see Table 3.2). However, computer simulations suggest that, while maintaining the ordering of vehicles, almost-circular trajectories are achievable for lengthy time periods. Figure 5.10a shows a simulation of three unit-speed unicycles that start roughly in the $\{3 / 2\}$ configuration (the simulation ends after 45 s ). Despite the fact that they do not converge to a $\{3 / 2\}$ polygon, their motion maintains an almost- $\{3 / 2\}$ formation. Interestingly, among the six important eigenvalues associated with the system's linearization about the $\{3 / 2\}$ polygon (in relative coordinates; see Section 3.3 for details), there is only one complex-conjugate pair of unstable eigenvalues and these eigenvalues lie particularly close to the imaginary axis ( $\lambda \approx 0.0419 \pm j 1.5303$ ). If the simulation of Figure 5.10a is continued for more than 250 s , the unicycles eventually break their pattern of motion and rearrange themselves into a stable $\{3 / 1\}$ formation, as illustrated in the extended simulation of Figure 5.10b.

Figure 5.11 presents a sequence of captured images of three rovers in cyclic pursuit (using the same pursuit order as the rovers in Figure 5.7). Both the unicycles of Figure 5.10 and the rovers of Figure 5.11 started close to a $\{3 / 2\}$ formation. Consequently, their resulting trajectories appear qualitatively similar, maintaining the ordering of a $\{3 / 2\}$ polygon yet never actually converging to a stable formation. If allowed to run for long enough, the rover formation was also seen to "wobble," as in Figure 5.10a. However, even after several minutes, evolution of the rovers into a $\{3 / 1\}$-polygon formation was never observed, unlike


Figure 5.9: $\mathrm{A}\{4 / 1\}$ formation after $k_{\alpha}$ is changed from 0.3 to 0.1


Figure 5.10: Unicycles demonstrating an almost- $\{3 / 2\}$ formation with $k_{\alpha}=0.2$
what happens in simulation for unicycles (cf. Figure 5.10b). On the other hand, this type of maneuver was likely not even possible given the rovers' limited FOV and the fact that no protocol for collision avoidance among rovers was implemented. Nevertheless, it is clear from Figure 5.11 that the $\{3 / 2\}$ polygon is not asymptotically stable for rovers, as predicted by the theory for unicycles.

Similar to Figure 5.8, in Figure 5.12 the heading errors $\alpha_{i}, i=1,2,3$, have been plotted as a function of time for each of the simulated unicycles in Figure 5.10 (dotted lines) and for the actual rovers of Figure 5.11. The time axis in Fig. 5.12 corresponds almost exactly to the times noted for each image frame in Figure 5.11. Again, owing to their physical differences, the unicycle and rover trajectories should not be expected to match. However, Figure 5.12 shows how their behaviors are qualitatively consistent, with oscillations appearing in the heading errors of both the unicycles and the rovers.

### 5.5 Summary of Findings

In this chapter, details concerning the apparatus and results of multivehicle pursuit experiments have been presented. By adapting the hardware and developing software for the existing Argo Rovers, experiments were conducted using groups of two, three, and four rovers, the purpose of which was to determine whether the theoretical results obtained in Chapter 3 could be applied in practice to real systems distinct from ideal kinematic unicycles. Given the physical differences between unicycles and the Argo Rovers, and that there were delays in the system due to sensing and information processing not accounted for in the accompanying theory of Chapter 3, the presented results are very encouraging.

However, success is not to say there were not limitations. Firstly, owing to the difficulties in bringing multiple rovers into working order (i.e., free of hardware difficulties), experiments were limited to $n \leq 4$ rovers. Although it is likely that the reported results extend to $n>4$ rovers, no experiments were conducted to confirm this. Secondly, the rovers were severely limited by the FOV of their cameras. Even with the inclusion of camera servoing, for certain initial conditions the rovers inevitably lost their target, thus limiting the range of experiments that could be tried. On the other hand, computation of the control law was based solely on sensing and data processing carried out locally (i.e., without any explicit communication, nor the use of an overhead camera system or other GPS).


Figure 5.11: A $\{3 / 2\}$ semi-stable formation with $k_{\alpha}=0.1$


Figure 5.12: Kinematic unicycle (cf. Figure 5.10a, dotted lines) together with actual rover (cf. Figure 5.11, solid lines) heading errors

Finally, experiments were further restricted by the fact that no method of collision avoidance was employed, a practical issue not considered in this research. Experimentation, as such, and the analysis of its limitations also serve to indicate areas of future theoretical research. For example, under what conditions can it be analytically proved that the rover model is sufficiently similar to the unicycle that the stability results of Chapters 3 and 4 always hold?

In conclusion, the cyclic pursuit strategy developed for unicycles in Chapter 3 was found to be practical from the point of view of robustness to unmodelled dynamics, disturbances in the vehicle velocities, and delays in the system due to sensing and information processing. These findings not only bode well for continuing research on cooperative control strategies based on the notion of pursuit, but also for other cooperative control techniques employing similar local interactions.

## Chapter 6

## Symmetries of Pursuit

"It's a basic principle: Structure always affects function," says Steven Strogatz in his book entitled Sync (Strogatz, 2003, p. 237). "The structure of social networks affects the spread of information and disease; the structure of the power grid affects the stability of power transmission. The same must be true for species in an ecosystem, companies in the global marketplace, cascades of enzyme reactions in living cells. The layout of the web must profoundly shape its dynamics."

Surely the same principle must hold for engineered multiagent systems. This chapter deviates somewhat from the kinematic unicycle research reported in Chapters 3-5. By revisiting the linear integrator model introduced in Chapter 2, it explores how the interconnection structure among individuals of a multiagent system influences, in particular, the invariance of discrete symmetries in its trajectories.

### 6.1 Motivation and Background

As has been discussed in Chapters 1 and 2, a current research emphasis in the multiagent systems and cooperative control literature is to generalize: What are the connectivity conditions for achieving consensus (Beard and Stepanyan, 2003; Moreau, 2003; Z. Lin et al., 2005)? What happens if the interconnection topology between agents is dynamic (Tanner et al., 2003b; Olfati-Saber and Murray, 2004; Ren and Beard, 2005)? These are matters of fundamental theoretical significance. On the other hand, practical issues arise when designing autonomous agent systems required to perform specific tasks. For instance, consider the problem of dynamic target tracking using a team of $n>1$ autonomous mobile robots. This task requires that agents act as a mobile and reconfigurable sensor array. Suppose each agent is equipped with a target-tracking sensor (e.g., an ultrasonic
sensor, a laser range finder, or a CCD camera) that, when combined with the sensor readings of other agents, can be utilized by a central observer to estimate the location of a target. If the sensors measure distances to the target, then it can be shown that a configuration that optimizes the estimate is one in which the sensors are uniformly placed in a circular fashion around the target (Aranda et al., 2005). This optimal sensor placement is "symmetrical," in the sense that the configuration remains optimal under rotations by $2 \pi / n$ about the target.

The problem of achieving and maintaining symmetry in multiagent formations is not a new endeavour. For example, the work of Sugihara and Suzuki (1990) investigates distributed heuristic algorithms for the formation of geometric patterns in the plane (e.g., circles and polygons). Leonard and Fiorelli (2001) use artificial potentials to generate stable symmetric formations by inserting virtual leaders among the agents. A method for stabilizing multiple agents to rigidly constrained formations, while moving along a desired path, is examined by Egerstedt and Hu (2001). In the work of Fierro, Das, Kumar, and Ostrowski (2001), a hybrid control strategy is employed to achieve stability for a desired formation, irrespective of its symmetry. How information flow influences the stability of formations is studied by Fax and Murray (2004). Of relevance to the current work is that of Pogromsky, Santoboni, and Nijmeijer (2002), which exploits the symmetry in a network of coupled identical dynamical systems to classify invariant manifolds of the overall system dynamics with respect to their stability. Hence, "stability in the network descends from its topology" (Pogromsky et al., 2002, p. 67). Symmetry in the interconnection structure is also exploited by Recht and D'Andrea (2004), who study the problem of distributed controller synthesis for large arrays of spatially interconnected systems. Consider the cyclic pursuit strategy studied in Chapters $3-5$. The possible equilibrium formations are generalized regular polygons, which are distinctly "symmetric" formations. A basic question is: Would such symmetries have naturally emerged given a different interconnection topology?

The present research is especially influenced by the work of Bruckstein et al. (1995) and Richardson (2001b), wherein a circulant interconnection structure among multiple agents is utilized to deduce the overall steady-state behaviour of the agents. In particular, Bruckstein et al. (1995) study the asymptotic behaviour of a collection of agents in discrete-time circulant pursuit. Similarly, Richardson (2001b) studies the stability of certain geometric patterns for a collection of continuous-time fixed-speed agents in cyclic pursuit.

### 6.1.1 Purpose and Outline of the Chapter

This chapter generalizes the linear agents problem introduced in Chapter 2 and studies connectivity as it relates to the problem of choosing distributed controllers that inherently preserve symmetric formations. Designing or studying the stability of symmetric formations, as in Egerstedt and Hu (2001); Fierro et al. (2001); Pogromsky et al. (2002); Leonard and Fiorelli (2001); Fax and Murray (2004), is not expressly examined here. Rather, as a first step towards a more general approach to formation stabilization for interconnected systems, this chapter seeks to identify those interconnection structures that naturally result in invariant manifolds corresponding to formation symmetry. The problem of possibly stabilizing the system to these invariant manifolds is left as future research.

Thus, for a system of $n>1$ identical planar integrators, each endowed with only relative sensing capabilities, it is revealed how the information flow structure among agents influences symmetry in the multiagent system's trajectories. The principal findings of this chapter are as follows: (i) If the aggregate multiagent system matrix is circulant, then (under certain technical assumptions) rotation symmetries of any given formation are preserved under the system's dynamics; and, (ii) Circulant connectivity is also necessary if one wishes to make invariant the rotation group symmetry of order $n$, together with all its subgroups.

Finally, and in addition to the aforementioned central focus of this chapter, the evolution of formations generated by way of circulant pursuit is studied as a potentially useful extension to the results of Bruckstein et al. (1995) and Richardson (2001b). This is done merely as a first step towards understanding the steady-state behaviour of agents in circulant pursuit.

### 6.2 Symmetry Groups, Graphs, and Pursuit

This section introduces some basic terminology and useful background material relating to symmetry groups, for which a reference is Coxeter (1948). Also, details about the class of multiagent systems studied in this chapter are introduced.

### 6.2.1 Symmetry Groups in $\mathbb{R}^{2}$

It is perhaps best to start with a fundamental definition.

Definition 6.1 (Group): $A$ group, denoted $G$, is a set of elements together with an operation called multiplication, which associates with each ordered pair $g_{1}, g_{2} \in G$ a third element called their product, denoted $g_{1} g_{2}$, such that: (i) if the elements $g_{1}, g_{2}, g_{3} \in G$ then $g_{1}\left(g_{2} g_{3}\right)=\left(g_{1} g_{2}\right) g_{3}$; (ii) there exists a unique identity element $g_{I} \in G$ such that, for any $g \in G, g g_{I}=g_{I} g=g$; and, (iii) for any element $g \in G$, there exists a unique inverse element $g^{-1} \in G$ such that $g g^{-1}=g^{-1} g=g_{I}$.

If a group $G$ consists of a finite number of elements, then it is called a finite group, and the number of elements is its order. Otherwise, it is called an infinite group. A subset of $G$ whose products comprise the whole group is called a set of generators. A subgroup of $G$ is a group formed by a subset of the elements of $G$ with the same multiplication operation. A group is said to be trivial if it consists of only the identity element.

An isometry is a transformation that preserves distances. The set of isometries in $\mathbb{R}^{2}$ form a group under composition, denoted $I\left(\mathbb{R}^{2}\right)$. A subgroup $G$ of $I\left(\mathbb{R}^{2}\right)$ is called a symmetry group of a subset $\mathcal{U} \subset \mathbb{R}^{2}$ if $\mathcal{U}$ remains invariant under every element of $G$. A group is called cyclic when all its elements are powers $g^{k}$ of some one element $g$. For any element $g$ in a group $G$, the set $\left\{g^{k}: k \in \mathbb{Z}\right\}$ is the cyclic subgroup of $G$ generated by $g$. If $g^{m}=g_{I}$ for some positive integer $m$, then the group generated by $g$ is a finite group. If $m$ is the least positive integer for which this is true, then $m$ is the group's order.

Definition 6.2 (Rotation Group): The rotation group of order $m$, denoted $C_{m}$, is the cyclic group generated by a rotation through $2 \pi / m$ about the origin.

Therefore, it is said that a subset $\mathcal{U} \subset \mathbb{R}^{2}$ has symmetry $C_{m}$ if the rotation group $C_{m}$ is a symmetry group of $\mathcal{U}$. Every finite subgroup of $I\left(\mathbb{R}^{2}\right)$ leaves at least one point invariant (Coxeter, 1948, p. 44). In this thesis, it is assumed (without loss of generality) that this point is always the origin.

### 6.2.2 Agents in Pursuit

In this chapter, it is useful to view the agents as points in the complex plane, $\mathbb{C}$. Consider, as in Chapter 2, $n>1$ agents, $z_{1}(t), z_{2}(t), \ldots, z_{n}(t) \in \mathbb{C}$, evolving in time $t$. Suppose that each agent is a simple integrator; i.e., $\dot{z}_{i}(t)=u_{i}(t) \in \mathbb{C}$, $i=1,2, \ldots, n$, where $u_{i}(t)$ is the control input. Assume that the agents have only
relative sensing capabilities (i.e., there is no global reference frame) and, therefore, that the inputs $u_{i}(t)$ are of the type

$$
\begin{equation*}
u_{i}(t)=\sum_{k \neq i}^{n} a_{i k}\left(z_{k}(t)-z_{i}(t)\right), i=1,2, \ldots, n \tag{6.1}
\end{equation*}
$$

That is, the aggregate multiagent system is of the form

$$
\begin{equation*}
\dot{z}(t)=A z(t) \tag{6.2}
\end{equation*}
$$

where $z(t)=\left(z_{1}(t), z_{2}(t), \ldots, z_{n}(t)\right) \in \mathbb{C}^{n}$. A direct consequence of the relative sensing limitation is the following property (also mentioned in Section 2.1.3):

Property 6.1: The matrix $A$ has zero row-sums (i.e., $A[11 \ldots 1]^{\top}=0$ ).
Hence, if the agents are all collocated, then there is no motion. Recall from Section 2.1 that the information flow between agents can be modelled as a digraph, denoted $\Gamma(A)$ for the system (6.2).

This chapter concerns itself with the trajectories of (6.2), and the following question is addressed: What fixed interconnection topologies $\Gamma(A)$ and associated interconnection weights $A=\left[a_{i k}\right]$ preserve rotation group symmetries in multiagent formations $z(t) \in \mathbb{C}^{n}$ for all $t \geq 0$ ?

### 6.2.3 Circulant Interconnections

It will be shown in Sections 6.3 and 6.4 that of fundamental significance to the topic of symmetry invariance is a particular structure in the interagent sensing topology: namely, circulant connectivity. By circulant connectivity it is meant that, possibly after a relabeling ${ }^{1}$ of the agents, the multiagent system matrix $A$ is a circulant matrix (see Section 2.2.3); i.e., $A=\operatorname{circ}\left(a_{0}, a_{1}, \ldots, a_{n-1}\right)$. Let $\kappa \geq 0$ denote the cardinality of the set $\left\{a_{i}, i=1,2, \ldots, n-1: a_{i} \neq 0\right\}$. In other words, $\kappa$ represents the degree of coupling between vertices of the circulant digraph ${ }^{2} \Gamma(A)$.

Following a standard notion, e.g., as in C.-T. Lin (1974); Siljak (1991), a matrix $A_{1}$ is said to have the same structure as another matrix $A_{2}$, of the same

[^7]dimensions, if for every zero entry of $A_{1}$ the corresponding entry in $A_{2}$ is also zero, and vice versa. Accordingly, if a square matrix $A$ is such that there exists a circulant matrix $A_{c}$ of the same order and structure as $A$, then $A$ is called structurally circulant. Clearly, the topology of $\Gamma(A)$ is identical to that of $\Gamma\left(A_{c}\right)$.

If the degree of coupling between individuals is $\kappa=1$ and the off-diagonal element of a circulant matrix $A$ is positive, then (6.2) becomes the traditional cyclic pursuit problem (see Sections $2 \cdot 2.1$ and 2.2.2). If the degree of coupling $\kappa=n-1$, then this corresponds to what is referred to as "all-to-all" coupling, since every agent can sense every other agent. Although not every all-to-all coupled matrix $A$ is circulant, every all-to-all coupled $A$ is structurally circulant.

### 6.2.4 Permutations

Of particular utility when studying formations and symmetry is the theory of permutations. Let $\mathcal{N}:=\{1,2, \ldots, n\}$ and consider a bijection $\sigma: \mathcal{N} \rightarrow \mathcal{N}$, which is called a permutation of the set $\mathcal{N}$. In general, one can write

$$
\begin{aligned}
\sigma(1) & =i_{1} \\
\sigma(2) & =i_{2} \\
& \vdots \\
\sigma(n) & =i_{n}
\end{aligned}
$$

or as is often seen in the literature (Davis, 1994, pp. 24-25),

$$
\sigma:\left(\begin{array}{cccc}
1 & 2 & \cdots & n \\
i_{1} & i_{2} & \cdots & i_{n}
\end{array}\right)
$$

Associated with every permutation $\sigma$ is a square matrix, denoted $P_{\sigma}$, of order $n$. Given an $n \times n$ matrix $A=\left[a_{i k}\right], P_{\sigma}$ is such that $P_{\sigma} A=\left[a_{\sigma(i), k}\right]$ and, therefore, that $P_{\sigma} A P_{\sigma}^{\top}=\left[a_{\sigma(i), \sigma(k)}\right]$ (e.g., $\Pi_{n}$ is the matrix corresponding to $\sigma(i)=i+1$ ). Let $\sigma^{l}(i):=\sigma \circ \sigma \circ \cdots \circ \sigma(i)$, the permutation $\sigma$ applied $l$ times to element $i \in \mathcal{N}$. Every $i \in \mathcal{N}$ generates a subset of $\mathcal{N}$ called a cycle of length $l$, where $l$ is the least positive integer such that $\sigma^{l}(i)=i$. In general, a permutation $\sigma$ can be factored ${ }^{3}$

[^8]into a product of disjoint cycles. This factorization is unique up to the ordering of factors (which are disjoint cycles). A permutation is called primitive if it has only one factor (which has full length $n$ ).

To illustrate these concepts, consider a permutation of the set $\mathcal{N}$ with $n=4$. The permutation $\sigma(i)=i+1, i=1,2,3,4$, is primitive since it factors into only one cycle of full length 4 , hence denoted as a single factor $\sigma=(1,2,3,4)$. This representation says that $\sigma$ maps $1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 1$. The least positive integer such that $\sigma^{l}(i)=i$ is $l=4$ for every $i \in \mathcal{N}$. Next, consider a different permutation

$$
\sigma:\left(\begin{array}{llll}
1 & 2 & 3 & 4 \\
1 & 4 & 2 & 3
\end{array}\right)
$$

In this case, $\sigma$ has two factors, of lengths 1 and 3 , and is thus uniquely denoted as the product of two factors, namely (1) and $(2,4,3)$, as follows: $\sigma=(1)(2,4,3)$. As before, this representation says that $\sigma$ maps $1 \mapsto 1$ and $2 \mapsto 4 \mapsto 3 \mapsto 2$. For more about permutations, see Davis (1994, Section 2.4).

### 6.2.5 Formation Graphs

It is often convenient to represent a formation of agents together with their interconnection topology as a graph (cf. Section 2.1). At each instant $t$, one can define a set of locations $\mathcal{V}_{t}=\left\{z_{1}(t), z_{2}(t), \ldots, z_{n}(t)\right\}$ and a set $\mathcal{E}_{t}$ of edge vectors $e_{i k}(t): \mathcal{V}_{t} \times \mathcal{V}_{t} \rightarrow \mathbb{C}$ such that an edge $e_{i k}(t):=z_{k}(t)-z_{i}(t)$ exists in $\mathcal{E}_{t}$ only if there exists a corresponding edge in $\mathcal{E}$. Abusing terminology, it is convenient to refer to the pair $\left(\mathcal{V}_{t}, \mathcal{E}_{t}\right)=: \Gamma(A, z(t))$ as the formation graph (or often just graph for short). Figure 6.1 provides three example formation graphs. The graphs in Figures 6.1a and 6.1b have circulant connectivity.

### 6.3 Symmetric Formations and Invariance

Ignoring the interagent connections, the configuration of points $z(t) \in \mathbb{C}^{n}$ at time $t$ is referred to as a multiagent formation. The principal result of this section is Theorem 6.2, which states that if a system has circulant connectivity (see Section 6.2.3), then symmetric formations remain symmetric.


Figure 6.1: Example formation graphs $\Gamma(A, z(t))$

### 6.3.1 Formation Symmetry

Recall that $j:=\sqrt{-1}$.
Definition 6.3 (Formation Symmetry): The formation $z(t) \in \mathbb{C}^{n}$ is said to have symmetry $C_{m}$ at time $t$ if there exists a permutation $\sigma: \mathcal{N} \rightarrow \mathcal{N}$ such that

$$
\begin{equation*}
e^{j 2 \pi / m} z(t)=P_{\sigma} z(t) . \tag{6.3}
\end{equation*}
$$

That is, by rotating the agents $z(t) \in \mathbb{C}$ through angle $2 \pi / m$ one obtains the same set of points in $\mathbb{C}$, but (generally) with a different labeling.

Henceforth, it will simply be said that a formation $z(t) \in \mathbb{C}^{n}$ has symmetry $C_{m}$ "with $P_{\sigma}$ " if the vector $z(t)$ satisfies Definition 6.3 with associated permutation matrix $P_{\sigma}$. Following Definition 6.3, several remarks are in order. Let $n_{0} \geq 0$ denote the number of agents located at the origin.

Remark 6.1: If at time $t$ a formation $z(t)$, with $n_{0}=0$, has symmetry $C_{m}$, then $m$ divides $n$. This can be seen by applying the constraint (6.3) $m$ times, yielding

$$
e^{j 2 \pi m / m} z(t)=z(t)=P_{\sigma}^{n} z(t)
$$

(i.e., $\sigma^{m}(i)=i$ for every $i \in \mathcal{N}$ ). Thus, the factorization of $\sigma$ factors into $n / m$ disjoint cycles, each of length $m$. Equivalently, $C_{m}$ is a subgroup of $C_{n}$.

Remark 6.2: If a formation $z(t)$, with $n_{0}=0$, has symmetry $C_{n}$, then the associated permutation $\sigma$ is primitive. For if not (i.e., $\sigma$ has a cycle of length $l<n$ ),
then one obtains at the $l$-th iteration (as in Remark 6.1)

$$
e^{j 2 \pi l / n} z(t)=P_{\sigma}^{l} z(t)=z(t),
$$

which can only be true for $l<n$ if $z(t) \equiv 0$.
Remark 6.3: Suppose a formation $z(t)$, with $n_{0}=0$, has symmetry $C_{m}$, where $m<n$. If there are collocated agents, then it is possible that there exist more than one permutation $\sigma$ such that (6.3) is satisfied. For instance, the $n=8$ agents in Figure 6.1c have symmetry $C_{4}$ with the primitive permutation

$$
\sigma:\left(\begin{array}{ccccc}
1 & 2 & \cdots & 7 & 8 \\
2 & 3 & \cdots & 8 & 1
\end{array}\right)
$$

or, equivalently, $\sigma=(1,2, \ldots, 8)$. However, the constraint (6.3) also holds with

$$
\sigma:\left(\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 3 & 4 & 1 & 6 & 7 & 8 & 5
\end{array}\right)
$$

which has two factors and can be denoted $\sigma=(1,2,3,4)(5,6,7,8)$. Following Remark 6.1, it is clear from the geometry of symmetry $C_{m}$ that any factors of $\sigma$ must have a length that is a multiple of $m$.

Consequent to Remark 6.3, it is assumed in this thesis that if a formation $z(t)$ has symmetry $C_{m}$ according to Definition 6.3, then its associated permutation $\sigma$ is one that factors into exactly $n / m$ cycles of length $m$. Let $\operatorname{gcd}(n, q)$ denote the greatest common divisor of the integers $n$ and $q$. The following is a useful fact.

Remark 6.4: If $m$ divides $n$, then there always exists an integer $q \in\{1,2, \ldots, n-$ $1\}$ such that $\operatorname{gcd}(n, q)=n / m$ since one can always choose $q=n / m$.

### 6.3.2 Canonical Ordering

Before discussing symmetry invariance, this section establishes a connection between formation symmetry $C_{m}$ and a canonical ordering of the agents. It is shown that agents satisfying the formation symmetry constraint (6.3) can always be reordered such that (6.3) holds with $P_{\sigma}=\Pi_{n}^{q}$, for an appropriate choice of $q$. Note that the permutation corresponding to $P_{\sigma}=\Pi_{n}^{q}$ is $\sigma(i)=i+q, i=1,2, \ldots, n$.

This choice of ordering is not new; indeed, some textbooks assume it from the outset when discussing rotation group symmetry (e.g., Coxeter, 1948, 1989). A proof is offered here, since ordering is crucial to the results that follow.

Theorem 6.1: Consider a formation $\tilde{z}(t)$ that has no agents at the origin. Suppose that $\tilde{z}(t)$ has symmetry $C_{m}$ at time $t$ and let $q \in\{1,2, \ldots, n-1\}$ satisfy $\operatorname{gcd}(n, q)=n / m$ (cf. Remark 6.4). Then, there exists a permutation $\tau$ of the agent locations $z(t)=P_{\tau} \tilde{z}(t)$ such that (6.3) holds with $P_{\sigma}=\Pi_{n}^{q}$.

Before giving a proof, an example is provided to help clarify.
Example 6.1: Consider the formation of agents in Figure 6.2a, which has symmetry $C_{5}$ since the constraint (6.3) holds with $m=5$ and

$$
P_{\sigma}=\left[\begin{array}{llllllllll}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}\right] .
$$

In Figure 6.2, the angle between adjacent dotted lines of equal length is $2 \pi / 5$.
Let $q=4$, which satisfies $\operatorname{gcd}(10,4)=10 / 5=2$. By Theorem 6.1, there exists a permutation $\tau$ of the agent locations such that the constraint (6.3) holds with the permutation matrix $\Pi_{10}^{4}$, as is demonstrated by Figure 6.2b. After this relabeling $\tau$, the new permutation $\tilde{\sigma}(i)=i+4$ factors as $\tilde{\sigma}=(1,5,9,3,7)(2,6,10,4,8)$.

The proof of Theorem 6.1 requires a lemma.
Lemma 6.1: Let $\sigma: \mathcal{N} \rightarrow \mathcal{N}$ be a permutation of the form $\sigma(i)=i+q$, for some $q \in\{1,2, \ldots, n-1\}$ and $i=1,2, \ldots, n$. Then, the factorization of $\sigma$ has $p$ cycles of length $n / p$ if and only if $\operatorname{gcd}(n, q)=p$.


Figure 6.2: A formation of $n=10$ agents with $C_{5}$ symmetry

Proof: $(\Leftarrow)$ Let $\operatorname{gcd}(n, q)=p$, and define $k_{n}:=n / p$ and $k_{q}:=q / p$. It must be shown that $\sigma$ has exactly $p$ cycles, each of length $k_{n}$. Observe that the indices $k=1,2, \ldots, p$ generate lists of the form

$$
\begin{align*}
& \left(1,1+q, 1+2 q, \ldots, 1+\left(k_{n}-1\right) q\right) \\
& \left(2,2+q, 2+2 q, \ldots, 2+\left(k_{n}-1\right) q\right)  \tag{6.4}\\
& \quad \vdots \\
& \left(p, p+q, p+2 q, \ldots, p+\left(k_{n}-1\right) q\right) .
\end{align*}
$$

Notice that $i+k_{n} q=i+n q / p=i+n k_{q}=i$ (modulo $n$ ), which proves that the above lists are indeed cycles.

To prove each cycle is of length $k_{n}$, it must be shown that the indices within a cycle are distinct. Suppose there exists a cycle $i \in\{1,2, \ldots, p\}$ and integers $k, l \in\left\{0,1, \ldots, k_{n}-1\right\}, k>l$, such that two indices in cycle $i$ satisfy $i+k q=i+l q$ (modulo $n$ ). That is, $(k-l) q=\gamma n$ for some integer $\gamma \geq 1$. Then, $q=\gamma n /(k-l)$. However, one also has that $q=k_{q} p$. Equating these two expressions for $q$ yields $\gamma k_{n} / k_{q}=k-l$. Since $\operatorname{gcd}\left(k_{n}, k_{q}\right)=1, k_{q}$ must be a factor of $\gamma$. This implies that $k-l$ is a multiple of $k_{n}$. But $k-l<k_{n}$, a contradiction.

To prove that the cycles are disjoint, suppose there exist $i, k \in\{1,2, \ldots, p\}$, $i>k$, and $l_{i}, l_{k} \in\left\{0,1, \ldots, k_{n}-1\right\}$ such that $i+l_{i} q=k+l_{k} q$. Substituting $q=k_{q} p$ and rearranging, one has that $i-k=k_{q} p\left(l_{k}-l_{i}\right)$. Thus, $i-k$ is a multiple of $p$. But by definition, $i-k<p$, a contradiction.
$(\Rightarrow)$ Suppose $\sigma$ factors into $p$ cycles of length $n / p$. Since every permutation has a unique factorization, one obtains (by the proof arguments above) that $\operatorname{gcd}(n, q)=p$, completing the proof.

Proof of Theorem 6.1: Because $\tilde{z}(t)$ has symmetry $C_{m}$, it satisfies (6.3) with some permutation $\sigma: \mathcal{N} \rightarrow \mathcal{N}$. The proof is accomplished by performing a cyclic decomposition of $\sigma$ and then by revealing a new ordering such that (6.3) holds with $P_{\sigma}=\Pi_{n}^{q}$. Firstly, for $i=1,2, \ldots, n$, list the $m$ distinct agents generated by a rotation of each agent through $2 \pi / m:\left(i, \sigma(i), \sigma^{2}(i), \ldots, \sigma^{m-1}(i)\right)$. Any two of these lists are either the same, up to an ordering, or they are disjoint. Consider only the $n / m$ disjoint lists, denoted

$$
\begin{equation*}
\left(i_{k}, \sigma\left(i_{k}\right), \sigma^{2}\left(i_{k}\right), \ldots, \sigma^{m-1}\left(i_{k}\right)\right) \tag{6.5}
\end{equation*}
$$

for $k=1,2, \ldots, n / m$. By construction, each agent appears in only one of these lists and all $n$ agents are accounted for. Now, let $q \in\{1,2, \ldots, n-1\}$ be such that $\operatorname{gcd}(n, q)=n / m$ and consider a reordering of the agents to obtain $n / m$ cycles

$$
\begin{align*}
& (1,1+q, 1+2 q, \ldots, 1+(m-1) q) \\
& (2,2+q, 2+2 q, \ldots, 2+(m-1) q)  \tag{6.6}\\
& \quad \vdots \\
& (n / m, n / m+q, n / m+2 q, \ldots, n / m+(m-1) q)
\end{align*}
$$

By Lemma 6.1, this new permutation $\sigma(i)=i+q$, which is associated with the matrix $\Pi_{n}^{q}$, has $n / m$ cycles of length $m$. In other words, the reordering is a bijection. This implies that the reordering, which brings (6.5) into the form (6.6), is also a permutation.

Let $\tau$ denote the permutation of Theorem 6.1 that reorders the agents. Notice that by substituting $\tilde{z}(t)=P_{\tau}^{\top} z(t)$ into (6.3), one obtains

$$
e^{j 2 \pi / m} z(t)=P_{\tau} P_{\sigma} P_{\tau}^{\top} z(t)=\Pi_{n}^{q} z(t) .
$$

Remark 6.5: Clearly, if a formation $z(t)$ has symmetry $C_{m}$, then any permutation of the agent locations does not change this; it only changes the permutation $\sigma$ with which (6.3) holds. By simultaneously permuting the rows and columns of $A$
(i.e., computing $P_{\tau} A P_{\tau}^{\top}$ ) one can view this as just a transformation of coordinates given by $P_{\tau}$ or, equivalently, simply a relabeling of the agents.

Henceforth, it will simply be said that a formation $z(t) \in \mathbb{C}^{n}$ has symmetry $C_{m}$ "with $P_{\sigma}=\Pi_{n}^{q}$ " if the vector $z(t)$ satisfies Definition 6.3 with $P_{\sigma}=\Pi_{n}^{q}$ for some $q \in\{1,2, \ldots, n-1\}$ satisfying $\operatorname{gcd}(n, q)=n / m$ (cf. Lemma 6.1).

Finally, agents at the origin play no role in symmetry; they merely complicate the ordering. The following corollary emphasizes this fact.

Corollary 6.1: Suppose the conditions of Theorem 6.1 hold, but that there are $n_{0}>0$ agents located at the origin. Let $q \in\left\{1,2, \ldots, n-n_{0}-1\right\}$ satisfy $\operatorname{gcd}(n-$ $\left.n_{0}, q\right)=\left(n-n_{0}\right) / m$. Then, there exists a permutation $\tau$ of the agent locations $z(t)=P_{\tau} \tilde{z}(t)$ such that for all $z_{i}(t) \neq 0$,

$$
\begin{equation*}
z_{i+q}(t)=e^{j 2 \pi / m} z_{i}(t) \tag{6.7}
\end{equation*}
$$

If $z_{i}(t) \equiv 0$, then the constraint (6.7) holds with $q=n$.
Proof: The proof is the same as that of Theorem 6.1, with $n$ replaced everywhere by $n-n_{0}$. Moreover, if an agent $i \in \mathcal{N}$ appears more than once in a list (6.5) it must be that $z_{i}(t)=0$. To see this, suppose there exist $0 \leq \alpha<\beta<m$ such that $e^{j 2 \pi \alpha / m} z_{i}(t)=z_{\sigma^{\alpha}(i)}(t)=z_{\sigma^{\beta}(i)}=e^{j 2 \pi \beta / m} z_{i}(t)$. This implies that $\left(1-e^{j 2 \pi(\beta-\alpha) / m}\right) z_{i}(t)=0$, or equivalently that $z_{i}(t) \equiv 0$.

Hence, for the sake of simplicity, it is assumed that $n_{0}=0$ throughout the remainder of this chapter.

### 6.3.3 Symmetry Invariance

The focus of this chapter is on identifying certain interconnection structures that inherently result in invariant manifolds corresponding to formation symmetry. Following Section 6.3.2, this naturally leads to the following definition.

Definition 6.4 (Formation Symmetry Invariance): Let $m$ be a divisor of n. Formation symmetry $C_{m}$ is said to be invariant under the dynamics (6.2) if for every $q \in\{1,2, \ldots, n-1\}$ such that $\operatorname{gcd}(n, q)=n / m$ and for every initial formation $z(0) \in \mathbb{C}^{n}$ having symmetry $C_{m}$ with $P_{\sigma}=\Pi_{n}^{q}$, the formation $z(t)$ has formation symmetry $C_{m}$ with $P_{\sigma}=\Pi_{n}^{q}$ for all $t \geq 0$.

What follows is the first of two principal results of this chapter. It shows that given $n$ properly ordered agents, every possible rotation group symmetry of a formation is invariant when the dynamics are circulant.

Theorem 6.2: If $A$ is a circulant matrix, then formation symmetry $C_{m}$ is invariant under the dynamics (6.2) for every $m$ that divides $n$.

Proof: For every $m$ that divides $n$, associated with the constraint (6.3) at time $t=0$ is a complex linear subspace $\mathcal{M}=\left\{z \in \mathbb{C}^{n}: M z=0\right\} \subset \mathbb{C}^{n}$, where $M=\Pi_{n}^{q}-e^{j 2 \pi / m} I_{n}$. It is well known that the subspace $\mathcal{M}$ is $A$-invariant if $M A=A M$. Since $A$ is a circulant matrix, it can be written in the form (2.7) on page 28 , implying that

$$
\begin{aligned}
M A & =\left(\Pi_{n}^{q}-e^{j 2 \pi / m} I_{n}\right) \sum_{i=0}^{n-1} a_{i} \Pi_{n}^{i} \\
& =\sum_{i=0}^{n-1} a_{i} \Pi_{n}^{i+q}-e^{j 2 \pi / m} \sum_{i=0}^{n-1} a_{i} \Pi_{n}^{i} \\
& =\sum_{i=0}^{n-1} a_{i} \Pi_{n}^{i}\left(\Pi^{q}-e^{j 2 \pi / m} I_{n}\right) \\
& =A M
\end{aligned}
$$

Therefore, the subspace $\mathcal{M}$ is invariant under the dynamics (6.2), which means that the constraint (6.3) holds with $P_{\sigma}=\Pi_{n}^{q}$ for all $t \geq 0$.

Example 6.2: Consider the $n=8$ agents depicted in Figure 6.3a. This formation $z(0)$ has symmetry $C_{4}$ with associated permutation $\sigma=(1,3,5,7)(2,4,6,8)$. Let

$$
\begin{equation*}
A=\operatorname{circ}(-1,-1,0,0,0,0,2,0) \tag{6.8}
\end{equation*}
$$

be the corresponding multiagent system matrix. Thus, every agent $i \in \mathcal{V}$ is repelled from agent $i+1$, but doubly attracted to agent $i+6$. Figure 6.2 b shows the evolution of the formation starting at $z(0)$ under the dynamics (6.2) with (6.8). The fact that the agents converge to the origin is not of interest. Rather, the dashed lines connecting agents in the cycle $\{1,3,5,7\}$ at regular intervals during the simulation highlight that $C_{4}$ symmetry is preserved.

(a) Initial graph $\Gamma(A, z(0))$

(b) Simulation demonstrating symmetry invariance

Figure 6.3: Initial formation graph and simulation results for Example 6.2

The following corollary to Theorem 6.2 addresses the more general case when the formation is not initially ordered.

Corollary 6.2: Given a permutation $\sigma$, let $\tau$ be such that $P_{\tau} P_{\sigma} P_{\tau}^{\top}=\Pi_{n}^{q}$ (cf. Theorem 6.1). Let $m$ be any divisor of $n$ and suppose $z(0) \in \mathbb{C}^{n}$ has symmetry $C_{m}$ with permutation matrix $P_{\sigma}$. If $P_{\tau} A P_{\tau}^{\top}$ is a circulant matrix, then the formation $z(t)$ has symmetry $C_{m}$ with $P_{\sigma}$ for all $t \geq 0$.

Proof: If $z(0)$ has symmetry $C_{m}$ with $P_{\sigma}$, Theorem 6.1 says that there always exists a relabeling of the agents $\tilde{z}(0)=P_{\tau} z(0)$ such that

$$
\begin{equation*}
e^{j 2 \pi / m} \tilde{z}(0)=\Pi_{n}^{q} \tilde{z}(0) \tag{6.9}
\end{equation*}
$$

where $q \in\{1,2, \ldots, n-1\}$ satisfies $\operatorname{gcd}(n, q)=n / m$. Viewing $\tilde{z}(t)=P_{\tau} z(t)$ as a transformation of coordinates implies that (6.2) becomes

$$
\begin{equation*}
\dot{\tilde{z}}(t)=P_{\tau} A P_{\tau}^{\top} \tilde{z}(t) \tag{6.10}
\end{equation*}
$$

If the new system matrix $P_{\tau} A P_{\tau}^{\top}$ is circulant, then (6.9) and (6.10) satisfy the conditions of Theorem 6.2. Therefore, the formation $\tilde{z}(t)$ has symmetry $C_{m}$ with permutation $\Pi_{n}^{q}$ for all $t \geq 0$. If $\tilde{z}(t)$ has symmetry $C_{m}$ for all $t \geq 0$, then $z(t)=P_{\tau}^{\top} \tilde{z}(t)$ also has symmetry $C_{m}\left(\right.$ with $\left.P_{\sigma}\right)$ for all $t \geq 0$. This is because the change of coordinates given by $\tau$ is merely a permutation of the agent locations, which does not alter the symmetry of the formation (cf. Remark 6.5).

The following example illustrates the previous corollary.

Example 6.3: Consider a system (6.2) of $n=4$ agents with

$$
A=\left[\begin{array}{rrrr}
-2 & 2 & 1 & -1 \\
2 & -2 & -1 & 1 \\
-1 & 1 & -2 & 2 \\
1 & -1 & 2 & -2
\end{array}\right]
$$

which is not circulant. Suppose the graph $\Gamma(A, z(0))$ is the same as in Figure 6.7 a, but with agents 2 and 3 having swapped positions. Hence, the relabeling
that takes $P_{\sigma}$ into the form $\Pi_{n}$ is given by the permutation matrix

$$
P_{\tau}=\left[\begin{array}{llll}
1 & 0 & 0 & 0  \tag{6.11}\\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]
$$

Moreover, $P_{\tau} A P_{\tau}^{\top}=\operatorname{circ}(-2,1,2,-1)$. Therefore, following Corollary 6.2, the formation $z(t)$ has symmetry $C_{4}$ for all $t \geq 0$.

In conclusion, if the multiagent system (6.2) has an underlying circulant structure (possibly after a change of coordinates), then every rotation group symmetry, of any initial formation, is invariant under the system's dynamics.

### 6.3.4 Decomposition of the Dynamics

Before moving on to necessity, this section explores the complex linear subspace that corresponds to formation symmetry $C_{m}$ in the proof of Theorem 6.2.

It has been shown in the previous section that, given a canonical ordering, circulant systems preserve rotation group symmetries (Theorem 6.2). The complex linear subspace $\mathcal{M}=\left\{z \in \mathbb{C}^{n}: M z=0\right\} \subset \mathbb{C}^{n}$, where $M=\Pi_{n}^{q}-e^{j 2 \pi / m} I_{n}$ and which characterizes rotation group symmetries, corresponds to $n-n / m$ independent complex constraints on the motion of the multiagent system (6.2). In other words, $\operatorname{dim} \operatorname{Ker} M=n / m$. One way to see this is graphically. For example, consider the $n=10$ agents with symmetry $C_{5}$ in Figure 6.2b. The associated permutation factors into two disjoint cycles, namely $\sigma=(1,5,8,3,7)(2,6,10,4,8)$. By selecting only two agents, one from each cycle, one can determine the locations of all the remaining agents by performing rotations through $2 \pi / 5$. More generally, one can always write out the cycles generated by a given $C_{m}$ formation of agents $1,2, \ldots, n / m$, as in (6.6). These cycles are the disjoint factors of $\sigma(i)=i+q$, where $\operatorname{gcd}(n, q)=n / m$. Since one is allowed to independently specify the locations of the first $n / m$ agents, one has exactly $n / m$ complex degrees of freedom. Hence, there exist $n-n / m$ independent complex constraints on the system.

Let $p:=n / m$ and define $w(t):=\left(z_{1}(t), z_{2}(t), \ldots, z_{p}(t)\right)$. For every formation $z(t) \in \mathcal{M}, z(t)$ can be written as

$$
z(t)=\left[\begin{array}{lllll}
w^{\top}(t) & e^{j 2 \pi / m} w^{\top}(t) & \left(e^{j 2 \pi / m}\right)^{2} w^{\top}(t) & \cdots & \left(e^{j 2 \pi / m}\right)^{m-1} w^{\top}(t)
\end{array}\right]^{\top}
$$

Observe that if the system matrix $A$ is circulant and of order $n=p \cdot m$, it can be partitioned into precisely $m^{2}$ blocks, each of order $p$. This partitioning causes $A$ to become a block circulant matrix, denoted $A=\operatorname{circ}\left(A_{0}, A_{1}, \ldots, A_{m-1}\right)$, where the blocks $A_{0}, A_{1}, \ldots, A_{m-1}$ are of order $p$ (see Davis, 1994, Section 5.6). This partitioning allows one to write the $n / m$-dimensional dynamics on $\mathcal{M}$ as

$$
\dot{w}(t)=\left[\begin{array}{llll}
A_{0} & A_{1} & \cdots & A_{m-1} \tag{6.12}
\end{array}\right] z(t)=\sum_{i=0}^{m-1}\left(e^{j 2 \pi / m}\right)^{i} A_{i} w(t) .
$$

Example 6.4: Consider the special case of cyclic pursuit (see Section 2.2.3), with $A=\operatorname{circ}(-1,1,0, \ldots, 0)$. Suppose $z(t) \in \mathbb{C}^{n}$ has symmetry $C_{n}$ with $P_{\sigma}=\Pi_{n}$. Hence, the dimension of the complex dynamics on $\mathcal{M}$ is simply 1 (i.e., there are $n-1$ complex constraints). Let $w(t)=z_{1}(t)$, yielding the dynamics on $\mathcal{M}$,

$$
\dot{w}(t)=\left(e^{j 2 \pi / n}-1\right) w(t) .
$$

Next, consider the agents in Figure 6.1a. The formation $z(t)$ of $n=6$ agents has symmetry $C_{2}$ with $P_{\sigma}=\Pi_{n}^{3}$. Suppose the agents are in cyclic pursuit with $A=$ $\operatorname{circ}(-1,1,0, \ldots, 0)$, which is consistent with the graph $\Gamma(A, z(t))$ in the figure. In this case, $w(t)=\left(z_{1}(t), z_{2}(t), z_{3}(t)\right)$. The dynamics on $\mathcal{M}$ have dimension $6 / 2=3$, and are given by

$$
\dot{w}(t)=\left[\begin{array}{rrr}
-1 & 1 & 0 \\
0 & -1 & 1 \\
-1 & 0 & 1
\end{array}\right] w(t) .
$$

Using (6.12), the $3 \times 3$ matrix in the above equation is $A_{0}-A_{1}$, where $A_{0}$ is the upper-left $3 \times 3$ block of $A$ and $A_{1}$ is the upper-right $3 \times 3$ block of $A$.

### 6.4 Circulant Necessity

Thus far, it has been proved that circulant multiagent systems preserve rotation group symmetries. The question that is addressed in this section is: To what extent is circulant connectivity also necessary? It is revealed in Theorem 6.3, which follows, that circulant connectivity is necessary if symmetry $C_{m}$ is to be invariant under the multiagent system's dynamics for every $m$ that divides $n$.

### 6.4.1 A Counterexample

Firstly, the condition of Theorem 6.2 that $A$ be a circulant matrix is not necessary for any single $m$ dividing $n$. The following example illustrates this fact.

Example 6.5: Consider a system (6.2) of $n=4$ agents, where the inputs (6.1) are

$$
\begin{aligned}
& u_{1}(t)=z_{2}(t)-z_{1}(t) \\
& u_{2}(t)=z_{4}(t)-z_{2}(t)-\left(z_{1}(t)-z_{2}(t)\right) \\
& u_{3}(t)=z_{4}(t)-z_{3}(t) \\
& u_{4}(t)=z_{1}(t)-z_{4}(t)
\end{aligned}
$$

The corresponding system matrix $A$ is the non-circulant matrix

$$
A=\left[\begin{array}{rrrr}
-1 & 1 & 0 & 0  \tag{6.13}\\
-1 & 0 & 0 & 1 \\
0 & 0 & -1 & 1 \\
1 & 0 & 0 & -1
\end{array}\right]
$$

Consider the initial formation $z(0)$ and graph $\Gamma(A, z(0))$ given in Figure 6.4. The formation $z(0)$ has symmetry $C_{4}$ with $P_{\sigma}=\Pi_{4}$. It can be verified that, under the dynamics (6.2) with (6.13), the formation $z(t)$ has symmetry $C_{4}$ for all $t \geq 0$.

However, it can also be verified that there exists an initial formation having symmetry $C_{2}$ (a subgroup of $C_{4}$ ) with $P_{\sigma}=\Pi_{4}^{2}$ such that symmetry $C_{2}$ is not preserved for all $t \geq 0$ (e.g., let $z_{1}(0)=z_{3}(0)$ and $z_{2}(0)=z_{4}(0)$ ).

### 6.4.2 A Special Class of Formations

In studying the necessity of circulant connectivity, it is helpful to employ a special class of formations $z(t)$; namely, those given by the constraint

$$
\begin{equation*}
\omega^{q} z(t)=\Pi_{n} z(t) \tag{6.14}
\end{equation*}
$$

for some $q \in\{1,2, \ldots, n-1\}$ and where $\omega:=e^{j 2 \pi / n}$. Notice that the locations $z_{i}(t)$, $i=1,2, \ldots, n$, generated by the constraint (6.14) all have the same magnitude. The following lemma associates a formation satisfying (6.14) with its symmetry.


Figure 6.4: Non-circulant figure $\Gamma(A, z(0))$ for Example 6.5

Lemma 6.2: Suppose $\omega^{q} z(t)=\Pi_{n} z(t)$ holds for some $q \in\{1,2, \ldots, n-1\}$ and $z(t) \in \mathbb{C}^{n}$. Then, the formation $z(t)$ has symmetry $C_{m}$, where $m=n / \operatorname{gcd}(n, q)$.

Proof: Let $p:=\operatorname{gcd}(n, q)$ and define $m:=n / p$ and $k_{q}:=q / p$. To show the formation has symmetry $C_{m}$ one must show there exists a permutation matrix $P_{\sigma}$ such that (6.3) holds. From $\omega^{q} z(t)=\Pi_{n} z(t)$ one has

$$
\begin{equation*}
\left(e^{j 2 \pi / n}\right)^{q} z(t)=\left(e^{j 2 \pi / m}\right)^{k_{q}} z(t)=\Pi_{n} z(t) \tag{6.15}
\end{equation*}
$$

By Bézout's identity ${ }^{4}$, there exist integers $l_{q}$ and $l_{m}$ such that $1=\operatorname{gcd}\left(k_{q}, m\right)=$ $l_{q} k_{q}+l_{m} m$. This fact together with (6.15) yields

$$
\begin{equation*}
e^{j 2 \pi / m} z(t)=\left(e^{j 2 \pi / m}\right)^{l_{q} k_{q}} z(t)=\Pi_{n}^{l_{q}} z(t) . \tag{6.16}
\end{equation*}
$$

By letting $P_{\sigma}=\Pi_{n}^{l_{q}}$, one obtains the desired result.

Notice that the proof of Lemma 6.2 also reveals how formations satisfying the special constraint (6.14) have symmetry $C_{m}$ with the canonical ordering introduced in Section 6.3 .2 (i.e., (6.3) holds with $P_{\sigma}=\Pi_{n}^{l_{q}}$ ).

Example 6.6: Consider the example graphs $\Gamma(A, z(0))$ with $\omega^{q} z(0)=\Pi_{n} z(0)$ given in Figure 6.5, where $n=6$. In Figure 6.5a, $q=1$ and the formation has the symmetry group $C_{6}$ since $m=6 / \operatorname{gcd}(6,1)=6 / 1=6$. In Figure $6.5 \mathrm{~b}, q=2$ and the formation has the symmetry group $C_{3}$ since $m=6 / \operatorname{gcd}(6,2)=6 / 2=3$.

[^9]

Figure 6.5: Example graphs $\Gamma(A, z(0))$ with $\omega^{q} z(0)=\Pi_{6} z(0)$

Let $v_{q}:=\left(1, \omega^{q}, \omega^{2 q}, \ldots, \omega^{(n-1) q}\right)$, the $(q+1)$-th column of the Fourier matrix $F_{n}^{*}$ multiplied by $\sqrt{n}$ (the Fourier matrix defined in (2.8) on page 29).

Lemma 6.3: For every $q \in\{1,2, \ldots, n-1\}$, the vector $z \in \mathbb{C}^{n}$ satisfies $\omega^{q} z=\Pi_{n} z$ if and only if $z=v_{q} z_{1}$.

Proof: The statement $\omega^{q} z=\Pi_{n} z$ is equivalent to

$$
\begin{aligned}
z_{2} & =\omega^{q} z_{1} \\
z_{3} & =\omega^{q} z_{2}=\omega^{2 q} z_{1} \\
& \vdots \\
& \\
z_{n} & =\omega^{(n-1) q} z_{1}
\end{aligned}
$$

with $\omega^{n q} z_{1}=z_{1}$. Equivalently, $z=v_{q} z_{1}$.

### 6.4.3 Necessary Conditions for Invariance

The aim of this section is to show that circulant connectivity is necessary in order that symmetry $C_{m}$ be invariant for every $m$ that divides $n$. The following result is the second principal result of this chapter.

Theorem 6.3: If formation symmetry $C_{m}$ is invariant under the dynamics (6.2) for every $m$ that divides $n$, then $A$ is a circulant matrix.

Proof: Theorem 2.6 (from Davis, 1994) on page 29 says that an $n \times n$ matrix $A$ is circulant if and only if it commutes with the fundamental permutation matrix $\Pi_{n}$. Therefore, it suffices to show that $\Pi_{n} A-A \Pi_{n}=0$.

If formation symmetry $C_{m}$ is invariant for every $m$ that divides $n$ it must be that, in particular, initially symmetric formations satisfying (6.14) are symmetric for all $t \geq 0$, after Lemma 6.2. Let $q \in\{1,2, \ldots, n-1\}$ be arbitrary and pick an initial formation $z(0)=v_{q} z_{1}(0) \neq 0$. By Lemma 6.3, $z(0)$ satisfies $\omega^{q} z(0)=$ $\Pi_{n} z(0)$. By Lemma 6.2, $z(0)$ has symmetry $C_{m}$ with $m=n / \operatorname{gcd}(n, q)$. By assumption, the formation $z(t)$ has symmetry $C_{m}$ for all $t \geq 0$. By differentiating the formation constraint $\omega^{q} z(t)=\Pi_{n} z(t)$ in time, one obtains

$$
\begin{aligned}
\omega^{q} A z(t)=\Pi_{n} A z(t) & \stackrel{(6.14)}{\Longleftrightarrow}\left(\Pi_{n} A-A \Pi_{n}\right) z(t)=0 \\
& \Longleftrightarrow\left(\Pi_{n} A-A \Pi_{n}\right) v_{q} z_{1}(t)=0,
\end{aligned}
$$

for all $t \geq 0$. In particular, since $z_{1}(0) \neq 0,\left(\Pi_{n} A-A \Pi_{n}\right) v_{q}=0$.
By Property 6.1, $A$ has zero row-sums. Thus, $A v_{0}=0$. Also, because $v_{0}$ is an eigenvector of $\Pi_{n}$ with corresponding eigenvector $\lambda=1, \Pi_{n} v_{0}=v_{0}$ (Davis, 1994, pp. 72-73). Therefore, one finds

$$
\left(\Pi_{n} A-A \Pi_{n}\right) v_{0}=\Pi_{n} A v_{0}-A \Pi_{n} v_{0}=-A v_{0}=0
$$

Recall that, $\left[v_{0} v_{1} \cdots v_{n-1}\right]=\sqrt{n} F_{n}^{*}$, where $F_{n}$ is the Fourier matrix. Therefore, it has been shown that $\left(\Pi_{n} A-A \Pi_{n}\right) F_{n}^{*}=0$. Since $F_{n}^{*}$ is invertible, $\Pi_{n} A-A \Pi_{n}=0$. Therefore, $A$ is a circulant matrix.

The following example highlights the significance of the assumption that not only is symmetry $C_{n}$ invariant, but also all of its subgroups are invariant under the system's dynamics (further to Example 6.5).

Example 6.7: Consider $n=6$ agents initially configured such that $\omega z(0)=$ $\Pi_{6} z(0)$. Suppose $\Gamma(A, z(0))$ is coupled in an all-to-all fashion, as in Figure 6.5a. Let $\tilde{A}=\operatorname{circ}(-5,1,1,1,1,1)$ and let $A$ be the matrix $\tilde{A}$ but with its second row replaced by $(1 / 2,-4,1 / 2,1 / 2,2,1 / 2)$. For the initial formation $\omega z(0)=\Pi_{6} z(0)$, Figure 6.6a shows how the rotation group $C_{6}$ is invariant under the dynamics (6.2), despite the fact that $A$ is not circulant. In Figure 6.5a, the dashed lines connect agents $(1,2,3,4,5,6)$, in sequence, at regular intervals during the simulation.

However, consider a different initial formation $\omega^{2} z(0)=\Pi_{6} z(0)$, which has symmetry $C_{3}($ since $\operatorname{gcd}(6,2)=2$, implying that $m=6 / 2=3) . C_{3}$ is a subgroup of $C_{6}$. The associated formation graph is given in Figure 6.5b. Formation symmetry $C_{3}$ is not invariant under the dynamics (6.2), as one can see from the simulation results of Figure 6.6b, where the dashed lines connect agents $(1,2,3)$. As time evolves, the initial equilateral formation becomes only isosceles.

### 6.4.4 Summary and a Comment on Ordering

Therefore, by combining the sufficiency result of Theorem 6.2 with the necessity result of Theorem 6.3, one obtains the following necessary and sufficient result, which summarizes the significance of circulant connectivity.

Corollary 6.3: Formation symmetry $C_{m}$ is invariant under the dynamics (6.2) for every $m$ that divides $n$ if and only if $A$ is circulant.

Finally, one might naturally wonder about the necessity of the canonical labeling, introduced in Section 6.3.2 and assumed in the definition of invariance (Definition 6.4). Is this ordering assumption without loss of generality? Do there exist other classes of ordering for which there is symmetry invariance if and only if the system matrix is circulant? This remains an open question.

### 6.4.5 Graph Symmetry and Invariance

It has been shown that multiagent systems with circulant connectivity have the attractive property that formation symmetry $C_{n}$ and all of its subgroups are invariant under the system's evolution. Moreover, circulant connectivity among the agents is also necessary to obtain this invariance property. Although Theorems 6.2 and 6.3 make no mention of graph symmetry, the condition that $A$ is (structurally) circulant implies the graph (see Section 6.2.5) is also symmetric. This result is offered in Proposition 6.1, but a definition and example are required first.

Definition 6.5 (Graph Symmetry): The graph $\Gamma(A, z(t))=\left(\mathcal{V}_{t}, \mathcal{E}_{t}\right)$ is said to have the symmetry group $G$ at time $t$ if it has the property that for every element $g \in G$, if $v(t) \in \mathcal{V}_{t}$, then $g v(t) \in \mathcal{V}_{t}$, and if $e(t) \in \mathcal{E}_{t}$, then $g e(t) \in \mathcal{E}_{t}$. Moreover, the induced maps $v(t) \mapsto g v(t)$ and $e(t) \mapsto g e(t)$ are permutations.


Figure 6.6: Simulations for Example 6.7

The requirement that the maps in Definition 6.5 be permutations (or bijections) guarantees that two agents (respectively, edges), possibly collocated, cannot be mapped to the same agent (respectively, edge).

Notice that graph symmetry $C_{m}$ (Definition 6.5) implies formation symmetry $C_{m}$ (Definition 6.3), but not the converse. For instance, the formation $z(t)$ in Figure 6.1 b has formation symmetry $C_{4}$ (the constraint (6.3) holds with $\sigma=$ $(1,4,2,3))$, but the graph $\Gamma(A, z(t))$ has only symmetry $C_{1}$.

Example 6.8: Consider the example formation graphs $\Gamma(A, z(t))$ of Figure 6.1. Figure 6.1a has symmetry $C_{2}$ but not $C_{4}$, because a rotation through $\pi / 2$ does not map vertices to vertices. Figure 6.1b has symmetry $C_{1}$ but not $C_{2}$, because a rotation through $\pi$ reverses the edge directions. For the same reason, Figure 6.1c has only symmetry $C_{1}$, not symmetry $C_{2}$.

Proposition 6.1: Suppose $z(t)$ has symmetry $C_{m}$ with $P_{\sigma}=\Pi_{n}^{q}$. If $A$ is a structurally circulant matrix, then the graph $\Gamma(A, z(t))$ has symmetry $C_{m}$.

Proof: As per Definition 6.5, it is enough to show that the map induced by a generator of the cyclic group $C_{m}$ maps vertices in $\mathcal{V}_{t}$ (respectively, edge vectors in $\mathcal{E}_{t}$ ) to vertices in $\mathcal{V}_{t}$ (respectively, edge vectors in $\mathcal{E}_{t}$ ) by a bijection. Rotation through $2 \pi / m$ is a generator of the cyclic group $C_{m}$ (cf. Definition 6.2). Constraint (6.3) implies the map $z(t) \mapsto e^{j 2 \pi / m} z(t)$ is a bijection on $\mathcal{V}_{t}$, which means that vertices $z_{i}(t) \in \mathcal{V}_{t}$ are mapped to vertices in $\mathcal{V}_{t}$ by a bijection. Consider the rotation of an arbitrary edge vector $e_{i k}(t) \in \mathcal{E}_{t}$ through angle $2 \pi / m$, yielding

$$
e^{j 2 \pi / m} e_{i k}(t)=e^{j 2 \pi / m}\left(z_{k}(t)-z_{i}(t)\right)=z_{k+q}(t)-z_{i+q}(t)=e_{i+q, k+q}(t)
$$

Since $e_{i k} \in \mathcal{E}, a_{i k} \neq 0$. But, since $A$ is structurally circulant, $a_{i+q, k+q} \neq 0$, implying that $e_{i+q, k+q} \in \mathcal{E}$. Hence, by the constraint (6.3), edge vectors $e_{i k}(t) \in \mathcal{E}_{t}$ are mapped to edge vectors in $\mathcal{E}_{t}$ by a bijection ${ }^{5}$.

Example 6.9: Figure 6.7 gives two example graphs, each with a (structurally) circulant interconnection topology between agents. In each case, one can compare formation symmetry with graph symmetry. Both the formation and graph in Figure 6.7a have symmetry $C_{4}$. In this case, the associated permutation is $\sigma(i)=$

[^10]$i+1$, which is primitive. In the case of Figure 6.7b, the formation $z(t)$ has symmetry $C_{4}$, but the graph has only symmetry $C_{2}$. In this case, the permutation associated with $C_{2}$ symmetry is $\sigma(i)=i+4$ and $\sigma$ can be factored into exactly four distinct cycles $\sigma=(1,5)(2,6)(3,7)(4,8)$.


Figure 6.7: Circulant formation graphs $\Gamma(A, z(t))$

The following example illustrates the fact that graph symmetry is not sufficient to preserve cyclic group symmetries. It also highlights, once again, the importance of the canonical agent ordering described in Section 6.3.2.

Example 6.10: Consider a system (6.2) of $n=4$ agents with

$$
A=\left[\begin{array}{rrrr}
-2 & 1 & 2 & -1 \\
-1 & -2 & 1 & 2 \\
2 & -1 & -2 & 1 \\
1 & 2 & -1 & -2
\end{array}\right]=\operatorname{circ}(-2,1,2,-1)
$$

The information flow between agents together with their locations at time $t=0$ is illustrated by $\Gamma(A, z(0))$ in Figure 6.7 a . Notice that $\Gamma(A, z(0))$ has symmetry $C_{4}$. Clearly, (6.3) is satisfied with $P_{\sigma}=\Pi_{n}$. Following Theorem 6.2, this formation's symmetry is invariant under the dynamics (6.2). But, consider a new initial formation, given by a permutation of the original one, $\tilde{z}(0)=P_{\tau} z(0)$, where $P_{\tau}$ is given by (6.11). Since the coupling is all-to-all, the new graph $\Gamma(A, \tilde{z}(0))$ still has symmetry $C_{4}$ (any permutation of the agent locations leaves the graph
unchanged). However, (6.3) does not hold with $P_{\sigma}=\Pi_{n}^{q}$ for any $q$, since

$$
P_{\tau} \Pi_{n} P_{\tau}^{\top}=\left[\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array}\right]
$$

is not of the form $\Pi_{n}^{q}$. It can be shown via simulation that $C_{4}$ symmetry in the formation starting at $\tilde{z}(0)$ is not invariant under the dynamics $\dot{\tilde{z}}(t)=A \tilde{z}(t)$, despite the fact that the graph $\Gamma(A, \tilde{z}(0))$ has symmetry $C_{4}$.

### 6.5 Shape Evolution for Circulant Pursuit

Conditions for the invariance of rotation group symmetry in multiagent formations were studied in the previous (main) sections of this chapter. From this work, it is clear that circulant connectivity plays a fundamental role in symmetry. The next step is to understand how circulant systems asymptotically behave. This section takes a first step in this direction by examining multiagent systems that have a system matrix $A$ which is both circulant and Metzler (i.e., its off-diagonal elements are nonnegative; see Section 2.1.3). How do the agents evolve? What basic shapes might emerge and are they symmetric? It should be emphasized that, strictly speaking, the stability of rotation group symmetries (e.g., whether the invariant subspace $\mathcal{M}$ of Section 6.3.3 is attractive or not) is not studied here.

A common approach to studying the evolution of formations is to formulate appropriate (nonlinear) shape dynamics that are invariant under rotation, scaling, and translation (cf. Richardson, 2001b). In contrast, this section exploits the linearity of (6.2) by developing a continuous-time analog to the discrete-time ideas of Bruckstein et al. (1995). It is proved that agents in circulant pursuit exponentially converge to a point that is the centroid of their initial conditions. At the same time, following the logic originally proposed by Bruckstein et al. (1995), the formation tends towards an elliptical shape whose radii and orientation depend only on the initial formation.

### 6.5.1 Consensus

To simplify the calculations, it is assumed (without loss of generality) that $A=$ $\operatorname{circ}\left(a_{0}, a_{1}, \ldots, a_{n-1}\right)$ is scaled such that $a_{0}=-1$. Recall that it is possible to diagonalize every circulant matrix by $A=F_{n}^{*} \Lambda F_{n}$, where $F_{n}$ is the $n \times n$ Fourier matrix (see Theorem 2.8 on page 29). The diagonal matrix $\Lambda=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right)$ are the eigenvalues of $A$. The Fourier matrix (2.8) on page 29 can be written as

$$
\begin{aligned}
F_{n}^{*} & =\frac{1}{\sqrt{n}}\left[\begin{array}{llll}
f_{1} & f_{2} & \cdots & f_{n}
\end{array}\right] \\
f_{i} & =v_{i-1}=\left(\omega^{0}, \omega^{i-1}, \omega^{2(i-1)}, \ldots, \omega^{(n-1)(i-1)}\right)
\end{aligned}
$$

Moreover, the eigenvalues of $A$ are given by (see Corollary 2.1 on page 29) $\lambda_{i}=$ $p_{A}\left(\omega^{i-1}\right)$, where $p_{A}(\lambda)=-1+a_{1} \lambda+a_{2} \lambda+\cdots+a_{n-1} \lambda^{n-1}$. Recall that the polynomial $p_{A}(\lambda)$ is called the circulant's representer.

The purpose of this section is to study the geometric evolution of the formation as $t \rightarrow \infty$. To understand how the agents evolve, examine the eigenvalues of $A$.

Lemma 6.4: Let $A$ be circulant, Metzler, and satisfy Property 6.1 (i.e., have zero row-sums). Then it has one zero eigenvalue $\lambda_{1}=0$, and the remaining eigenvalues have $\operatorname{Re}\left(\lambda_{i}\right)<0, i=2,3, \ldots, n$.

Proof: The proof is by explicit computation of the eigenvalues:

$$
\begin{equation*}
\lambda_{i}=p_{A}\left(\omega^{i-1}\right)=-1+\sum_{k=1}^{n-1} \omega^{k(i-1)} a_{k} \tag{6.17}
\end{equation*}
$$

When $i=1$, one obtains $\lambda_{1}=-1+\sum_{k=1}^{n-1} a_{k}=0$. The real part of the remaining $n-1$ eigenvalues is given by

$$
\operatorname{Re}\left(\lambda_{i}\right)=-1+\sum_{k=1}^{n-1} a_{k} \cos (2 \pi k(i-1) / n)
$$

Since $\cos (2 \pi k(i-1) / n)<1$ for every $i \in\{2,3, \ldots, n\}$,

$$
\operatorname{Re}\left(\lambda_{i}\right)<-1+\sum_{k=1}^{n-1} a_{k}<0, i=2,3, \ldots, n
$$

concluding the proof.

The following result extends Theorem 2.5 on page 27 to the more general case of circulant systems.

Proposition 6.2: Consider the multiagent system (6.2) and let $A$ be both circulant and Metzler. Then, for every initial formation $z(0) \in \mathbb{C}^{n}$ the agents (6.2) exponentially converge to their centroid.

Proof: First, solve the linear system (6.2):

$$
\begin{equation*}
z(t)=e^{A t} z(0)=e^{F_{n}^{*} \Lambda F_{n} t} z(0)=F_{n}^{*} e^{\Lambda t} F_{n} z(0) \tag{6.18}
\end{equation*}
$$

Next, rewrite the solution (6.18) as the sum of its modal elements:

$$
\begin{align*}
z(t) & =\frac{1}{n}\left[\begin{array}{llll}
f_{1} & f_{2} & \cdots & f_{n}
\end{array}\right] e^{\Lambda t}\left[\begin{array}{llll}
f_{1} & f_{2} & \cdots & f_{n}
\end{array}\right]^{*} z(0) \\
& =\frac{1}{n} \sum_{i=1}^{n} e^{\lambda_{i} t} f_{i} f_{i}^{*} z(0) \tag{6.19}
\end{align*}
$$

Given Lemma 6.4, the steady-state behaviour of the multiagent system is determined by the eigenspace associated with the eigenvalue $\lambda_{1}=0$. In other words, this part of the solution dominates as $t \rightarrow \infty$ and the remaining modal components die away. Therefore, we only need to look at the term $i=1$ in (6.19). Define the function (employing a similar notation to Bruckstein et al., 1995)

$$
\begin{aligned}
z^{\infty}(t) & :=\frac{1}{n} e^{\lambda_{1} t} f_{1} f_{1}^{*} z(0)=\frac{1}{n} e^{0 t}\left[\begin{array}{c}
1 \\
1 \\
\vdots \\
1
\end{array}\right]\left[\begin{array}{llll}
1 & 1 & \cdots & 1
\end{array}\right] z(0) \\
& =\frac{1}{n}\left[\begin{array}{cccc}
1 & 1 & \cdots & 1 \\
1 & 1 & & 1 \\
\vdots & \vdots & & \vdots \\
1 & 1 & \cdots & 1
\end{array}\right] z(0)
\end{aligned}
$$

Thus, the agents converge their centroid, $\bar{z}:=(1 / n) \sum_{i=1}^{n} z_{i}(0)$.

### 6.5.2 Ellipses

In light of Proposition 6.2, the question of interest becomes: Do the agents converge to any particular arrangement while at the same time contracting? To this end, the solution (6.19) can be decomposed as follows:

$$
z(t)=z^{\infty}(t)+e^{\gamma t} w(t), \text { where } \gamma:=\max _{i \neq 1}\left\{\operatorname{Re}\left(\lambda_{i}\right)\right\}
$$

Following Proposition 6.2, $z^{\infty}(t)$ represents the steady-state behaviour of the agents. By subtracting $z^{\infty}(t)$ from $z(t)$, one is left with the modal elements of $z(t)$ that determine its behaviour in the transient. However, this collection of remaining modal elements can itself be decomposed into the slowest modes, which dominate $z(t)$ as $t \rightarrow \infty$, and those that die away more quickly. These latter slowest modes determine the geometric shape $z(t)$ exponentially takes as $t \rightarrow \infty$ (Bruckstein et al., 1995). The function $w(t)$ in the above decomposition is

$$
\begin{equation*}
w(t):=\frac{1}{n e^{\gamma t}} \sum_{i \neq 1} e^{\lambda_{i} t} f_{i} f_{i}^{*} z(0) \tag{6.20}
\end{equation*}
$$

Therefore, $w(t)$ represents the aforementioned collection of transient modal elements of $z(t)$ scaled by the rate of decay of the slowest modes among them.

By the above arguments, it is the behaviour of $w(t)$ as $t \rightarrow \infty$ that determines the exponentially stable geometric shape assumed by the agents. Define the set $\mathcal{S}=\left\{i: \operatorname{Re}\left(\lambda_{i}\right)=\gamma\right\}$. Since it is the eigenvalues with indices in the set $\mathcal{S}$ that dominate (6.20) as $t \rightarrow \infty$, define

$$
\begin{equation*}
w^{\infty}(t):=\frac{1}{n e^{\gamma t}} \sum_{i \in \mathcal{S}} e^{\lambda_{i} t} f_{i} f_{i}^{*} z(0) \tag{6.21}
\end{equation*}
$$

For simplicity's sake, the case of cyclic pursuit, with $A=\operatorname{circ}(-1,1,0, \ldots, 0)$, is first studied. Define the angle $\psi_{i}:=2 \pi(i-1) / n$. Therefore, the eigenvalues of $A$ are $\lambda_{i}=-1+e^{j \psi_{i}}$, which are simply the $n$ roots of unity shifted by -1 along the real axis. Moreover, $\mathcal{S}=\{2, n\}$ with corresponding eigenvalues $\lambda_{2, n}=-1+\omega^{ \pm 1}$. Let $z_{i}(0)=r_{i} e^{j \theta_{i}}$, the initial conditions in polar form. Finally, independent of $i$, define the following quantities:

$$
f_{2}^{*} z(0)=\sum_{i=1}^{n}\left(\omega^{i-1}\right)^{*} z_{i}(0)=\sum_{i=1}^{n} r_{i} e^{j\left(\theta_{i}-\psi_{i}\right)}=: a e^{j \psi_{a}}
$$

and, by employing $\omega^{n-1}=\omega^{-1}$,

$$
f_{n}^{*} z(0)=\sum_{i=1}^{n}\left(\omega^{1-i}\right)^{*} z_{i}(0)=\sum_{i=1}^{n} r_{i} e^{j\left(\theta_{i}+\psi_{i}\right)}=: b e^{j \psi_{b}} .
$$

Proposition 6.3: For agents in cyclic pursuit with $A=\operatorname{circ}(-1,1,0, \ldots, 0)$, the points $w_{i}^{\infty}(t)$ belong to an ellipse of radii $(a+b) / n$ and $(a-b) / n$.

Proof: In this case, $\gamma=-1+\cos (2 \pi / n)$ and

$$
\begin{aligned}
w^{\infty}(t) & =\frac{1}{n e^{\gamma t}}\left(e^{\gamma t} e^{j t \sin (2 \pi / n)} f_{2} f_{2}^{*}+e^{\gamma t} e^{-j t \sin (2 \pi / n)} f_{n} f_{n}^{*}\right) z(0) \\
& =\frac{1}{n}\left(e^{j t \sin (2 \pi / n)} f_{2} f_{2}^{*}+e^{-j t \sin (2 \pi / n)} f_{n} f_{n}^{*}\right) z(0)
\end{aligned}
$$

Now, for each element of $w^{\infty}(t)$ write

$$
w_{i}^{\infty}(t)=\frac{1}{n} e^{j t \sin (2 \pi / n)} \omega^{i-1} a e^{j \psi_{a}}+\frac{1}{n} e^{-j t \sin (2 \pi / n)} \omega^{1-i} b e^{j \psi_{b}} .
$$

Let $\beta:=\sin (2 \pi / n)$, constant for a given $n$. Then,

$$
w_{i}^{\infty}(t)=\frac{a}{n} e^{j\left(\beta t+\psi_{a}\right)} \omega^{i-1}+\frac{a}{n} e^{j\left(\beta t-\psi_{b}\right)} \omega^{1-i} .
$$

Define $\phi:=\left(\psi_{a}-\psi_{b}\right) / 2$ and $\bar{\phi}:=\left(\psi_{a}+\psi_{b}\right) / 2$. Then, $a e^{j \psi_{a}}=a e^{j \phi} e^{j \bar{\phi}}$ and $b e^{j \psi_{b}}=b e^{-j \phi} e^{j \bar{\phi}}$. Simplifying yet again, one obtains

$$
\begin{align*}
w_{i}^{\infty}(t) & =\frac{a}{n} e^{j \beta t} e^{j \phi} e^{j \bar{\phi}} \omega^{i-1}+\frac{b}{n} e^{-j \beta t} e^{-j \phi} e^{j \bar{\phi}} \omega^{1-i} \\
& =e^{j \bar{\phi}}\left(\frac{a}{n} e^{j\left(\beta t+\phi+\psi_{i}\right)}+\frac{b}{n} e^{-j\left(\beta t+\phi+\psi_{i}\right)}\right) \\
& =e^{j \bar{\phi}}\left(\frac{a+b}{n} \cos \left(\beta t+\phi+\psi_{i}\right)+j \frac{a-b}{n} \sin \left(\beta t+\phi+\psi_{i}\right)\right) . \tag{6.22}
\end{align*}
$$

The fixed rotation $e^{j \bar{\phi}}$ can be ignored (it only depends on the initial formation). To see that the points (6.22) belong to an ellipse of radii $r_{1}:=(a+b) / n$ and $r_{2}:=(a-b) / n$, one needs only to recognize that they satisfy the equation of an ellipse (in $\mathbb{C}$ ), given by $\left(\operatorname{Re}\left(\omega_{i}^{\infty}\right)^{/} r_{1}\right)^{2}+\left(\operatorname{Im}\left(\omega_{i}^{\infty}\right)^{/} r_{2}\right)^{2}=1, i=1,2, \ldots, n$.

Note that if $a<b$, then the points are ordered and travel in the clockwise (rather than counterclockwise) direction around the steady-state ellipse.

Remark 6.6: Suppose that the agents are initially positioned on a line. Assume it is the real axis (without loss of generality). In this case,

$$
f_{2}^{*} z(0)=\sum_{i=1}^{n} e^{-j \psi_{i}} z_{i}(0)=\sum_{i=1}^{n} z_{i}(0)\left(\cos \psi_{i}-j \sin \psi_{i}\right)=: a e^{j \psi_{a}} .
$$

Likewise,

$$
f_{n}^{*} z(0)=\sum_{i=1}^{n} e^{j \psi_{i}} z_{i}(0)=\sum_{i=1}^{n} z_{i}(0)\left(\cos \psi_{i}+j \sin \psi_{i}\right)=a e^{-j \psi_{a}} .
$$

Therefore, $\phi=\psi_{a}, \bar{\phi}=0$, and (6.22) reduces to

$$
w_{i}^{\infty}(t)=\frac{2 a}{n} \cos \left(\beta t+\phi+\psi_{i}\right)
$$

suggesting that the agents remain on the real axis for all time.
The cyclic pursuit result of Proposition 6.2 is now extended to the more general case of circulant pursuit. Recall that, in this case, the eigenvalues of $A$ are given by (6.17). Notice that

$$
\operatorname{Re}\left(\sum_{k=1}^{n-1} a_{k} e^{j 2 \pi k(i-1) / n}\right)=\operatorname{Re}\left(\sum_{k=1}^{n-1} a_{k} e^{j 2 \pi k(n-i+1) / n}\right) .
$$

for every $k \in\{1,2, \ldots, n-1\}$ and $i \in\{1,2, \ldots, n\}$. Therefore, the eigenvalues of $A$ come in complex conjugate pairs $\lambda_{i}=\bar{\lambda}_{n-i+2}$, (6.19) and (6.20) still hold, and the set $\mathcal{S}=\left\{i: \operatorname{Re}\left(\lambda_{i}\right)=\gamma\right\}$, where the scalar $\gamma=\max _{i \neq 1}\left\{\operatorname{Re}\left(\lambda_{i}\right)\right\}$, as before. To simply notation, define $l:=n-k+2$ and suppose that $\mathcal{S}=\{k, l\}$. Thus,

$$
\begin{aligned}
w^{\infty}(t) & =\frac{1}{n e^{\gamma t}}\left(e^{\gamma t} e^{j \operatorname{Im}\left(\lambda_{k}\right) t} f_{k} f_{k}^{*}+e^{\gamma t} e^{-j \operatorname{Im}\left(\lambda_{k}\right) t} f_{l} f_{l}^{*}\right) z(0) \\
& =\frac{1}{n}\left(e^{j \operatorname{Im}\left(\lambda_{k}\right) t} f_{k} f_{k}^{*}+e^{-j \operatorname{Im}\left(\lambda_{k}\right) t} f_{l} f_{l}^{*}\right) z(0)
\end{aligned}
$$

Therefore, as before, compute

$$
f_{k}^{*} z(0)=\sum_{i=1}^{n}\left(\omega^{(i-1)(k-1)}\right)^{*} z_{i}(0)=: a e^{j \psi_{a}} .
$$

Likewise, compute

$$
f_{l}^{*} z(0)=\sum_{i=1}^{n}\left(\omega^{(i-1)(l-1)}\right)^{*} z_{i}(0):=b e^{-j \psi_{b}}
$$

Proposition 6.4: For agents in pursuit with $A=\operatorname{circ}\left(-1, a_{1}, a_{2}, \ldots, a_{n-1}\right)$, the points $w_{i}^{\infty}(t)$ belong to an ellipse of radii $(a+b) / n$ and $(a-b) / n$.

Proof (sketch): Let $\beta:=\operatorname{Im}\left(\lambda_{k}\right)$ and $l:=n-k+2$. This yields,

$$
\begin{aligned}
w_{i}^{\infty}(t) & =\frac{1}{n}\left(e^{j \beta t} \omega^{(i-1)(k-1)} a e^{j \psi_{a}}+e^{-j \beta t} \omega^{(i-1)(l-1)} b e^{j \psi_{b}}\right) \\
& =\frac{1}{n}\left(e^{j \beta t} e^{j \psi_{i}(k-1)} a e^{j \psi_{a}}+e^{-j \beta t} e^{-j \psi_{i}(k-1)} b e^{j \psi_{b}}\right)
\end{aligned}
$$

which when simplified in the same manner as (6.22) gives

$$
w_{i}^{\infty}(t)=e^{j \bar{\phi}}\left(\frac{a+b}{n} \cos \left(\beta t+\phi+\psi_{i}(k-1)\right)+j \frac{a-b}{n} \sin \left(\beta t+\phi+\psi_{i}(k-1)\right)\right) .
$$

By employing the same arguments as in the proof of Proposition 6.3, these points belong an ellipse of radii $(a+b) / n$ and $(a-b) / n$.

Notice that the ordering of the agents around the steady-state ellipse depends on the value of $k$ (i.e., on the eigenvalues with magnitude $\gamma$ ). Hence, this ordering depends on the number of agents being pursued. In the case of cyclic pursuit $k=2$, and thus the agents are ordered contiguously around the ellipse.

In summary, agents in circulant pursuit exponentially converge to a point that is the centroid of their initial conditions. At the same time, the agents tend towards elliptical trajectories (but this behaviour is slower than the rate at which they converge to a point) whose shape and orientation depends only on the initial formation, $z(0)$. Moreover, as they approach an elliptical shape they become ordered and equally spaced in a radial sense, since $\psi_{i}=2 \pi(i-1) / n$ changes as $i$ is incremented from 1 to $n$. Also, the ellipse does not rotate with time because the angle of the ellipse with respect to the real axis is defined by the angle $\bar{\phi}$, which again depends only on the initial conditions. However, the agents do rotate around the ellipse at a frequency given by $\beta$. Finally, if the agents start on a line, they stay on a line since one radius of the ellipse is equal to zero.

## Chapter 7

## Summary and Conclusions

By linking coordination control problems in engineering with pursuit problems found in the mathematics and physics literature, this thesis studies the geometric formations of autonomous agent systems consisting of individuals programmed to locally pursue one another. Broadly speaking, this thesis emphasizes the importance of structure in the interconnection topology among agents, and illustrates how structure can be exploited towards analysis. This final chapter contains a concise summary of the thesis' primary contributions and ends with a brief discussion about questions that have arisen over the course of this research, followed by some closing remarks.

### 7.1 Summary of Contributions

This section identifies the primary contributions of this work. These contributions are organized into categories: those pertaining to agents in pursuit, nonlinear vehicles in cyclic pursuit, and the invariance of symmetries in multiagent formations.

## Agents in Pursuit

Chapter 2 offers the following principal contributions.

1. The notion of pursuit is proposed as a multiagent coordination technique, and a connection is drawn between existing literature on positive systems and the proposed coordination strategy (Section 2.1.3).
2. Details are given of an important connection between multiagent coordination, the traditional pursuit problem (e.g., see Theorem 2.5), and the theory of circulant matrices (as in Davis, 1994). This relationship is exemplified by

Section 2.2.6 and its significance is highlighted by the fact that it supplies the foundation for subsequent contributions in Chapters 3, 4, and 6.

It is worth mentioning that the relevance and utility of this last contribution has recently been acknowledged by other researchers. For example, Aranda et al. (2005) use circulant connectivity to stabilize multiagent formations that are optimal for target tracking. Smith, Broucke, and Francis (2005) study hierarchies of cyclic pursuit with the aim of improving convergence rates for consensus.

## Vehicles in Cyclic Pursuit

An extension of traditional cyclic pursuit (see Section 2.2) to $n>1$ nonholonomic vehicles is carried out in Chapters 3 and 4. Its most important theoretical contributions are summarized below, all of which correspond to the coordination law (3.4) for fixed-speed vehicles and (4.1) for varying-speed vehicles.

1. In relative coordinates, for both fixed- and varying-speed unicycles, every equilibrium formation corresponds to a generalized regular polygon, denoted $\{n / d\}$, of density $1 \leq d<n$ (Theorems 3.1 and 4.1).
(a) In the case of fixed-speed vehicles, the vehicles traverse a circle of radius $\rho=v_{R} n /(k \pi d)$, where $v_{R}$ is the vehicles' speed (Corollary 3.1).
(b) In the case of varying-speed vehicles, an equilibrium formation exists only for a specific gain $k=k^{\star}$, given by (4.3).
2. Local stability analyses are carried out that reveal, for every $n>1$, exactly which equilibrium formations are asymptotically stable. By exploiting the circulant structure of the control laws (3.4) and (4.1), a technique for reducing the problem from $3 n$ dimensions to only 3 is developed.
(a) In the case of fixed-speed vehicles Theorem 3.3 and Corollaries 3.3 through 3.6 yield that: (i) every $\{n / 1\}$ formation is locally asymptotically stable; (ii) every $\{n / d\}$ formation with $d \geq 6$ is unstable; and, (iii) in the remaining possible cases, Table 3.2 identifies all possible equilibrium formations and gives their stability.
(b) In the case of varying-speed vehicles (with $k=k^{\star}$ ), Theorem 4.4 states that the only locally asymptotically stable equilibrium formations are the ordinary regular polygons, $\{n / 1\}$.
3. For varying-speed vehicles, when $k \neq k^{\star}$ the vehicles do not converge to an equilibrium formation (see Sections 4.3 and 4.6).
(a) In general, depending on whether $k=k^{\star}-\epsilon$ or $k=k^{\star}+\epsilon$, for sufficiently small $\epsilon$, the vehicles converge or diverge, respectively, while at the same time approaching an $\{n / 1\}$ polygon formation (Theorems 4.5 and 4.6).
(b) In the special case when $n=2$, a global stability analysis is offered (see Section 4.3), revealing for what gains and which initial conditions the vehicles either converge to a point, diverge, or converge to equally spaced motion around a circle (Theorem 4.2).
4. For both fixed- and varying-speed vehicles, every equilibrium formation is stationary, meaning that the formation's centroid does not drift over time (see Sections 3.4 and 4.7 for fixed- and varying-speed cases, respectively).
5. For both fixed- and varying-speed vehicles, simulations suggest that stable periodic solutions also exist for vehicles subject the coordination laws (3.4) and (4.1). These solutions are collectively referred to as the $n$-vehicle "weave" (see Sections 3.5 and 4.8). An invariant subspace is revealed in Section 3.5.2, which contains no equilibrium points, and it is conjectured that the periodic solution corresponding to the weave is contained in this subspace. To the author's best knowledge, the problem of achieving periodic multivehicle trajectories of this sort has not yet been studied in the multiagent and cooperative control engineering literature.

Other researchers have already begun to generalize the contributions mentioned above. For example, Sinha and Ghose (2005) extend the results of Chapter 3 to the case of vehicles with different speeds and different controller gains. Jeanne, Leonard, and Paley (2005) utilize the results of Chapter 3 in their study of ringcoupled planar particles modelled as coupled oscillators (cf. Section 1.1.2). They prove that the equilibrium formations of their system are also generalized regular polygons. In their work, the stability of each polygon depends on one's choice of controller gain (which they refer to as the "coupling strength parameter").

Although there is a growing amount of theoretical work on multiagent systems that employ local interactions, there are at present relatively few instances of experimental research specifically validating their practicality. In this thesis, the
theoretical contributions of Chapters 3 were validated experimentally, as reported in Chapter 5. These contributions are as follows.

1. Experiments using the four-wheeled UTIAS Argo Rovers confirmed that the theoretical results of Chapter 3 can be applied in practice to real vehicles that are in many respects different from ideal kinematic unicycles. The proposed multivehicle pursuit strategy was observed to be robust in the presence of unmodelled dynamics, disturbances in the vehicle velocities, and delays in the system due to sensing and information processing.
2. Computation of the control law was based solely on sensing and data processing carried out locally, without explicit communication between the vehicles, the use of a global positioning system, nor a central supervisor. This is unlike many cooperative control experiments reported in the literature.

These contributions are, in particular, significant in that the findings are not only encouraging for pursuit-based coordination strategies, but also for other cooperative control techniques that employ similar local interaction techniques.

## Symmetry and Invariance

Chapter 6 looks at linear multiagent systems of the form (6.1)-(6.2) from a unique perspective: What fixed decentralized controllers inherently preserve symmetric formations? Its primary contributions are the following.

1. Of central relevance to the topic of rotation symmetry invariance in multiagent formations is circulant structure in the interconnection topology among agents. It is revealed to what extent structure influences symmetry.
(a) If the system matrix $A$ of (6.2) is circulant (possibly after a permutation of coordinates), then formation symmetry $C_{m}$ of any initial formation is invariant under the system's dynamics for every $m$ that divides $n$ (Theorem 6.2 and Corollary 6.2).
(b) When canonically ordered, if formation symmetry $C_{m}$ is invariant under the system's dynamics for every $m$ that divides $n$, then the system matrix $A$ is necessarily circulant (Theorem 6.3).
(c) If the system matrix $A$ of (6.2) has a circulant structure, but is not necessarily circulant, then the induced information flow graph has symmetry $C_{m}$ (Proposition 6.1).
2. Agents in circulant pursuit converge to a point that is the centroid of the agents. At the same time, they tend towards an elliptical arrangement whose shape and orientation depend on the initial conditions (see Section 6.5).

### 7.2 Questions Arising

The work described in this thesis generates several possible directions for future research. A few of these ideas are briefly described in this section.

Firstly, complete analysis of the weaving behaviour described in Sections 3.5 and 4.8 remains an open problem. Still, the observations made in this thesis offer a point of departure. For instance, it is conjectured that the periodic solution of interest is contained in the subspace $\mathcal{W} \subset \mathbb{R}^{3 n}$ described in Section 3.5.2. Furthermore, (3.26) and Figure 3.11 suggest that it may be possible to reduce the problem from $3 n$ dimensions to only three, like in the local stability analyses of Chapters 3 and 4. Lastly, a similar phenomenon has been noted in the physics literature on coupled oscillators. Researchers in this field have described a periodic solution in coupled oscillator equations that they refer to as the "splay-phase" state, "in which all oscillators have the same waveform, but are shifted by a fixed fraction of a wavelength" (Tsang, Mirollo, Strogatz, and Wiesenfeld, 1991, p. 105). This very same characteristic is apparent in the trajectories of Figure 3.11a, hinting that a connection may exist.

A second direction for future research is the design of multivehicle systems with dynamic gains. In Chapter 3, for every gain $k$ there exist $2(n-1)$ equilibrium formations of fixed radius defined by the value $k>0$. But, not every stable equilibrium formation is desirable (e.g., the $\{4 / 2\}$ formation is stable and has collocated vehicles). On the other hand, in Chapter 4, the only stable formations are those of the appealing form $\{n / 1\}$. However, it is proved that an equilibrium formation exists only when the gain $k=k^{\star}$ (see Theorem 4.1), a property which is not, in theory, robust. What is more, the formation's radius depends on the initial vehicle locations. This begs the question: Is it possible to locally and dynamically choose $k$, so as to stabilize an $\{n / 1\}$-polygon formation of a desired radius?


Figure 7.1: Block diagram for a simple dynamic gain $k_{i}(t)$ compensator

Consider, for example, a desired radius $\bar{\rho}$, which (by simple geometry) corresponds to an equilibrium distance $\bar{r}=2 \bar{\rho} \sin (\pi / n)$ between vehicles at equilibrium. Let $k_{i}(t)$ denote the local gain of vehicle $i$ and define $e_{i}(t):=\bar{r}-r_{i}(t)$. Given the results of Sections 4.5 and 4.6, an intuitive approach would be to set $k_{i}(t)>k^{\star}$ if $e_{i}(t)>0$ (i.e., increase the formation radius), and $k_{i}(t)<k^{\star}$ if $e_{i}(t)<0$ (i.e., shrink the formation radius). A block diagram for this simple decentralized design is given in Figure 7.1. It can be shown that, when the number of vehicles is $n=2$, the analysis is simplified and the linearized system's transfer function from $u_{i}(t)$ to $r_{i}(t)$ in Figure 7.1 is given by

$$
G(s)=\frac{\pi \bar{r}}{s(s+1)}
$$

which is a type 1 system. Thus, proportional control is sufficient to locally drive the error $e_{i}(t)$ to zero. When $n>2$, the problem becomes more complicated. However, some preliminary calculations suggest that, for local formation stabilization, a dynamic compensator $D(s)$ is required. Further to the local stabilization problem, is it possible to design a nonlinear compensator that yields results of a more global nature? Broadly speaking, this idea of designing decentralized dynamic gain controllers seems worthy of further investigation.

Chapter 6 demonstrates that circulant connectivity plays a fundamental role when the task is preserving symmetry in multiagent formations. However, it remains to be determined whether the canonical ordering introduced in Section 6.3.2 is without loss of generality. Do there exist other classes of ordering for which there is symmetry invariance if and only if the system matrix is circulant? Additionally, the obvious next step is to study the stabilization of symmetries. The preliminary investigation of Section 6.5 hints that, for the studied linear system,
rotation group symmetries are not, in general, asymptotically stable. What kind of local feedback is necessary to stabilize a given rotation group symmetry?

Multiagent systems design is often presented as the problem of synthesizing local control strategies that generate desired global behaviours for the system. Instead, the contributions of this thesis emphasize the importance of structure. For example, the stability analyses of Chapters 3 and 4 exploit the circulant structure in cyclic pursuit, thus reducing the problem from one in $3 n$ states to one in only three states. The relevance of structure is again emphasized in Chapter 6 , where symmetric formations are of interest. Naturally, one might wonder: Can structure be exploited towards design? For instance, given a set of fixed agent behaviors, is it possible to control a multiagent system's function (e.g., steadystate and transient behaviours) by switching the agent interconnection topology?

### 7.3 Closing Remarks

It is arguable that the systematic (versus heuristic) study of multivehicle systems is relatively new to engineers, not to mention of recent and growing popularity. In the nearly four years that have passed since the research reported in this thesis was first started, the number of papers appearing in the systems and control literature on this general topic has steadily grown. Today, almost every major systems and control engineering conference has a full track, sometimes multiple tracks, dedicated to multiagent systems, or cooperative control, or some variation thereupon (Spong, 2005). The ability to predict the global outcome of local and distributed coordination algorithms is certainly of importance to engineers charged with the task of designing reliable autonomous agent systems. In this sense, it is hoped that the work of this thesis will serve as a basis for continuing research in this direction, and that the presented ideas and techniques will augment the set of tools available to scientists and engineers studying interconnected systems and problems of coordinated autonomy.

## Bibliography

Abraham, R., and Marsden, J. E. (1977). Foundations of Mechanics (2nd ed.). Cambridge, Massachusetts: Westview Press.

Academy of Motion Picture Arts and Sciences. (2004). The Official Academy Awards Database. Web site: http://www.oscars.org/awardsdatabase/.

Aranda, S., Martínez, S., and Bullo, F. (2005, April). On optimal sensor placement and motion coordination for target tracking. In Proceedings of the IEEE Conference of Robotics and Automation. Barcelona, Spain. (To appear)

Arkin, R. C. (1998). Behavior-based Robotics. Cambridge, Massachusetts: The MIT Press, Inc.

Balch, T. (2003, June). "Emergent" is not a Four-letter Word. (Abstract accompanying the author's talk at the 2003 Block Island Workshop on Cooperative Control, Block Island, Rhode Island)

Balch, T., and Arkin, R. C. (1998, December). Behavior-based formation control for multirobot teams. IEEE Transactions on Robotics and Automation, 14(6), 926-939.

Barfoot, T. D. (2002). Stochastic Decentralized Systems. Ph.D. Thesis, University of Toronto, Toronto, Canada.

Barfoot, T. D. (2003, May). Odometry for Four-wheeled Vehicles (Tech. Rep.). Toronto, Canada: University of Toronto Institute for Aerospace Studies.

Barfoot, T. D., Earon, E. J. P., and D'Eleuterio, G. M. T. (1999, October). Controlling the masses: Control concepts for multi-agent mobile robotics. In Second Canadian Space Exploration Workshop. Calgary, Canada.

Barnett, S. (1983). Polynomials and Linear Control Systems. New York: Marcel Dekker, Inc.

Beard, R. W., Lawton, J., and Hadaegh, F. Y. (2001, November). A coordination
architecture for spacecraft formation control. IEEE Transactions on Control Systems Technology, 9(6), 777-790.

Beard, R. W., and Stepanyan, V. (2003, December). Information consensus in distributed multiple vehicle coordinated control. In Proceedings of the $42 n d$ IEEE Conference on Decision and Control (pp. 2029-2034). Maui, Hawaii.

Behroozi, F., and Gagnon, R. (1979, November). Cylcic pursuit in a plane. Journal of Mathematical Physics, 20(11), 2212-2216.

Belta, C., and Kumar, V. (2004, October). Abstraction and control for groups of robots. IEEE Transactions on Robotics, 20(5), 865-875.

Berman, A., and Plemmons, R. J. (1994). Nonnegative Matrices in the Mathematical Sciences (2nd ed., Vol. 9). Philadephia: Society for Industrial and Applied Mathematics.

Bernhart, A. (1959). Polygons of pursuit. Scripta Mathematica, 24, 23-50.
Boi, S., Couzin, I. D., Buono, N. D., Franks, N. R., and Britton, N. F. (1999). Coupled oscillators and activity waves in ant colonies. Proceedings of the Royal Society: Biological Sciences, 266, 371-378.

Braitenberg, V. (1984). Vehicles: Experiments in Synthetic Psychology. Cambridge, Massachusetts: The MIT Press.

Breder, Jr., C. M. (1954, July). Equations descriptive of fish schools and other animal aggregations. Ecology, 35(3), 361-370.

Brooks, R. A. (1986, March). A robust layered control system for a mobile robot. IEEE Journal of Robotics and Automation, RA-2(1), 14-23.

Bruckstein, A. M. (1993). Why the ant trails look so straight and nice. The Mathematical Intelligencer, 15(2), 59-62.

Bruckstein, A. M., Cohen, N., and Efrat, A. (1991, July). Ants, Crickets and Frogs in Cyclic Pursuit (Center for Intelligent Systems technical report \#9105). Haifa, Israel: Technion-Israel Institute of Technology.

Bruckstein, A. M., Sapiro, G., and Shaked, D. (1995). Evolutions of planar polygons. International Journal of Pattern Recognition and Artificial Intelligence, 9(6), 991-1014.

Camazine, S., Deneubourg, J. L., Franks, N. R., Sneyd, J., Theraulz, G., and

Bonabeau, E. (2001). Self-organization in biological systems. Princeton, New Jersey: Priceton University Press.

Campion, G., Novel, B. d'Andrea, and Bastin, G. (1990, Nobember). Controllability and state feedback stabilizability of nonholonomic mechanical systems. In C. Canudas de Wit (Ed.), Advanced Robot Control: Proceedings of the International Workshop on Nonlinear and Adaptive Control: Issues in Robotics. Grenoble, France.

Cao, Y. U., Fukunaga, A. S., and Kahng, A. B. (1997). Cooperative mobile robotics: Antecedents and directions. Autonomous Robots, 4, 1-23.

Chapra, S. C., and Canale, R. P. (1988). Numerical Methods for Engineers (2nd ed.). New York: McGraw-Hill, Inc.

Cole, B. J. (1991, February). Short-term activity in ants: Generation of periodicity by worker interaction. The American Naturalist, 137(2), 244-259.

Couzin, I. D., Krause, J., James, R., Ruxton, G. D., and Franks, N. R. (2002). Collective memory and spacial sorting in animal groups. Journal of Theoretical Biology, 218(1), 1-11.

Coxeter, H. S. M. (1948). Regular Polytopes. London: Methuen \& Co. Ltd.
Coxeter, H. S. M. (1989). Introduction to Geometry (2nd ed.). Toronto: John Wiley \& Sons, Inc.

Davis, P. J. (1994). Circulant Matrices (2nd ed.). New York: Chelsea Publishing.
Desai, J. P., Ostrowski, J. P., and Kumar, V. (2001, December). Modeling and control of formations of nonholonomic mobile robots. IEEE Transactions on Robotics and Automation, 17(6), 905-908.

Earon, E. J. P., Barfoot, T. D., and D'Eleuterio, G. M. T. (2001, July). Development of a multiagent robotic system with application to space exploration. In 2001 IEEE/ASME International Conference on Advanced Intelligent Mechatronics Proceedings (pp. 1267-1272). Como, Italy.

Egerstedt, M., and Hu, X. (2001, December). Formation constrained multi-agent control. IEEE Transactions on Robotics and Automation, 17(6), 947-951.

Ellis, J. R. (1969). Vehicle Dynamics. London: Business Books Ltd.
Farina, L., and Rinaldi, S. (2000). Positive Linear Systems: Thoery and Applica-
tions. New York: John Wiley \& Sons, Inc.
Fax, J. A., and Murray, R. M. (2004, September). Information flow and cooperative control of vehicle formations. IEEE Transactions on Automatic Control, 49(9), 1465-1476.

Fiedler, M. (1986). Special Matrices and their Applications in Numerical Mathematics. Boston, Massachusetts: Kluwer Boston, Inc.

Fierro, R., Das, A. K., Kumar, V., and Ostrowski, J. P. (2001, May). Hybrid control of formations of robots. In Proceedings of IEEE International Conference on Robotics and Automation (pp. 157-162). Seoul, Korea.

Gazi, V., and Passino, K. M. (2002a, May). Stability analysis of swarms. In Proceedings of the American Control Conference (pp. 1813-1818). Anchorage, Alaska.

Gazi, V., and Passino, K. M. (2002b, May). Stability analysis of swarms in an environment with an attractant/repellent profile. In Proceedings of the American Control Conference (pp. 1819-1824). Anchorage, Alaska.

Grünbaum, D., Viscido, S., and Parrish, J. K. (2004). Extracting interactive control algorithms from group dynamics of schooling fish. In V. Kumar, N. E. Leonard, and A. S. Morse (Eds.), Cooperative Control: A Post-workshop Volume: 2003 Block Island Workshop on Cooperative Control (Vol. 309, pp. 103-117). Spring-Verlag, Inc.

Halmos, P. R. (1958). Finite-dimensional Vector Spaces. Princeton, New Jersey: D. Van Nostrand Company, Inc.

Horn, R. A., and Johnson, C. R. (1985). Matrix Analysis. Cambridge, Massachusetts: Cambridge University Press.

Isidori, A. (1995). Nonlinear Control Systems (3rd ed.). London: Springer-Verlag.
Isidori, A. (1999). Nonlinear Control Systems II. London: Springer-Verlag.
Jadbabaie, A., Lin, J., and Morse, A. S. (2003, June). Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Transactions on Automatic Control, 48(6), 988-1001.

Jadbabaie, A., Motee, N., and Barahona, M. (2004, June). On the stability of the Kuramoto model of coupled nonlinear oscillators. In Proceedings of the 2004

American Control Conference (pp. 4296-4301). Boston, Massachusetts.
Jeanne, J., Leonard, N. E., and Paley, D. (2005, March). Collective motion of ringcoupled planar particles. In Proceedings of the Joint 44 th IEEE Conference on Decision and Control and European Control Conference. Seville, Spain. (To appear)

Jin, Z., Waydo, S., Wildanger, E., Lammers, M., Scholze, H., Foley, P., et al. (2004, July). MVWT-II: The second generation Caltech multi-vehicle wireless testbed. In Proceedings of the 2004 American Control Conference (pp. 5321-5326). Boston, Massachusetts.

Jones, G. A., and Jones, J. M. (1998). Elementary Number Theory. London: Springer-Verlag.

Justh, E. W., and Krishnaprasad, P. S. (2002). A simple control law for UAV formation flying (Technical report No. TR 2002-38). Collge Park, Maryland: Institute for Systems Research.

Justh, E. W., and Krishnaprasad, P. S. (2003, December). Steering laws and continuum models for planar formations. In Proceedings of the 42nd IEEE Conference on Decision and Control (pp. 3609-3614). Maui, Hawaii.

Justh, E. W., and Krishnaprasad, P. S. (2004). Equilibria and steering laws for planar formations. Systems 8 Control Letters, 52, 25-38.

Khalil, H. K. (2002). Nonlinear Systems (3rd ed.). New Jersey: Prentice-Hall.
King, E., Kuwata, Y., Alighanbari, M., Bertucelli, L., and How, J. P. (2004, June). Coordination and control experiments on a multi-vehicle testbed. In Proceedings of the 2004 American Control Conference (pp. 5315-5320). Boston, Massachusetts.

Klamkin, M. S., and Newman, D. J. (1971, June/July). Cyclic pursuit or "The three bugs problem". The American Mathematical Monthly, 78(6), 631-639.

Kolmanovsky, I., and McClamroch, N. H. (1995, December). Developments in nonholonomic control problems. IEEE Control Systems Magazine, 15, 2036.

Kozyreff, G., Vladimirov, A. G., and Mandel, P. (2000, October). Global coupling with time delay in an array of semiconductor lasers. Physical Review Letters,

85(18), 3809-3812.
Kuramoto, Y. (1975). In H. Arakai (Ed.), International Symposium on Mathematical Problems in Theoretical Physics (Vol. 39, p. 420). New York: Springer.

Kuramoto, Y. (1984). Chemical Oscillations, Waves, and Turbulence. Berlin: Springer-Verlag, Inc.

Leonard, N. E., and Fiorelli, E. (2001). Virtual leaders, artificial potentials and coordinated control of groups. In Proceedings of the 40 th IEEE Conference on Decision and Control (pp. 2968-2973). Orlando, Florida.

Lin, C.-T. (1974, June). Structural controllability. IEEE Transactions on Automatic Control, AC-19(3), 201-208.

Lin, Z., Broucke, M. E., and Francis, B. A. (2004, April). Local control strategies for groups of mobile autonomous agents. IEEE Transactions on Automatic Control, 49(4), 622-629.

Lin, Z., Francis, B. A., and Maggiore, M. (2005, January). Necessary and sufficient graphical conditions for formation control of unicycles. IEEE Transactions on Automatic Control, 50(1), 121-127.

Luenberger, D. G. (1979). Introduction to dynamic systems: Theory, models, and applications. New York: John Wiley \& Sons, Inc.

Marshall, J. A., Broucke, M. E., and Francis, B. A. (2003, December). A pursuit strategy for wheeled-vehicle formations. In Proceedings of the $42 n d$ IEEE Conference on Decision and Control (pp. 2555-2560). Maui, Hawaii.

Marshall, J. A., Broucke, M. E., and Francis, B. A. (2004a, November). Formations of vehicles in cyclic pursuit. IEEE Transactions on Automatic Control, 49(11), 1963-1974.

Marshall, J. A., Broucke, M. E., and Francis, B. A. (2004b, June). Unicycles in cyclic pursuit. In Proceedings of the 2004 American Control Conference (pp. 5344-5349). Boston, Massachusetts.

Marshall, J. A., Broucke, M. E., and Francis, B. A. (2005). Pursuit formations of unicycles. Automatica, 41 (12). (To appear)

Marshall, J. A., Fung, T., Broucke, M. E., D'Eleuterio, G. M. T., and Francis, B. A. (2005, June). Experimental validation of mulit-vehicle coordination
strategies. In Proceedings of the 2005 American Control Conference (pp. 1090-1095). Portland, Oregon.

Matarić, M. J. (1995). Issues and approaches in the design of collective autonomous agents. Robotics and Autonomous Systems, 16, 321-331.

McLain, T. W., and Beard, R. W. (2004, June). Unmanned air vehicle testbed for cooperative control experiments. In Proceedings of the 2004 American Control Conference (pp. 5327-5331). Boston, Massachusetts.

Mirza, M. A., Beach, D. M., Earon, E. J. P., and D’Eleuterio, G. M. T. (2004, June). Development of a next-generation autonomous robotic network and experimental testbed. In Proceedings of the Planetary and Terrestrial Mining Sciences Symposium. Sudbury, Canada.

Moreau, L. (2003, December). Leaderless coordination via bidirectional and unidirectional time-dependent communication. In Proceedings of the 42 nd IEEE Conference on Decision and Control (pp. 3070-3075). Maui, Hawaii.

Moreau, L. (2004, December). Stability of continuous-time distributed consensus algorithms. In Proceedings of the 43rd IEEE Conference on Decision and Control (pp. 3998-4003). Atlantis, Paradise Island, Bahamas.

Moreau, L. (2005). Stability of multi-agent systems with time-dependent communication links. IEEE Transactions on Automatic Control. (To appear)

Muratori, S., and Rinaldi, S. (1991). Excitability, stability, and sign of equilibria in positive linear systems. Systems \& Control Letters, 16, 59-63.

Murray, R. M., Li, Z., and Sastry, S. S. (1994). A Mathematical Introduction to Robotic Manipulation. Boca Raton: CRC Press.

Neĭmark, J. I., and Fufaev, N. A. (1972). Dynamics of Nonholonomic Systems. Providence, Rhode Island: American Mathematical Society.

Ögren, P., Fiorelli, E., and Leonard, N. E. (2004, August). Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment. IEEE Transactions on Automatic Control, 49(8), 1292-1302.

Olfati-Saber, R., and Murray, R. M. (2003a, December). Agreement problems in networks with directed graphs and switching toplogy. In Proceedings of the 42nd IEEE Conference on Decision and Control (p. 4126-4132). Maui,

Hawaii.
Olfati-Saber, R., and Murray, R. M. (2003b, December). Flocking with obstacle avoidance: Cooperation with limited communication in mobile networks. In Proceedings of the $42 n d$ IEEE Conference on Decision and Control (pp. 2022-2028). Maui, Hawaii.

Olfati-Saber, R., and Murray, R. M. (2004, September). Consensus problems in networks of agents with switching toplogy and time-delays. IEEE Transactions on Automatic Control, 49(9), 1520-1533.

Paley, D., Leonard, N. E., and Sepulchre, R. (2004, December). Collective motion: Bistability and trajectory tracking. In Proceedings of the 43 rd IEEE Conference on Decision and Control (pp. 1932-1937). Atlantis, Paradise Island, Bahamas.

Parrish, J. K., Viscido, S. V., and Grünbaum, D. (2002, June). Self-organized fish schools: An examination of emergent properties. Biological Bulletin, 202(3), 296-305.

Peterson, I. (2001, July). Art of pursuit. Science News Online, 160 (3).
Piccardi, C., and Rinaldi, S. (2002). Remarks on excitability, stability, and sign of equilibria in cooperative systems. Systems \& Control Letters, 46, 153-163.

Pogromsky, A., Santoboni, G., and Nijmeijer, H. (2002). Partial synchronization: From symmetry towards stability. Physica D, 172, 65-87.

Recht, B., and D'Andrea, R. (2004, September). Distributed control of systems over discrete groups. IEEE Transactions on Automatic Control, 49 (9), 1446-1452.

Ren, W., and Beard, R. W. (2005, May). Consensus seeking in multi-agent systems under dynamically changing interaction topologies. IEEE Transactions on Automatic Control, 50(5), 655-661.

Ren, W., Beard, R. W., and McLain, T. W. (2004). Coordination variables and consensus building in multiple vehicle systems. In V. Kumar, N. E. Leonard, and A. S. Morse (Eds.), Cooperative Control: A Post-workshop Volume: 2003 Block Island Workshop on Cooperative Control (Vol. 309, pp. 171-188). Spring-Verlag, Inc.

Reynolds, C. W. (1987, July). Flocks, herds, and schools: A distributed behavioural model. Computer Graphics, 21(4), 25-34.

Richardson, T. J. (2001a). Non-mutual captures in cyclic pursuit. Annals of Mathematics and Artificial Intelligence, 31, 127-146.

Richardson, T. J. (2001b). Stable polygons of cyclic pursuit. Annals of Mathematics and Artificial Intelligence, 31, 147-172.

Sepulchre, R., Paley, D., and Leonard, N. (2004). Collective motion and oscillator synchronization. In V. Kumar, N. E. Leonard, and A. S. Morse (Eds.), Cooperative Control: A Post-workshop Volume: 2003 Block Island Workshop on Cooperative Control (Vol. 309, pp. 189-205). Spring-Verlag, Inc.

Siljak, D. D. (1991). Decentralized control of complex systems. Boston, Massachussetts: Academic Press.

Sinha, A., and Ghose, D. (2005, June). Generalization of the cyclic pursuit problem. In Proceedings of the 2005 American Control Conference (pp. 4997-5002). Portland, Oregon.

Smith, S. L., Broucke, M. E., and Francis, B. A. (2005). A hierarchical cyclic pursuit scheme for vehicle networks. Automatica, 41 (6), 1045-1053.

Spong, M. W. (2005, April). Relevancy and no-shows. IEEE Control Systems Magazine, 25(2), 8-10.

Strogatz, S. H. (2000). From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled nonlinear oscillators. Physica D, 143, 1-20.

Strogatz, S. H. (2001, March). Exploring complex networks. Nature, 410, 268-276.
Strogatz, S. H. (2003). SYNC: How Order Emerges from Chaos in the Universe, Nature, and Daily Life. New York: Hyperion.

Sugihara, K., and Suzuki, I. (1990). Distributed motion coordination of multiple mobile robots. In Proceedings of the 5th IEEE International Symposium on Intelligent Control (pp. 138-143).

Suzuki, I., and Yamashita, M. (1999). Distributed anonymous mobile robots: Formation of geometric patterns. SIAM Journal of Computing, 28(4), 13471363.

Swaroop, D., and Hedrick, J. K. (1996, March). String stability of interconnected systems. IEEE Transactions on Automatic Control, 41 (3), 349-357.

Tabuada, P., Pappas, G. J., and Lima, P. (2001, June). Feasible formations of multi-agent systems. In Proceedings of the American Control Conference (pp. 56-61). Arlington, Virginia.

Tanner, H. G., Jadbabaie, A., and Pappas, G. J. (2003a, December). Stable flocking of mobile agents, part I: Fixed topology. In Proceedings of the 42 nd IEEE Conference on Decision and Control (pp. 2010-2015). Maui, Hawaii.

Tanner, H. G., Jadbabaie, A., and Pappas, G. J. (2003b, December). Stable flocking of mobile agents, part II: Dynamic topology. In Proceedings of the 42nd IEEE Conference on Decision and Control (pp. 2016-2021). Maui, Hawaii.

Tanner, H. G., Pappas, G. J., and Kumar, V. (2004, June). Leader-to-formation stability. IEEE Transactions on Robotics and Automation, 20(3), 443-455.

Terzopoulous, D., Tu, X., and Grzeszczuk, R. (1994). Artificial fishes: Autonomous locomotion, perception behavior, and learning in a simulated phyiscal world. Artificial Life, 1 (4), 327-351.

Tsang, K. Y., Mirollo, R. E., Strogatz, S. H., and Wiesenfeld, K. (1991). Dynamics of a globally coupled oscillator array. Physica D, 48, 102-112.

Vladimerou, V., Stubbs, A., Rubel, J., Fulford, A., and Dullerud, G. E. (2004, June). A hovercraft testbed for decentrallized and cooperative control. In Proceedings of the 2004 American Control Conference (pp. 5332-5337). Boston, Massachusetts.

Wang, P. K. C. (1989, September). Navigation strategies for multiple autonomous mobile robots moving in formation. In IEEE/RSJ International Workshop on Intelligent Robots and Systems (pp. 486-493). Tsukuba, Japan.

Watanabe, S., and Strogatz, S. H. (1994, July). Constants of motions for superconducting Josephson arrays. Physica D, 74, 197-253.

Watton, A., and Kydon, D. W. (1969). Analytical aspects of the $n$-bug problem. American Journal of Physics, 37, 220-221.

## Appendix A

## Supplementary Proofs

This appendix presents the outcome of algebraic computations deemed to lengthly to be printed in the text of Chapters 3 and 4.

## A. 1 Verification of Coordinates Transformations

## A.1.1 Transformation of Section 3.3.2

In this section, it is shown that the change of coordinates $\varphi=\Phi(\xi)$ given by (3.14) on page 51 is proper by verifying that its Jacobian matrix, evaluated at $\bar{\xi} \in \mathcal{M}$, is nonsingular (cf. Section 2.2.5). Firstly, compute

$$
\left.\frac{\partial \Phi(\xi)}{\partial \xi}\right|_{\bar{\xi}}=\left[\begin{array}{cc}
I_{3 n-3} & 0_{(3 n-3) \times 3} \\
* & G
\end{array}\right]
$$

where $G$ is the $3 \times 3$ block given by

$$
G=\left.\frac{\partial g(\xi)}{\partial \xi_{n}}\right|_{(\bar{r}, \bar{\alpha}, \bar{\beta})}=\left[\begin{array}{ccc}
\sin \left(\alpha_{n}+\gamma_{n}\right) & r_{n} \cos \left(\alpha_{n}+\gamma_{n}\right) & 0 \\
\cos \left(\alpha_{n}+\gamma_{n}\right) & -r_{n} \sin \left(\alpha_{n}+\gamma_{n}\right) & 0 \\
0 & 0 & 1
\end{array}\right]_{\left(\bar{r}, \bar{\alpha}, \bar{\gamma}_{n}\right)}
$$

where

$$
\begin{equation*}
\gamma_{i}:=(i-1) \pi-\sum_{j=1}^{i-1} \beta_{j} . \tag{A.1}
\end{equation*}
$$

Hence, $\Phi$ is a proper change of coordinates since $G$ is nonsingular for every $r_{n}>0$.

## A.1.2 Transformation of Section 4.6

In this section, it is shown that the coordinates transformation $\varphi=\Phi(\xi)$ given by (4.11) on page 98 is proper by verifying that its Jacobian matrix is nonsingular in the domain $\mathcal{D}=\left\{\xi \in \mathbb{R}^{3 n}: \alpha_{i}, \beta_{i} \in \mathbb{R}, r_{i}>0, i=1,2, \ldots, n\right\} \subset \mathbb{R}^{3 n}$. Firstly, it is useful to strategically reorder the new coordinates so that those involving only $\alpha_{i}$ 's and $\beta_{i}$ 's appear as the first $2 n$ coordinates, followed by $\varphi_{1}, \varphi_{4}, \varphi_{7}, \ldots, \varphi_{3 n-2}$. Let this "shuffled" version of the coordinates transformation be denoted $\tilde{\Phi}$. Hence, the Jacobian of $\tilde{\Phi}$ has the block diagonal form

$$
\left.\frac{\partial \tilde{\Phi}(\xi)}{\partial \xi}\right|_{\xi \in \mathcal{D}}=\left[\begin{array}{cc}
I_{2 n} & 0 \\
0 & G
\end{array}\right]
$$

where $G$ is the $n \times n$ matrix

$$
G=\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & r_{3}^{-1} & 0 & 0 & \cdots & 0 \\
0 & -r_{2} r_{3}^{-2} & r_{4}^{-1} & 0 & \cdots & 0 \\
\vdots & & & \ddots & & \\
0 & \cdots & 0 & -r_{n-2} r_{n-1}^{-2} & r_{n}^{-1} & 0 \\
-r_{n} r_{1}^{-2} & 0 & 0 & \cdots & 0 & r_{1}^{-1}
\end{array}\right]
$$

Since $G$ is clearly nonsingular for $\xi \in \mathcal{D}$, so is the Jacobian of $\Phi$.

## A. 2 Proof of Equivalence (A)

In this section, the equivalence (A) used in the proof of Lemma 3.5 on page 52 is derived. A consequence of this result is the identity

$$
\begin{equation*}
\frac{\partial g(\xi)}{\partial \xi} \hat{f}(\xi)=0 \text { for every } \xi \in \mathcal{M} \tag{A.2}
\end{equation*}
$$

which is used to prove the invariance of $\mathcal{M}$ under $\hat{f}$ in Lemma 3.4. Recall that the problem of equivalence (A) is to find an expression for

$$
\begin{equation*}
\frac{\partial g(\xi)}{\partial \xi} \hat{f}(\xi) \tag{A.3}
\end{equation*}
$$

in the new coordinates $\varphi=\Phi(\xi)$. First, rewrite the constraint functions $g(\xi)$ using summation notation. By substituting (A.1) into the constraint equations defined in Section 3.1.3, one obtains

$$
g_{1}(\xi)=\sum_{i=1}^{n} r_{i} \sin \left(\alpha_{i}+\gamma_{i}\right), g_{2}(\xi)=\sum_{i=1}^{n} r_{i} \cos \left(\alpha_{i}+\gamma_{i}\right), g_{3}(\xi)=\sum_{i=1}^{n} \beta_{i}+n \pi .
$$

Next, rewrite (A.3) as

$$
\sum_{i=1}^{n} \frac{\partial g(\xi)}{\partial \xi_{i}} f\left(\xi_{i}, \xi_{i+1}\right) .
$$

That is, separate the problem into $n$ vehicle subsystems. The first constraint has

$$
\frac{\partial g_{1}(\xi)}{\partial \xi_{i}}=\left[\begin{array}{lll}
\sin \left(\alpha_{i}+\gamma_{i}\right) & r_{i} \cos \left(\alpha_{i}+\gamma_{i}\right) & -\sum_{j=i+1}^{n} r_{j} \cos \left(\alpha_{j}+\gamma_{j}\right)
\end{array}\right]
$$

Multiplying this by $f\left(\xi_{i}, \xi_{i+1}\right)$ from (3.5) on page 40 gives

$$
\begin{align*}
& \frac{\partial g_{1}(\xi)}{\partial \xi_{i}} f\left(\xi_{i}, \xi_{i+1}\right)=-\left(\cos \alpha_{i}+\cos \left(\alpha_{i}+\beta_{i}\right)\right) \sin \left(\alpha_{i}+\gamma_{i}\right) \\
& \quad+\left(\sin \alpha_{i}+\sin \left(\alpha_{i}+\beta_{i}\right)\right) \cos \left(\alpha_{i}+\gamma_{i}\right)-k r_{i} \alpha_{i} \cos \left(\alpha_{i}+\gamma_{i}\right) \\
& \quad-k\left(\alpha_{i}-\alpha_{i+1}\right) \sum_{j=i+1}^{n} r_{j} \cos \left(\alpha_{j}+\gamma_{j}\right) \tag{A.4}
\end{align*}
$$

Simplifying the first two terms of (A.4) using trigonometric identities yields

$$
\begin{aligned}
& \left(\sin \alpha_{i}+\sin \left(\alpha_{i}+\beta_{i}\right)\right) \cos \left(\alpha_{i}+\gamma_{i}\right)-\left(\cos \alpha_{i}+\cos \left(\alpha_{i}+\beta_{i}\right)\right) \sin \left(\alpha_{i}+\gamma_{i}\right) \\
& \quad=\sin \left(\beta_{i}-\gamma_{i}\right)-\sin \gamma_{i} .
\end{aligned}
$$

The sum of these $n$ terms is

$$
\begin{align*}
& \sum_{i=1}^{n}\left(\sin \left(\beta_{i}-\gamma_{i}\right)-\sin \gamma_{i}\right) \\
& \quad=\left(\sin \beta_{1}+\sin \left(\beta_{1}+\beta_{2}-\pi\right)+\cdots\right. \\
& \left.\quad \quad \cdots+\sin \left(\beta_{1}+\cdots+\beta_{n}-(n-1) \pi\right)\right) \\
& \quad-\left(0+\sin \left(\pi-\beta_{1}\right)+\sin \left(2 \pi-\beta_{1}-\beta_{2}\right)+\cdots\right. \\
& \left.\quad \cdots+\sin \left((n-1) \pi-\beta_{1}-\cdots-\beta_{n-1}\right)\right) \\
& =\sin \left(\sum_{i=1}^{n} \beta_{i}-(n-1) \pi\right) \\
& = \\
& =\sin \left(g_{3}(\xi)-\pi\right)  \tag{A.5}\\
& = \\
& \quad-\sin \left(g_{3}(\xi)\right)
\end{align*}
$$

Now, look at the remaining two terms of (A.4). Summing these terms gives

$$
\begin{align*}
\sum_{i=1}^{n}( & \left.-k r_{i} \alpha_{i} \cos \left(\alpha_{i}+\gamma_{i}\right)-k\left(\alpha_{i}-\alpha_{i+1}\right) \sum_{j=i+1}^{n} r_{j} \cos \left(\alpha_{j}+\gamma_{j}\right)\right) \\
= & -k \sum_{i=1}^{n} r_{i} \alpha_{i} \cos \left(\alpha_{i}+\gamma_{i}\right)-k\left(\alpha_{1} \sum_{j=2}^{n} r_{j} \cos \left(\alpha_{j}+\gamma_{j}\right)\right. \\
& \left.+\alpha_{2} \sum_{j=3}^{n} r_{j} \cos \left(\alpha_{j}+\gamma_{j}\right)+\cdots+\alpha_{n-1} r_{n} \cos \left(\alpha_{n}+\gamma_{n}\right)\right) \\
& +k\left(\alpha_{2} \sum_{j=2}^{n} r_{j} \cos \left(\alpha_{j}+\gamma_{j}\right)+\cdots+\alpha_{n} r_{n} \cos \left(\alpha_{n}+\gamma_{n}\right)\right) \\
= & -k \sum_{i=1}^{n} r_{i} \alpha_{i} \cos \left(\alpha_{i}+\gamma_{i}\right)-k \alpha_{1} \sum_{j=2}^{n} r_{j} \cos \left(\alpha_{j}+\gamma_{j}\right) \\
& +k \sum_{j=2}^{n} \alpha_{j} r_{j} \cos \left(\alpha_{j}+\gamma_{j}\right) \\
= & -k \alpha_{1} \sum_{i=1}^{n} r_{i} \cos \left(\alpha_{i}+\gamma_{i}\right) \\
= & -k \alpha_{1} g_{2}(\xi) . \tag{A.6}
\end{align*}
$$

Combining equations (A.5) and (A.6) yields

$$
\begin{equation*}
\frac{\partial g_{1}(\xi)}{\partial \xi} \hat{f}(\xi)=-k \alpha_{1} g_{2}(\xi)-\sin \left(g_{3}(\xi)\right) \tag{A.7}
\end{equation*}
$$

Repeat this process for the second constraint, which has

$$
\frac{\partial g_{2}(\xi)}{\partial \xi_{i}}=\left[\begin{array}{lll}
\cos \left(\alpha_{i}+\gamma_{i}\right) & -r_{i} \sin \left(\alpha_{i}+\gamma_{i}\right) & \sum_{j=i+1}^{n} r_{j} \sin \left(\alpha_{j}+\gamma_{j}\right)
\end{array}\right]
$$

Multiplying this by $f\left(\xi_{i}, \xi_{i+1}\right)$ gives

$$
\begin{align*}
& \frac{\partial g_{2}(\xi)}{\partial \xi_{i}} f\left(\xi_{i}, \xi_{i+1}\right)=-\left(\cos \alpha_{i}+\cos \left(\alpha_{i}+\beta_{i}\right)\right) \cos \left(\alpha_{i}+\gamma_{i}\right) \\
& \quad-\left(\sin \alpha_{i}+\sin \left(\alpha_{i}+\beta_{i}\right)\right) \sin \left(\alpha_{i}+\gamma_{i}\right)+k r_{i} \alpha_{i} \sin \left(\alpha_{i}+\gamma_{i}\right) \\
& \quad+k\left(\alpha_{i}-\alpha_{i+1}\right) \sum_{j=i+1}^{n} r_{j} \sin \left(\alpha_{j}+\gamma_{j}\right) . \tag{A.8}
\end{align*}
$$

Simplifying the first two terms of (A.8) using trigonometric identities yields

$$
\begin{aligned}
& -\left(\cos \alpha_{i}+\cos \left(\alpha_{i}+\beta_{i}\right)\right) \cos \left(\alpha_{i}+\gamma_{i}\right)-\left(\sin \alpha_{i}+\sin \left(\alpha_{i}+\beta_{i}\right)\right) \sin \left(\alpha_{i}+\gamma_{i}\right) \\
& \quad=-\cos \left(\beta_{i}-\gamma_{i}\right)-\cos \gamma_{i} .
\end{aligned}
$$

Summing $n$ of these terms gives

$$
\begin{align*}
\sum_{i=1}^{n}( & \left.-\cos \left(\beta_{i}-\gamma_{i}\right)-\cos \gamma_{i}\right) \\
= & -\left(\cos \beta_{1}+\cos \left(\beta_{1}+\beta_{2}-\pi\right)+\cdots\right. \\
& \left.\cdots+\cos \left(\beta_{1}+\cdots+\beta_{n}-(n-1) \pi\right)\right) \\
& -\left(1+\cos \left(\pi-\beta_{1}\right)+\cos \left(2 \pi-\beta_{1}-\beta_{2}\right)+\cdots\right. \\
& \left.\cdots+\cos \left((n-1) \pi-\beta_{1}-\cdots-\beta_{n-1}\right)\right) \\
= & -\cos \left(\sum_{i=1}^{n} \beta_{i}-(n-1) \pi\right)-1 \\
= & -\cos \left(g_{3}(\xi)-\pi\right)-1 \\
= & \cos \left(g_{3}(\xi)\right)-1 . \tag{A.9}
\end{align*}
$$

To obtain the remaining terms of (A.8), carry out a computation that exactly parallels the one resulting in (A.6), yielding

$$
\sum_{i=1}^{n}\left(k r_{i} \alpha_{i} \sin \left(\alpha_{i}+\gamma_{i}\right)+k\left(\alpha_{i}-\alpha_{i+1}\right) \sum_{j=i+1}^{n} r_{j} \sin \left(\alpha_{j}+\gamma_{j}\right)\right)=k \alpha_{1} g_{1}(\xi)
$$

which, together with (A.9), gives

$$
\begin{equation*}
\frac{\partial g_{2}(\xi)}{\partial \xi} \hat{f}(\xi)=k \alpha_{1} g_{1}(\xi)+\cos \left(g_{3}(\xi)\right)-1 \tag{A.10}
\end{equation*}
$$

The last constraint has $\partial g_{3}(\xi) / \partial \xi_{i}=\left[\begin{array}{lll}0 & 0 & 1\end{array}\right]$. Multiplying by $f\left(\xi_{i}, \xi_{i+1}\right)$ yields

$$
\frac{\partial g_{3}(\xi)}{\partial \xi} f\left(\xi_{i}, \xi_{i+1}\right)=k\left(\alpha_{i}-\alpha_{i+1}\right)
$$

such that

$$
\begin{equation*}
\frac{\partial g_{3}(\xi)}{\partial \xi} \hat{f}(\xi)=\sum_{i=1}^{n} k\left(\alpha_{i}+\alpha_{i+1}\right)=0 \tag{A.11}
\end{equation*}
$$

since the indices $i+1$ are taken modulo $n$.
The equations (A.7), (A.10), and (A.11) together provide the equivalence (A) used in the proof of Lemma 3.5. Furthermore, notice that the identity (A.2) is easily verified by choosing $g(\xi)=0$, or equivalently $\xi \in \mathcal{M}$.

## A. 3 Proof of Equivalence (B)

Parallel to the proof of equivalence (A) in Appendix A.2, in this section the equivalence (B) used in the proof of Lemma 4.2 on page 88 is derived. A consequence of this result is the identity (A.2), which is used to prove the invariance of $\mathcal{M}$ under $\hat{f}$ in Lemma 4.1. Let $\gamma_{i}$ be defined as in (A.1) and compute

$$
\begin{align*}
& \frac{\partial g_{1}(\xi)}{\partial \xi_{i}} f\left(\xi_{i}, \xi_{i+1}\right)=-k\left(r_{i} \cos \alpha_{i}+r_{i+1} \cos \left(\alpha_{i}+\beta_{i}\right)\right) \sin \left(\alpha_{i}+\gamma_{i}\right) \\
& \quad+k\left(r_{i} \sin \alpha_{i}+r_{i+1} \sin \left(\alpha_{i}+\beta_{i}\right)\right) \cos \left(\alpha_{i}+\gamma_{i}\right) \\
& \quad-r_{i} \alpha_{i} \cos \left(\alpha_{i}+\gamma_{i}\right)-\left(\alpha_{i}-\alpha_{i+1}\right) \sum_{j=i+1}^{n} r_{j} \cos \left(\alpha_{j}+\gamma_{j}\right) . \tag{A.12}
\end{align*}
$$

Simplifying the first two terms of (A.12) using trigonometric identities yields

$$
k r_{i+1} \sin \left(\beta_{i}-\gamma_{i}\right)-k r_{i} \sin \gamma_{i} .
$$

Summing $n$ of these terms gives

$$
\begin{aligned}
& k \sum_{i=1}^{n}\left(r_{i+1} \sin \left(\beta_{i}-\gamma_{i}\right)-r_{i} \sin \gamma_{i}\right) \\
& \quad=k\left(r_{2} \sin \beta_{1}+r_{3} \sin \left(\beta_{1}+\beta_{2}-\pi\right)+\cdots\right. \\
& \left.\quad \cdots+r_{1} \sin \left(\beta_{1}+\cdots+\beta_{n}-(n-1) \pi\right)\right) \\
& \quad-k\left(0+r_{2} \sin \left(\pi-\beta_{1}\right)+r_{3} \sin \left(2 \pi-\beta_{1}-\beta_{2}\right)+\cdots\right. \\
& \left.\quad \cdots+r_{n} \sin \left((n-1) \pi-\beta_{1}-\cdots-\beta_{n-1}\right)\right) \\
& = \\
& =k r_{1} \sin \left(\sum_{i=1}^{n} \beta_{i}-(n-1) \pi\right) \\
& = \\
& =k r_{1} \sin \left(g_{3}(\xi)-\pi\right) \\
& =
\end{aligned} k_{1} \sin \left(g_{3}(\xi)\right) . .
$$

Following Appendix A.2, the remaining terms of (A.12) sum to $-\alpha_{1} g_{2}(\xi)$. Together with this fact, the above results yield

$$
\begin{equation*}
\frac{\partial g_{1}(\xi)}{\partial \xi} \hat{f}(\xi)=-\alpha_{1} g_{2}(\xi)-k r_{1} \sin \left(g_{3}(\xi)\right) \tag{A.13}
\end{equation*}
$$

Repeat this process for the second constraint, giving

$$
\begin{align*}
& \frac{\partial g_{2}(\xi)}{\partial \xi_{i}} f\left(\xi_{i}, \xi_{i+1}\right)=-k\left(r_{i} \cos \alpha_{i}+r_{i+1} \cos \left(\alpha_{i}+\beta_{i}\right)\right) \cos \left(\alpha_{i}+\gamma_{i}\right) \\
& \quad-k\left(r_{i} \sin \alpha_{i}+r_{i+1} \sin \left(\alpha_{i}+\beta_{i}\right)\right) \sin \left(\alpha_{i}+\gamma_{i}\right) \\
& \quad+r_{i} \alpha_{i} \sin \left(\alpha_{i}+\gamma_{i}\right)+\left(\alpha_{i}-\alpha_{i+1}\right) \sum_{j=i+1}^{n} r_{j} \sin \left(\alpha_{j}+\gamma_{j}\right) \tag{A.14}
\end{align*}
$$

Simplifying the first two terms of (A.14) using trigonometric identities yields

$$
-k r_{i+1} \cos \left(\beta_{i}-\gamma_{i}\right)-k r_{i} \cos \gamma_{i}
$$

Summing $n$ of these terms gives

$$
\begin{aligned}
k \sum_{i=1}^{n}( & \left.-\cos \left(\beta_{i}-\gamma_{i}\right)-\cos \gamma_{i}\right) \\
= & -k\left(r_{2} \cos \beta_{1}+r_{3} \cos \left(\beta_{1}+\beta_{2}-\pi\right)+\cdots\right. \\
& \left.\cdots+r_{1} \cos \left(\beta_{1}+\cdots+\beta_{n}-(n-1) \pi\right)\right) \\
\quad & -k\left(r_{1}+r_{2} \cos \left(\pi-\beta_{1}\right)+r_{3} \cos \left(2 \pi-\beta_{1}-\beta_{2}\right)+\cdots\right. \\
& \left.\cdots+r_{n} \cos \left((n-1) \pi-\beta_{1}-\cdots-\beta_{n-1}\right)\right) \\
= & -k r_{1} \cos \left(\sum_{i=1}^{n} \beta_{i}-(n-1) \pi\right)-k r_{1} \\
= & -k r_{1} \cos \left(g_{3}(\xi)-\pi\right)-k r_{1} \\
= & k r_{1} \cos \left(g_{3}(\xi)\right)-k r_{1} .
\end{aligned}
$$

Following Appendix A.2, the remaining terms of (A.14) sum to $\alpha_{1} g_{1}(\xi)$. Together with this fact, the above results yield

$$
\begin{equation*}
\frac{\partial g_{2}(\xi)}{\partial \xi} \hat{f}(\xi)=\alpha_{1} g_{1}(\xi)+k r_{1} \cos \left(g_{3}(\xi)\right)-k r_{1} \tag{A.15}
\end{equation*}
$$

Finally, the last constraint has

$$
\begin{equation*}
\frac{\partial g_{3}(\xi)}{\partial \xi} \hat{f}(\xi)=\sum_{i=1}^{n}\left(\alpha_{i}+\alpha_{i+1}\right)=0 \tag{A.16}
\end{equation*}
$$

since the indices $i+1$ are taken modulo $n$.
The equations (A.13), (A.15), and (A.16) together provide the equivalence (B) used in the proof of Lemma 4.2. Furthermore, notice that the identity (A.2) is easily verified by choosing $g(\xi)=0$, or equivalently $\xi \in \mathcal{M}$.

## Appendix B

## Modelling of the Argo Rovers

This appendix is intended to supplement Section 5.3 .1 of Chapter 5 and, in particular, detail the developed Argo Rover models (5.1) and (5.2). For simplicity's sake, it is assumed that each wheel of the body-centred-axis rover model of Figure 5.3 rolls without laterally slipping. Although this assumption is not a perfectly realistic description of the situation, neither is it altogether unreasonable given the near-ideal conditions of the UTIAS laboratory floor. This assumption manifests itself as two independent nonholonomic constraints on the vehicle's motion.

## B. 1 Nonholonomic Systems Background

This section offers, as background, a differential geometric approach to modelling kinematic systems with nonholonomic motion constraints. The purpose here is to provide only a brief overview of this (very rich) topic. The interested reader is referred to Kolmanovsky and McClamroch (1995) and references therein.

Let $\mathcal{M}$ be an $n$-dimensional manifold that describes the set of all possible configurations for a mechanical system. For the purposes of this thesis, the system's configuration is represented in local coordinates, which live in an open set $Q \subset \mathbb{R}^{n}$. Recall that when there are geometric constraints on the motion of a system there exists a $k$-dimensional (with $k<n$ ) vector function $c(q): Q \rightarrow \mathbb{R}^{k}$ such that $c(q)=0$ for all $q \in Q$. These kinds of constraints imply that $n-k$ coordinates are sufficient to describe all possible configurations of the system (Campion, Novel, and Bastin, 1990). Recall that kinematic constraints on the motion of a system are constraints imposed on the coordinate velocities and are often of the form

$$
\begin{equation*}
\omega_{i}(q)=0, \tag{B.1}
\end{equation*}
$$

where $q \in Q$ and $\omega_{1}(q), \omega_{2}(q), \ldots, \omega_{k}(q)$ are smooth $n$-dimensional covector fields or one-forms on the configuration space $Q$. Constraints of this form are sometimes called Pfaffian constraints (Murray, Li, and Sastry, 1994). These $k$ constraints can also be written in matrix form such that $A^{\top}(q) \dot{q}=0$, where $A(q)$ has dimension $n \times k$ and columns $\omega_{1}^{\top}(q), \omega_{2}^{\top}(q), \ldots, \omega_{k}^{\top}(q)$.

If a constraint is integrable, then there exists a function $h_{i}: Q \rightarrow \mathbb{R}$ such that $h_{i}(q)=0$ is equivalent to $\omega_{i}(q) \dot{q}=0$. If a particular constraint is integrable it is said to be a holonomic constraint and the constraint causes a reduction in the dimension of the configuration space (i.e., similar to a geometric constraint). Otherwise, if the constraint is not integrable then it is said to be nonholonomic. Further details about integrability can be found in Murray et al. (1994).

## B.1.1 Modelling Kinematic Systems

At this point, an example is useful.
Example B.1: Consider the kinematic unicycle (a single rolling wheel), illustrated in Figure 2.5 on page 35. It is assumed that the unicycle's wheel cannot laterally slip. This no-slipping assumption results in a kinematic constraint of the form

$$
\begin{equation*}
\dot{x} \sin \theta-\dot{y} \cos \theta=0, \tag{B.2}
\end{equation*}
$$

where $q=(x, y, \theta) \in Q$ is the vector of system coordinates. In vector form, the constraint (B.2) has the form $A^{\top}(q) \dot{q}=0$ where

$$
A(q)=\left[\begin{array}{lll}
\sin \theta & -\cos \theta & 0
\end{array}\right]^{\top}
$$

As it turns out, this constraint is not integrable and is thus nonholonomic.
One can assume that all redundant coordinates of the system due to geometric constraints have been eliminated and that only independent kinematic constraints of the form (B.1) remain. These constraints are said to be independent if $A(q)$ has full rank $k$ for all $q \in Q$. The set of all possible tangent vectors to $Q$ at any point $q \in Q$ is called the tangent space and is denoted $T_{q} Q$. The tangent bundle of $Q$ is given by the union

$$
T Q=\bigcup_{q \in Q} T_{q} Q
$$

The cotangent space and cotangent bundle are denoted $T_{q}^{\star} Q$ and $T^{\star} Q$, respectively. Recall that a distribution assigns a subspace of the tangent space to each point $q \in Q$ in a smooth way. Let the annihilator of the codistribution

$$
\Omega(q)=\operatorname{span}\left\{\omega_{1}(q), \omega_{2}(q), \ldots, \omega_{k}(q)\right\} \subset T_{q}^{\star} Q
$$

defined by the kinematic constraints be denoted $\Delta \subset T Q$, where for each $q \in Q$

$$
\Delta(q)=\operatorname{span}\left\{g_{1}(q), g_{2}(q), \ldots, g_{n-k}(q)\right\} \subset T_{q} Q
$$

is $(n-k)$-dimensional, smooth, and $\Delta=\Omega^{\perp}$. Thus, if $m=n-k$, then the allowable trajectories of the system can be written as solutions of

$$
\begin{equation*}
\dot{q}=g_{1}(q) v_{1}+g_{2}(q) v_{2}+\cdots+g_{m}(q) v_{m} \tag{B.3}
\end{equation*}
$$

where $v(t)=\left(v_{1}(t), v_{2}(t), \ldots, v_{m}(t)\right) \in \mathbb{R}^{m}$ is the vector of control inputs. The system (B.3) can be written in matrix form $\dot{q}=G(q) v$, where $G(q)$ is $n \times m$ and has full rank $m$ since the constraints were assumed independent. Thus, it is clear that the allowable trajectories have $\dot{q} \in \operatorname{Img}(G(q))=\operatorname{Ker}\left(A^{\top}(q)\right)$.

Example B.2: Consider again the unicycle of Example B.1. The distribution $\Delta \subset T Q$ that annihilates the codistribution $\Omega \subset T^{\star} Q$ spanned by the columns of $A(q)$ (there is only one) is, for each $q \in Q$,

$$
\Delta(q)=\operatorname{span}\left\{g_{1}(q):=\left[\begin{array}{c}
\cos \theta \\
\sin \theta \\
0
\end{array}\right], g_{2}(q):=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]\right\} .
$$

This leaves one with the well-known kinematic unicycle model

$$
\left[\begin{array}{c}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{array}\right]=\left[\begin{array}{c}
\cos \theta \\
\sin \theta \\
0
\end{array}\right] v_{1}+\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right] v_{2},
$$

where $v_{1}, v_{2} \in \mathbb{R}$ are clearly linear and angular speed control inputs.
Thus the kinematic modelling approach taken in this appendix is to first write the kinematic constraints acting on the system and subsequently compute the admissible coordinate velocities via the kernel of $A^{\top}(q)$.

## B.1.2 Modelling Dynamic Systems

This section presents a Lagrangian approach to modelling dynamic systems subject to first-order (i.e., kinematic) nonholonomic constraints (Neĭmark and Fufaev, 1972; Campion et al., 1990). The Lagrangian method of dynamics is particularly useful for a number of reasons. Firstly, the technique essentially reduces the entire study of rigid body mechanics to a single procedure. Secondly, Lagrange's equation is valid in any set of coordinates that are suitable for describing the system's configuration. Finally, the approach is based on the scalar quantities: kinetic energy, potential energy, and virtual work.

Recall that $q=\left(q_{1}, q_{2}, \ldots, q_{n}\right) \in Q \subset \mathbb{R}^{n}$ denotes the vector of generalized coordinates that describe the system's configuration. Lagrange's equation (for systems without nonholonomic constraints) has the form

$$
\frac{d}{d t}\left(\frac{\partial T}{\partial \dot{q}_{i}}\right)-\frac{\partial T}{\partial q_{i}}+\frac{\partial V}{\partial q_{i}}=\tau_{i}(t, q)
$$

where $T(q, \dot{q})$ is the total kinetic energy of the system, $V(q)$ is the total potential energy of the system, $\tau_{i}(t, q)$ is the generalized force corresponding to a variation of the coordinate $q_{i}$ (e.g., an external or nonconservative force), and $i=1,2, \ldots, n$ (i.e., there are $n$ equations of motion; one for each degree of freedom or coordinate). Clearly, if the system is unforced then $\tau_{i}(t, q) \equiv 0$ for all $i \in\{1,2, \ldots, n\}$. Note that kinetic energy of a mechanical system can be defined as

$$
T(q, \dot{q})=\frac{1}{2} \dot{q}^{\top} M(q) \dot{q},
$$

where $M(q)$ is the $n \times n$ positive-definite symmetric inertia matrix. The kinetic and potential energies are typically combined into a single scalar called the Lagrangian, defined as $L=T-V$. Since the potential energy is not a function of the coordinate velocities $\dot{q}_{i}$, Lagrange's equation is most commonly written as

$$
\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{q}_{i}}\right)-\frac{\partial L}{\partial q_{i}}=\tau_{i}(t, q)
$$

Suppose that any redundant coordinates have been eliminated and that only independent kinematic constraints of the form (B.1) remain. In this case, Lagrange's equation can be extended to nonholonomic systems (see Neĭmark and Fufaev, 1972 for details) through a technique utilizing Lagrange multipliers. The
result is a set of equations of motion given by

$$
\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{q}_{i}}\right)-\frac{\partial L}{\partial q_{i}}=\tau_{i}(t, q)+\sum_{j=1}^{k} a_{i j}(q) \lambda_{j}
$$

where the coefficients $a_{i j}(q)$ are the elements of the $n \times k$ constraint matrix $A(q)$ and $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}\right)$ is the vector of Lagrange multipliers. Alternatively, the above Lagrange equations can be written in matrix form, yielding

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{\partial^{\top} L}{\partial \dot{q}}\right)-\frac{\partial^{\top} L}{\partial q}=\tau(t, q)+A(q) \lambda \tag{B.4}
\end{equation*}
$$

where $\tau=\left(\tau_{1}, \tau_{2}, \ldots, \tau_{n}\right)$. Thus, (B.4) and the constraint equation $A^{\top}(q) \dot{q}=0$, together, completely describe the nonholonomic system's dynamics.

Example B.3: Consider a unicycle, as in Examples B. 1 and B.2, actuated by a throttle force $f(t)$ acting in the direction of its wheel (like tractive effort) and a steering torque $\tau(t)$ at the wheel's axis $(x, y)$ in Figure 2.5. Suppose the unicycle has mass $m$, moment of inertia $I$ about the point $(x, y)$, and that one ignores the inertial effects due to the rotation of the wheel. In this case, for $q=(x, y, \theta)$,

$$
\begin{aligned}
T(q, \dot{q}) & =\frac{1}{2} m\left(\dot{x}^{2}+\dot{y}^{2}\right)+\frac{1}{2} I \dot{\theta}^{2} \\
V(q) & =0 .
\end{aligned}
$$

Recall, from Example B.1, that the constraint matrix is given by

$$
A(q)=\left[\begin{array}{lll}
\sin \theta & -\cos \theta & 0
\end{array}\right]^{\top}
$$

Applying Lagrange's equation (B.4) yields

$$
\begin{aligned}
m \ddot{x} & =f(t) \cos \theta+\lambda \sin \theta \\
m \ddot{y} & =f(t) \sin \theta-\lambda \cos \theta \\
I \ddot{\theta} & =\tau(t),
\end{aligned}
$$

or in matrix form

$$
\left[\begin{array}{ccc}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & I
\end{array}\right] \ddot{q}=\left[\begin{array}{cc}
\cos \theta & 0 \\
\sin \theta & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{c}
f(t) \\
\tau(t)
\end{array}\right]+\left[\begin{array}{c}
\sin \theta \\
-\cos \theta \\
0
\end{array}\right] \lambda,
$$

which together with $A^{\top}(q) \dot{q}=0$, completely describes the system's dynamics.

What does the vector of Lagrange multipliers $\lambda$ physically represent? These are the constraint forces required to maintain the imposed nonholonomic constraints. In the above example, it is easy to see that the Lagrange multiplier $\lambda$ is a force that always acts in a direction perpendicular to the wheel's direction of motion, keeping the wheel from laterally slipping.

As was illustrated in the above example, one often sees the Lagrange equations (B.4) rewritten in the standard matrix form

$$
\begin{equation*}
M(q) \ddot{q}+f(q, \dot{q})=\tau(t, q)+A(q) \lambda . \tag{B.5}
\end{equation*}
$$

One of the problems with this representation, and that of (B.4), is the presence of the vector of Lagrange multipliers $\lambda$. However, recall that by definition $G^{\top}(q) A(q)=0$, since the distribution $\Delta=\Omega^{\perp}$, where $G(q)$ is defined by the set of vector fields $\left\{g_{1}, g_{2}, \ldots, g_{m}\right\}$ that span $\Delta$. Thus, to eliminate the Lagrange multipliers, one can premultiply (B.5) by $G^{\top}(q)$, yielding $n-m$ equations

$$
\begin{equation*}
G^{\top}(q) M(q) \ddot{q}+G^{\top}(q) f(q, \dot{q})=G^{\top}(q) \tau(t, q) . \tag{B.6}
\end{equation*}
$$

Moreover, recall that the constraints $A^{\top}(q) \dot{q}=0$ imply that the coordinate velocities must satisfy $\dot{q} \in \operatorname{Img}(G(q))$ or, equivalently, $\dot{q}=G(q) v$ for some $v \in \mathbb{R}^{m}$. Differentiating this relationship gives

$$
\ddot{q}=G(q) \dot{v}+\sum_{i=1}^{n} \frac{\partial G(q)}{\partial q_{i}} \dot{q}_{i} v=G(q) \dot{v}+H(q, \dot{q}) v,
$$

which can be substituted into (B.6) with $\dot{q}=G(q) v$ to yield

$$
\begin{equation*}
G^{\top}(q) M(q)(G(q) \dot{v}+H(q, G(q) v) v)+G^{\top}(q) f(q, G(q) v)=G^{\top}(q) \tau(t, q) \tag{B.7}
\end{equation*}
$$

In other words, the $n-m$ equations (B.7) together with the $m$ equations $\dot{q}=G(q) v$ completely describe the system's dynamics.

Example B.4: This example continues where Example B. 3 finished. To eliminate
the Lagrange multiplier $\lambda$ one premultiplies by $G^{\top}(q)$, which gives

$$
\left[\begin{array}{ccc}
m \cos \theta & m \sin \theta & 0  \tag{B.8}\\
0 & 0 & I
\end{array}\right] \ddot{q}=\left[\begin{array}{c}
f(t) \\
\tau(t)
\end{array}\right]
$$

In place of $\ddot{q}$, substitute

$$
\ddot{q}=G(q) \dot{v}+\left[\begin{array}{cc}
-\sin \theta & 0 \\
\cos \theta & 0 \\
0 & 0
\end{array}\right] \dot{\theta} v
$$

which, when substituted into (B.8), yields

$$
\left[\begin{array}{cc}
m & 0 \\
0 & I
\end{array}\right] \dot{v}=\left[\begin{array}{l}
f(t) \\
\tau(t)
\end{array}\right]
$$

In other words, the dynamical equations of motion are

$$
\begin{aligned}
& \dot{q}=G(q) v \\
& \dot{v}=\left[\begin{array}{cc}
1 / m & 0 \\
0 & 1 / I
\end{array}\right]\left[\begin{array}{l}
f(t) \\
\tau(t)
\end{array}\right],
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
\dot{x} & =v_{1} \cos \theta \\
\dot{y} & =v_{1} \sin \theta \\
\dot{\theta} & =v_{2} \\
\dot{v_{1}} & =f(t) / m \\
\dot{v_{2}} & =\tau(t) / I,
\end{aligned}
$$

which is simply the kinematic model but extended using Newton's second law.

There exist more general results than the ones given above. For example, it can be shown that such systems can always be represented, via an appropriate feedback linearizing control law, by its kinematic model extended by a system of integrators (Campion et al., 1990; Kolmanovsky and McClamroch, 1995).

## B. 2 Kinematic Rover Model

Following the procedure outlined in Section B.1.1, the kinematic rover model (5.1) of Chapter 5 is now derived. Recall that the rover has generalized coordinates $q=(x, y, \theta, \phi) \in Q$. Since it has been assumed that the body-centred-axis model (see Figure 5.3 on page 111) rolls without laterally slipping, the vehicle's motion is kinematically constrained at each point $p_{f}$ and $p_{r}$. In fact, these are the same constraints (B.2) given for the unicycle (so, they are not integrable); namely

$$
\begin{align*}
\dot{x}_{f} \sin \left(\theta+\phi_{f}\right)-\dot{y}_{f} \cos \left(\theta+\phi_{f}\right) & =0  \tag{B.9}\\
\dot{x}_{r} \sin \left(\theta+\phi_{r}\right)-\dot{y}_{r} \cos \left(\theta+\phi_{r}\right) & =0 .
\end{align*}
$$

Therefore, by simple geometry, on obtains

$$
\begin{aligned}
x_{f} & =x+l \cos \theta \\
y_{f} & =y+l \sin \theta \\
x_{r} & =x-l \cos \theta \\
y_{f} & =y-l \sin \theta .
\end{aligned}
$$

Combining these with the fact that $\phi=\phi_{f}=-\phi_{r}$ gives, from (B.9), constraints that may be written in the form $A^{\top}(q) \dot{q}=0$, where

$$
A(q)=\left[\begin{array}{cccc}
\sin (\theta+\phi) & -\cos (\theta+\phi) & -l \cos \phi & 0 \\
\sin (\theta-\phi) & -\cos (\theta-\phi) & l \cos \phi & 0
\end{array}\right]^{\top} .
$$

Following the approach previously outlined, the admissible coordinate velocities are contained in the kernel of $A^{\top}(q)$. However, one actually has some choice about how to assign the distribution $\Delta$ that annihilates $\Omega$. In other words, one can choose our inputs $v_{1}$ and $v_{2}$ to suit the actual vehicle inputs. For example, suppose one views the vehicle as being driven by a velocity input $\tilde{v}_{1}$ acting at point $p=(x, y)$ in the direction of $\theta$ and steered by an angular steering velocity input $\tilde{v}_{2}$. Then, one obtains the kinematic rover model

$$
\left[\begin{array}{c}
\dot{x}  \tag{B.10}\\
\dot{y} \\
\dot{\theta} \\
\dot{\phi}
\end{array}\right]=\left[\begin{array}{c}
\cos \theta \\
\sin \theta \\
\frac{1}{l} \tan \phi \\
0
\end{array}\right] \tilde{v}_{1}+\left[\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right] \tilde{v}_{2} .
$$

Notice that there is a singularity in the model at $\phi= \pm \pi / 2$, where $g_{1}(q)$ has a discontinuity. This corresponds to the case when the driving velocity input at $p$, acting in the direction of $\theta$, is normal to the direction of the wheels. However, the actual steering range for the rovers is likely such that $|\phi|<\pi / 2$.

Fortunately, the real rovers are actually driven at their wheels, not at the centre point $p$ of the model. In this case, a more realistic input $v_{1}$ acts in the direction of the front wheel (or real wheel - it does not matter). In other words, $\tilde{v}_{1}=v_{1} \cos \phi$ and $\tilde{v}_{2}=v_{2}$, where $\tilde{v}_{1}$ and $\tilde{v}_{2}$ are as in (B.10). This gives

$$
\left[\begin{array}{c}
\dot{x}  \tag{B.11}\\
\dot{y} \\
\dot{\theta} \\
\dot{\phi}
\end{array}\right]=\left[\begin{array}{c}
\cos \phi \cos \theta \\
\cos \phi \sin \theta \\
\frac{1}{l} \sin \phi \\
0
\end{array}\right] v_{1}+\left[\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right] v_{2} .
$$

Notice that the singularity in (B.10) does not occur in (B.11) since the rover could, in theory, pivot about the point $p$ by actuating its wheels when $\phi= \pm \pi / 2$.

Another point worth noting is that the instantaneous radius of curvature $\rho$ of the trajectory in the plane traced out by the points $p_{f}$ and $p_{r}$ is given by $\rho=l / \sin \phi$, as can be seen in Figure 5.3. Therefore, specifying the steering angle $\phi$ is equivalent to prescribing a radius of curvature for the rover's motion.

## B. 3 Dynamic Rover Model

Following the procedure outlined in Section B.1.2, the dynamic rover model (5.2) of Chapter 5 is now derived. Ignoring friction and inertial effects due to rotation of the wheels, kinetic and potential energy expressions for the rover are

$$
\begin{align*}
T(q, \dot{q}) & =\frac{1}{2} m\left(\dot{x}^{2}+\dot{y}^{2}\right)+\frac{1}{2} I_{p} \dot{\theta}^{2}+\frac{1}{2} I_{s} \dot{\phi}^{2}  \tag{B.12}\\
V(q) & =0,
\end{align*}
$$

where $m$ is the rover's total mass (approximately 15 kg ), $I_{p}$ is the rover body's moment of inertia about the point $p$ in Figure 5.2 on page 111, and $I_{s}$ represents the inertia that needs to be overcome by the steering angle actuator (assumed constant). Applying Lagrange's equation (B.4) yields

$$
M(q) \ddot{q}=\tau(t, q)+A(q) \lambda,
$$

where $M(q)$ is the inertia matrix

$$
M(q)=\left[\begin{array}{cccc}
m & 0 & 0 & 0 \\
0 & m & 0 & 0 \\
0 & 0 & I_{p} & 0 \\
0 & 0 & 0 & I_{s}
\end{array}\right]
$$

Since the rovers are driven at their wheels, one can model this as a throttle force input $f(t)$ applied at the front wheel axis, point $p_{f}=(x+l \cos \theta, y+l \sin \theta)$ in Figure 5.2, acting in the direction of the wheels. Similarly, suppose the steering mechanism is actuated by a steering torque input $\tau(t)$, which acts at the front wheel axis (however, in reality this torque actuates both wheel pairs simultaneously). Therefore, the system model's generalized force vector $\tau(t, q)$ is

$$
\begin{aligned}
\tau_{x}(t, q) & =f(t) \cos (\theta+\phi) \\
\tau_{y}(t, q) & =f(t) \sin (\theta+\phi) \\
\tau_{\theta}(t, q) & =f(t) l \sin \phi \\
\tau_{\phi}(t, q) & =\tau(t) .
\end{aligned}
$$

Thus, the resulting equations of motion are

$$
\left[\begin{array}{cccc}
m & 0 & 0 & 0 \\
0 & m & 0 & 0 \\
0 & 0 & I_{p} & 0 \\
0 & 0 & 0 & I_{s}
\end{array}\right] \ddot{q}=\left[\begin{array}{cc}
\cos (\theta+\phi) & 0 \\
\sin (\theta+\phi) & 0 \\
l \sin \phi & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
f(t) \\
\tau(t)
\end{array}\right]+A(q)\left[\begin{array}{l}
\lambda_{1} \\
\lambda_{2}
\end{array}\right] .
$$

To eliminate the Lagrange multipliers $\lambda_{1}$ and $\lambda_{2}$, premultiply the above by $G^{\top}(q)$, where $G(q)$ is defined by the vector fields of (B.11). This yields

$$
\left[\begin{array}{cccc}
m \cos \phi \cos \theta & m \cos \phi \sin \theta & \frac{1}{l} I_{p} \sin \phi & 0  \tag{B.13}\\
0 & 0 & 0 & I_{s}
\end{array}\right] \ddot{q}=\left[\begin{array}{c}
f(t) \\
\tau(t)
\end{array}\right] .
$$

Now, in place of $\ddot{q}$, substitute

$$
\ddot{q}=G(q) \dot{v}+\left[\begin{array}{cc}
-\cos \phi \sin \theta & 0 \\
\cos \phi \cos \theta & 0 \\
0 & 0 \\
0 & 0
\end{array}\right] \dot{\theta} v+\left[\begin{array}{cc}
-\sin \phi \cos \theta & 0 \\
-\sin \phi \sin \theta & 0 \\
\frac{1}{l} \cos \phi & 0 \\
0 & 0
\end{array}\right] \dot{\phi} v
$$

which, when substituted into (B.13), yields

$$
\left[\begin{array}{cc}
m \cos ^{2} \phi+\frac{1}{l^{2}} I_{p} \sin ^{2} \phi & 0 \\
0 & I_{s}
\end{array}\right] \dot{v}-\left[\begin{array}{cc}
v_{2}\left(m-\frac{1}{l^{2}} I_{p}\right) \cos \phi \sin \phi & 0 \\
0 & 0
\end{array}\right] v=\left[\begin{array}{c}
f(t) \\
\tau(t)
\end{array}\right] .
$$

In other words, the dynamical equations of motion (5.2) are

$$
\begin{aligned}
\dot{x} & =v_{1} \cos \phi \cos \theta \\
\dot{y} & =v_{1} \cos \phi \sin \theta \\
\dot{\theta} & =v_{1} \frac{1}{l} \sin \phi \\
\dot{\phi} & =v_{2} \\
\dot{v}_{1} & =\left(m \cos ^{2} \phi+\frac{1}{l^{2}} I_{p} \sin ^{2} \phi\right)^{-1}\left(v_{1} v_{2}\left(m-\frac{1}{l^{2}} I_{p}\right) \cos \phi \sin \phi+f(t)\right) \\
\dot{v}_{2} & =\tau(t) / I_{s}
\end{aligned}
$$

If the steering angle $\phi=0$, then the forward velocity $v_{1}$ has the simple dynamics $\dot{v}_{1}=f(t) / m$, which is similar to the unicycle in Example B.4. Also, one could add frictional resistance terms to the dynamics. For example, (5.2f) could be replaced with $\dot{v}_{2}=\left(\tau(t)-b v_{2}\right) / I_{s}$, where $b$ is a coefficient of friction.

## B. 4 Throttle Actuator Dynamics

Although the rover dynamic model (5.2) on page 112 employs the force $f$ as its throttle input, this force is actually generated by a DC motor, which drives the rover wheels by way of a geared transmission mechanism. If one assumes that the transmission has a torque ratio of $\mu$, then the output torque at the wheels is $\mu$ times the input torque to the transmission and the output speed is approximately $1 / \mu$ times the input speed.

Let $\tau_{a}$ denote the torque generated by the DC motor and $\omega_{a}$ the armature's
angular speed. Then, it can be determined that the developed torque is given by

$$
\tau_{a} \approx \frac{k_{t}}{R_{a}}\left(v_{a}-k_{e} \omega_{a}\right)
$$

where $v_{a}$ is the armature voltage input and the constants $k_{t}, k_{e}$, and $R_{a}$ are the motor's torque constant, "back emf" constant, and in-line resistor value, respectively. Given the torque ratio $\mu$, the torque at the rover wheels is $\tau_{w} \approx \mu \tau_{a}$. Let $\rho$ denote the radius of the wheels. Therefore, the throttle force $f_{\tau}$ generated by the motor is approximately

$$
f_{\tau}=\tau_{w} / \rho \approx \frac{\mu k_{t}}{R_{a}}\left(v_{a}-k_{e} \omega_{a}\right)
$$

There is likely a significant amount of friction and resistance in the system, which can be added to the model as a resistive force $b v_{f}$, where $b$ is a coefficient of friction. Thus, the force $f$ in (5.2e) can be modelled by

$$
\begin{equation*}
f \approx \frac{\mu k_{t}}{R_{a}}\left(v_{a}-\frac{k_{e} \mu}{\rho} v_{f}\right)-b v_{f} \tag{B.14}
\end{equation*}
$$

## B. 5 Compensator Selection

Suppose (5.2e) is linearized about a constant steering angle $\phi \equiv \bar{\phi}$. Then the forward velocity $v_{f}$ has the simple linear dynamics $\dot{v}_{f}=f /\left(m \cos ^{2} \bar{\phi}+\frac{1}{l^{2}} I_{p} \sin ^{2} \bar{\phi}\right)$. To simplify the equations, suppose $\bar{\phi}=0$, so that (B.14) yields

$$
\dot{v}_{f} \approx \frac{\mu k_{t}}{m R_{a}}\left(v_{a}-\frac{k_{e} \mu}{\rho} v_{f}\right)-\frac{b}{m} v_{f}=: a_{1}\left(v_{a}+a_{2} v_{f}\right)-a_{3} v_{f}
$$

Thus, taking the Laplace transform of both sides and solving for the transfer function $G(s):=V_{f}(s) / V_{a}(s)$ yields

$$
G(s)=\frac{a_{1}}{s+\left(a_{1} a_{2}+a_{3}\right)}
$$

Even though the parameters $a_{1}, a_{2}$, and $a_{3}$ are not known, the form of $G(s)$ suggests the rover behaves approximately like a type 0 system (assuming a fixed
steering angle). Thus, for each rover a PI compensator of the form

$$
D(s)=k_{P}+\frac{k_{I}}{s}
$$

was designed to regulate its speed to a desired constant level. Thus, the final closed-loop transfer function from $v_{a}$ to $v_{f}$ is of the form

$$
T(s)=\frac{k_{P} a_{1} s+k_{I} a_{1}}{s^{2}+\left(k_{P} a_{1}+a_{1} a_{2}+a_{3}\right) s+k_{I} a_{1}} .
$$

Since $D(s) G(s)$ is type I, the closed loop steady-state tracking error tends to zero when the system is subject to a step input in the reference speed.


[^0]:    ${ }^{1}$ Reynolds' application was to computer simulated animation. Interestingly, his contributions in this field later earned him a Scientific and Engineering Award for "pioneering contributions to the development of three-dimensional computer animation for motion picture production" at the 70th Academy Awards in 1997 (Academy of Motion Picture Arts and Sciences, 2004).

[^1]:    ${ }^{2}$ Couzin et al. (2002) actually reported four (not three) behaviours. However, for the purposes of this discussion, their distinction between two kinds of parallel motion is unnecessary.

[^2]:    ${ }^{3}$ Each Josephson junction is a kind of superconductor "sandwich," capable of acting like an extremely high-frequency electronic switch (Strogatz, 2003, pp. 148-150).

[^3]:    ${ }^{1}$ Of course, the period of $R$, call it $l$, is merely the number of times $R$ must be applied to a point $z_{1} \neq 0$ before one obtains $z_{l}=R^{l} z_{1}=z_{1}$.

[^4]:    ${ }^{1}$ Recall, from Theorem 4.1, that $\bar{\alpha}= \pm \pi d / n$. The affine subspace $\mathcal{F}_{d}$ is defined here only for $\bar{\alpha}=\pi d / n$, since the affine subspace defined for $\bar{\alpha}=-\pi d / n$ has the same properties as $\mathcal{F}_{d}$.

[^5]:    ${ }^{1}$ The allusion being to the Greek myth of Jason, the Argonauts, and the Golden Fleece, since names belonging to the Argonauts have been bestowed on rovers of the fleet.

[^6]:    ${ }^{2}$ The the algorithm used to convert RGB values to hue values is based on sample code from Eugene Vishnevsky's colour conversion algorithms, which could be found on the world-wide-web at the time of printing at http://www.cs.rit.edu/~ncs/color/t_convert.html.

[^7]:    ${ }^{1}$ Strictly speaking, if there exists a relabeling of the agent indices such that $A$ is subsequently a circulant matrix, then the system is also said to have circulant connectivity. Further details about relabeling and connectivity are provided in Section 6.3.2.
    ${ }^{2}$ Notice that when $\kappa>0, \Gamma(A)$ is a strongly connected digraph. Therefore, the consensus condition (C) of Theorem 2.4 on page 22 is satisfied for every (nontrivial) circulant digraph.

[^8]:    ${ }^{3}$ This commonly used terminology (cf. Coxeter, 1948; Davis, 1994) stems from the more general theory of permutation groups, which is not discussed here. Here, one can interpret the process of factorization as equivalent to a partitioning of the index set $\mathcal{N}$.

[^9]:    ${ }^{4}$ Given two nonzero integers $a$ and $b$, Bézout's identity says that there exist integers $c$ and $d$ such that $\operatorname{gcd}(a, b)=a c+b d$ (Jones and Jones, 1998, Section 1.2, Theorem 1.7).

[^10]:    ${ }^{5}$ Let $\sigma: \mathcal{N} \rightarrow \mathcal{N}$ be a bijection and define $\tau: \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N} \times \mathcal{N}$ such that $\tau(i, j) \mapsto(\sigma(i), \sigma(j))$, $i, j \in \mathcal{N}$. Then, $\tau$ must also be a bijection.

