MARKED LENGTH SPECTRAL DETERMINATION OF ANALYTIC CHAOTIC BILLIARDS WITH AXIAL SYMMETRIES

JACOPO DE SIMOI, VADIM KALOSHIN, MARTIN LEGUIL

Abstract. We consider billiards obtained by removing from the plane finitely many strictly convex analytic obstacles satisfying the non-eclipse condition. The restriction of the dynamics to the set of non-escaping orbits is conjugated to a subshift, which provides a natural labeling of periodic orbits. We show that under suitable symmetry and genericity assumptions, the Marked Length Spectrum determines the geometry of the billiard table.

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Introduction

In this paper, we study the problem of length-spectral determination for a class of domains obtained by removing from the plane $m \geq 3$ strictly convex analytic obstacles which satisfy a standard non-eclipse condition.

Billiard dynamics on such domains studies the long-term behavior of a point particle which moves freely on the domain and undergoes elastic reflections with the domain boundary. Such systems were first considered in [GR], where the authors studied the classical scattering of a point particle from three circular disks on the plane. Strict convexity of the obstacles implies that the corresponding billiard dynamics enjoys strong hyperbolicity properties. Hyperbolicity, together with the non-eclipse condition, allows to encode the dynamics of non-escaping trajectories as a subshift of finite type on m symbols (see, e.g. [Mor] or [PS2, Section 2.2]). This observation provides, in particular, a natural marking of each periodic orbits with the associated encoding. The *Marked Length Spectrum* is then defined as the set of all lengths of periodic orbits together with their marking (see Definition 1.1). If two billiard tables have same Marked Length Spectrum, we say that they are *marked-length-isospectral*.

Of course, when two billiards are isometric, they are necessarily isospectral. On the other hand, it is a fascinating problem to characterize marked-length-isospectral billiards modulo isometries. We refer to this problem as the *dynamical inverse spectral problem*. In order to describe our results in some context, let us present some related classical problems and corresponding results.

The Laplace inverse spectral problem. The dynamical inverse spectral problem introduced above is tightly related to the question that M. Kac (see [K]) famously phrased as: *"Can one hear the shape of a drum?"*, i.e. is the shape of a planar domain determined by its Laplace Spectrum? The relation between the dynamical and the Laplace problem is apparent, for instance, in the trace formula proved by Andersson–Melrose (see [AM]): generalizing previous results by Chazarain, Duistermaat–Guillemin, they showed that, for strictly convex C^∞ domains, the singular support of the wave trace is contained in (and generically equals) the Length Spectrum. In particular, in this setting, the Laplace Spectrum generically determines the Length Spectrum. Similarly, there is a connection between Laplace Spectrum and Length Spectrum in hyperbolic situations: indeed, the Selberg trace formula shows that the Laplace Spectrum determines the Length Spectrum on hyperbolic manifolds, or for generic Riemannian metrics.

Spectral determination and spectral rigidity for convex domains with symmetries. In this subsection we recall a few results related to the question of Laplace spectral determination of convex domains. It has been famously proven by Zelditch in a series of papers (see [Z1, Z2, Z3]) that the Laplace Spectrum completely determines (modulo isometries) the domain in a generic class of analytic $\mathbb{Z}_2$-symmetric (i.e., symmetric with respect to some axis of reflection) planar convex domains. Hezari–Zelditch [HZ2] have obtained a higher dimensional analog of this result: bounded analytic domains in $\mathbb{R}^n$ with reflection symmetries across all coordinate axes, and with one axis height fixed (satisfying some generic non-degeneracy conditions) are spectrally determined among other such domains. Results of this kind are, currently, far beyond reach in the smooth category, although, in the last decade, interesting results have appeared in the weaker setting of spectral rigidity[1] properties. In [HZ1], Hezari–Zelditch have shown the following result: given a domain bounded by an ellipse, any one-parameter isospectral C^∞ deformation which additionally preserves the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group of the ellipse is necessarily flat (i.e., all derivatives have to vanish at the initial parameter).

The problem of dynamical spectral determination was studied by Colin de Verdière; in [CdV], he has shown that, in the class of convex analytic billiards with the symmetries of the ellipse, the Marked Length Spectrum determines the domain geometry. In the smooth category, in [DKW], the authors proved that any sufficiently smooth $\mathbb{Z}_2$-symmetric strictly convex domain sufficiently close to a circle is dynamically spectrally rigid, i.e., all deformations among domains in the same class which preserve the length of all periodic orbits of the associated billiard flow must necessarily be isometries.

The Laplace Inverse Resonance Problem A dual formulation of the inverse spectral problem is the inverse resonance problem, in which one attempts to reconstruct an unbounded domain (e.g. the complement of a finite number of convex scatterers) by the resonances (i.e. the poles) of the resolvent $(\Delta - z^2)^{-1}$ (see e.g. [PS2, Zw2, Z4]). From the dynamical point of view, these systems are described by the theory of *Dispersing Billiards*.

In [Z4], Zelditch showed that a $\mathbb{Z}_2$-symmetric configuration of two convex analytic obstacles in the plane $\mathbb{R}^2$ is determined by its Dirichlet or Neumann resonance poles. It is the analog for exterior domains of the proof that a $\mathbb{Z}_2$-symmetric bounded simply connected analytic plane domain is determined by its Dirichlet eigenvalues. The proof is based on the fact that wave invariants of an exterior domain are resonance invariants and on the method of [Z2, Z3] for calculating the wave invariants explicitly in terms of the boundary defining function. In [ISZ], the authors gave another proof of the inverse result with two symmetries using Birkhoff Normal Forms of the billiard map and quantum monodromy operator rather than the Laplacian, applying some results on semi-classical trace formulae and on quantum Birkhoff normal forms for semi-classical Fourier integral operators to inverse problems. Generalizing results of [G], they showed that the classical Birkhoff Normal Form can be recovered from semi-classical spectral invariants, and in fact, that the full quantum Birkhoff Normal Form of the quantum Hamiltonian near a closed orbit, and infinitesimally with respect to the energy can be recovered.

Dynamical inverse spectral problems for hyperbolic billiards. We finally come to the setting that we will explore in this paper. Our previous work [BDKL], joint with P. Bálint, shows that for chaotic billiards obtained by removing $m \geq 3$ strictly convex finitely smooth obstacles from $\mathbb{R}^2$, the Marked Length Spectrum determines the curvature at the collision points of every 2-periodic orbit. We have also shown that the Marked Length Spectrum also determines the Lyapunov exponent of each periodic orbit (see Subsection 1.4 for more details).

It is an important observation that, unlike billiards inside convex domains, billiards under consideration are *open systems*, i.e. there exist initial conditions for which the point particle escapes to infinity. As a consequence, periodic trajectories

[1] Recall that a domain is said to be *spectrally rigid* if any Laplace isospectral *continuous deformation* is necessarily isometric.

will not sample some regions in the configuration space. This implies that spectral data will not, in general, suffice to recover the geometry of the unexplored region. Hence, one can either attempt to recover the full geometry under additional assumptions (e.g. analyticity), or consider the question of determination restricted to the explored region.

In this paper, we pursue the first strategy and assume that all scatterers have real analytic curves as boundary. Our goal is to recover the full jet of the curvature at some point on each scatterer. Due to analyticity, this entirely determines the geometry of the scatterers. Our main result, stated below as our Main Theorem asserts that this is indeed possible, provided that two scatterers have some symmetries (similar to the "bi-symmetric" setting of [Z1]).

In our proof, we reconstruct from spectral data the classical (hyperbolic) Birkhoff Normal Form of a specific two-periodic orbit. This can be done, provided that a genericity condition is satisfied, by analyzing some asymptotics in the Marked Length Spectrum relative to periodic orbits that approximate homoclinic orbits of the two-periodic orbit.

Once the normal form has been obtained, exploiting the symmetries of our system, and some extra information that can be obtained by the Marked Length Spectrum, it is possible to reconstruct the geometry of the billiard. A more detailed explanation of the proof will be given in Section 1.2, after introducing some necessary preliminaries.

Our results are an analog of those presented in [CdV] for the class of chaotic billiards under consideration, or an analog in terms of the Marked Length Spectrum of [Z1] (see also [ISZ]).

Note that due to the convexity of the obstacles, the presence of more than two scatterers in our case is crucial to guarantee the existence of a large set of periodic orbits. On the other hand, billiard trajectories in the exterior of only two strictly convex domains in the plane. were considered by Stoyanov in [Sto].

1. Definitions and statement of our main results

In the present paper, we consider billiard tables $\mathcal{D} \subset \mathbb{R}^2$ given by $\mathcal{D} = \mathbb{R}^2 \backslash \bigcup_{i=1}^m \mathcal{O}_i$, for some integer $m \geq 3$, where each $\mathcal{O}_i$ is a convex domain with analytic boundary $\partial\mathcal{O}_i$. We refer to each of the $\mathcal{O}_i$'s as *obstacle* or *scatterer*. We let $\ell_i := |\partial\mathcal{O}_i|$ be the corresponding lengths, set $\mathbb{T}_i := \mathbb{R}/\ell_i\mathbb{Z}$, and parametrize each $\partial\mathcal{O}_i$ in arc-length, for some analytic map $\Upsilon_i \in C^\omega(\mathbb{T}_i, \mathbb{R}^2)$, $s \mapsto \Upsilon_i(s)$. We assume that the following condition holds:

Non-eclipse condition: The convex hull of any two scatterers is disjoint from the other $m-2$ scatterers.

The set of all billiard tables obtained by removing from the plane m strictly convex analytic obstacles satisfying the non-eclipse condition will be denoted by $\mathbf{B}(m)$.

Fix $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^m \mathcal{O}_i \in \mathbf{B}(m)$. We denote the collision space by

$$\mathcal{M} = \bigcup_i \mathcal{M}_i, \qquad \mathcal{M}_i = \{(q,v),\ q \in \partial\mathcal{O}_i,\ v \in \mathbb{R}^2,\ \|v\| = 1,\ \langle v, n\rangle \geq 0\},$$

where n is the unit normal vector to $\partial\mathcal{O}_i$ pointing inside $\mathcal{D}$. For each $x = (q,v) \in \mathcal{M}$, q is associated with the arclength parameter $s \in [0, \ell_i]$ for some $i \in \{1, \cdots, m\}$, i.e., $q = \Upsilon_i(s)$. We let $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ be the oriented angle between n and v and set

$r := \sin(\varphi)$. In other words, each $\mathcal{M}_i$ can be seen as a cylinder $\mathbb{T}_i \times [-1,1]$ endowed with coordinates (s,r). In the following, given a point $x=(q,v) \in \mathcal{M}$ associated with the pair (s,r), we also denote by $\Upsilon(s) := q$ the point of the table defined as the projection of x onto the q-coordinate. Moreover, for each pair $(s,r),(s',r') \in \mathcal{M}$, we denote by

$$h(s,s') := \|\Upsilon(s) - \Upsilon(s')\| \tag{1.1}$$

the Euclidean length of the segment connecting the associated points of the table.

Set $\Omega := \{(q,v) \in \mathcal{D} \times \mathbb{S}^1\}$. Denote by $\Phi^t \colon \Omega \to \Omega$ the flow of the billiard and let

$$\mathcal{F} = \mathcal{F}(\mathcal{D}) \colon \mathcal{M} \to \mathcal{M}, \quad x \mapsto \Phi^{\tau(x)+0}(x)$$

be the associated billiard map, where $\tau \colon \mathcal{M} \to \mathbb{R}_+ \cup \{+\infty\}$ is the first return time. For any point $x=(s,r) \in \mathcal{M}$ such that $(s',r') := \mathcal{F}(s,r)$ is well-defined, we denote by $\mathscr{L} := h(s,s')$ the distance between the two points of collision, we let $\mathcal{K} := \mathcal{K}(s)$, $\mathcal{K}' := \mathcal{K}(s')$ be the respective curvatures, and set $\nu := \sqrt{1-r^2}$, $\nu' := \sqrt{1-(r')^2}$. It follows from formula 2.26 on p. 35 of [CM] that

$$D_{(s,r)}\mathcal{F} = -\begin{pmatrix} \frac{1}{\nu'}(\mathscr{L}\mathcal{K}+\nu) & \frac{\mathscr{L}}{\nu\nu'} \\ \mathscr{L}\mathcal{K}\mathcal{K}' + \mathcal{K}\nu' + \mathcal{K}'\nu & \frac{1}{\nu}(\mathscr{L}\mathcal{K}'+\nu') \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}), \tag{1.2}$$

and the map $\mathcal{F}$ is symplectic for the form $ds \wedge dr$.

Due to the convexity of the obstacles, for each $i \in \{1,\cdots,m\}$, for $* \in \{1,\cdots,m\} \setminus \{i\}$, there exist $0 \le a_i^* \le b_i^* \le \ell_i$, and for each parameter $s \in [a_i^*, b_i^*]$, there exists a non-empty closed interval $I_i^*(s) \subset [-1,1]$ such that $\tau(x) < +\infty$, if $x=(s,r) \in \widetilde{\mathcal{M}}_i := \cup_{j\neq i} \widetilde{\mathcal{M}}_i^j$, and $\tau(x) = +\infty$, if $x \in \mathcal{M}_i \setminus \widetilde{\mathcal{M}}_i$, where

$$\widetilde{\mathcal{M}}_i^* := \{(s,r) \in \mathcal{M}_i : s \in [a_i^*, b_i^*],\ r \in I_i^*(s)\} = \mathcal{M}_i \cap \mathcal{F}^{-1}(\mathcal{M}_*).$$

In particular, the set of trajectories that do not escape to infinity is given by

$$\bigcap_{k\in\mathbb{Z}} \mathcal{F}^{-k}(\widetilde{\mathcal{M}}), \quad \widetilde{\mathcal{M}} := \bigcup_{j\neq i} \widetilde{\mathcal{M}}_j,$$

and is homeomorphic to a Cantor set. The restriction of the dynamics to this set is conjugated to a subshift of finite type associated with the transition matrix

$$\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}.$$

In other words, any word $(\varsigma_j)_j \in \{1,\cdots,m\}^{\mathbb{Z}}$ such that $\varsigma_{j+1} \neq \varsigma_j$ for all $j \in \mathbb{Z}$ can be realized by an orbit, and by hyperbolicity of the dynamics, this orbit is unique. Such a word is called *admissible*. Besides, this marking is unique provided that we fix a starting point in the orbit and an orientation.

In particular, any periodic orbit of period p (observe that necessarily $p \geq 2$) can be labeled by a finite admissible word $\sigma = (\sigma_1\sigma_2\ldots\sigma_p) \in \{1,\cdots,m\}^p$, such that the infinite word $\sigma^\infty := \ldots\sigma\sigma\sigma\ldots$ is admissible (or equivalently, such that $\sigma_j \neq \sigma_{j+1 \bmod p}$, for all $j \in \{1,\cdots,p\}$). We denote by Adm the set of finite admissible words $\sigma \in \cup_{p\geq 2}\{1,\cdots,m\}^p$.

Given any word $\sigma \in \mathrm{Adm}$, we also let $\overline{\sigma}$ be the transposed word

$$\overline{\sigma} := (\sigma_p \sigma_{p-1} \dots \sigma_1).$$

The word $\overline{\sigma}$ encodes the same periodic trajectory as σ, but with opposite orientation.

As explained above, for any $j \in \{1, \cdots, p\}$, the j^{th} symbol σ_j of σ corresponds to a point $x(j)$ in the trajectory, where $x(j) = (s(j), r(j)) \in \mathcal{M}_{\sigma_j}$ is represented by position and angle coordinates. For all $k \in \mathbb{Z}$, we also extend the previous notation by setting $\sigma_k := \sigma_{k \mod p}$, and similarly for $x(k)$, $s(k)$ and $r(k)$.

Definition 1.1. *The Marked Length Spectrum $\mathcal{MLS}(\mathcal{D})$ of $\mathcal{D}$ is defined as the function*

$$\mathcal{L}\colon \mathrm{Adm} \to \mathbb{R}_+, \quad \sigma \mapsto \mathcal{L}(\sigma), \tag{1.3}$$

where $\mathcal{L}(\sigma)$ is the length of the periodic orbit identified by σ, obtained by summing the lengths of all the line segments that compose it.

*In the following, an object is said to be a $\mathcal{MLS}$-*invariant *if it can be obtained by the sole knowledge of the Marked Length Spectrum.*

For any periodic orbit $(x_1, \cdots, x_p)$ encoded by a word σ of length $p \geq 2$, we have $D_{x_j}\mathcal{F}^p \in \mathrm{SL}(2, \mathbb{R})$, for $j \in \{1, \cdots, p\}$. Due to the strict convexity of the obstacles, $D_{x_j}\mathcal{F}^p$ is hyperbolic, and we denote by $\lambda(\sigma) < 1 < \lambda(\sigma)^{-1}$ its eigenvalues. The *Lyapunov exponent* of this orbit is defined as

$$\mathrm{LE}(\sigma) := -\frac{1}{p} \log \lambda(\sigma) > 0. \tag{1.4}$$

Definition 1.2. *The Marked Lyapunov Spectrum of the billiard table $\mathcal{D}$ is defined as the function*

$$\mathrm{LE}\colon \mathrm{Adm} \to \mathbb{R}_+, \quad \sigma \mapsto \mathrm{LE}(\sigma). \tag{1.5}$$

To conclude this section, let us recall an important symmetry of the billiard dynamics, which will be crucial in the following. Let us denote by $\mathcal{I}$ the involution map $\mathcal{I}\colon (s, r) \mapsto (s, -r)$. It conjugates the billiard map $\mathcal{F}$ with its inverse $\mathcal{F}^{-1}$, according to the time-reversal property of the billiard dynamics:

$$\mathcal{I} \circ \mathcal{F} \circ \mathcal{I} = \mathcal{F}^{-1}.$$

In the following, a periodic orbit of period $p = 2q \geq 2$ is called *palindromic* if it can be labeled by an admissible word $\sigma \in \{1, \cdots, m\}^p$ such that $\sigma = (\sigma_1 \dots \sigma_{q-1} \sigma_q \sigma_{q-1} \dots \sigma_1 \sigma_0)$ for certain symbols $(\sigma_0, \sigma_1, \cdots, \sigma_q) \in \{1, \cdots, m\}^{q+1}$. As we shall see later, there is a connection between the palindromic symmetry and the time-reversal property recalled above. In particular, by the palindromic symmetry and by expansiveness of the dynamics, the associated trajectory hits the billiard table perpendicularly at the points with symbols σ_0 and σ_q.

For more details about chaotic billiards and inverse spectral problems, we refer the reader to the books of Chernov–Markarian [CM] and Petkov–Stoyanov [PS2].

1.1. **Spectral determination.** Recall that $\mathbf{B}(m)$ is the set of all billiard tables $\mathcal{D}$ formed by $m \geq 3$ convex analytic obstacles satisfying the non-eclipse condition, that $\mathcal{F}(\mathcal{D})$ denotes the associated billiard map, and that $\mathcal{K}$ is the curvature function. We introduce a class of tables with two additional symmetries. Without loss of generality, we assume that those symmetries are associated with the obstacles $\mathcal{O}_1, \mathcal{O}_2$.

Definition 1.3. *We let $\mathbf{B}_{\mathrm{sym}}(m) \subset \mathbf{B}(m)$ be the subset of all billiard tables $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^m \mathcal{O}_i$ which are symmetric in the following sense:*

- *the jets of $\mathcal{K}$ are the same at the endpoints of the 2-periodic orbit (12);*
- *the jets of $\mathcal{K}|_{\mathbb{T}_1}$, $\mathcal{K}|_{\mathbb{T}_2}$ are even, assuming that $0_1 \in \mathbb{T}_1$, $0_2 \in \mathbb{T}_2$ are the arc-length parameters of the endpoints of the orbit (12).*

In particular, by analyticity, the pair of obstacles $\mathcal{O}_1, \mathcal{O}_2$ has some $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry: $\mathcal{O}_1, \mathcal{O}_2$ are images of each other by the reflection along the line segment bisector of the trace of the orbit (12), and each of them is symmetric with respect to the line through the endpoints of (12).

The reason for requiring the two symmetries will be clarified in Remark 1.8 below. In the following, we let $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$, and let $C^\omega(\mathbb{T}, \mathbb{R})$ be the Banach space of 2π-periodic real analytic functions endowed with the norm $|\cdot|_{\mathbb{T}}$, where $|f|_{\mathbb{T}} := \sup_{\theta \in \mathbb{T}} |f(\theta)|$, for $f \in C^\omega(\mathbb{T}, \mathbb{R})$. We denote by $C^\omega(\mathbb{T}, \mathbb{R}^2)$ the Banach space of analytic functions $f\colon \theta \mapsto (f_1(\theta), f_2(\theta))$, with $f_1, f_2 \in C^\omega(\mathbb{T}, \mathbb{R})$, endowed with the norm $\|\cdot\|_{\mathbb{T}}$, where $\|f\|_{\mathbb{T}} := \max(|f_1|_{\mathbb{T}}, |f_2|_{\mathbb{T}})$.

Definition 1.4 (Topology on $\mathbf{B}_{\mathrm{sym}}(m)$)**.** *Let $\mathbf{Conv} \subset C^\omega(\mathbb{T}, \mathbb{R}^2)$ be the set of all functions $f \in C^\omega(\mathbb{T}, \mathbb{R}^2)$ such that $f(\mathbb{T})$ is a simple closed curve and the interior region bounded by $f(\mathbb{T})$ is convex. We denote by $\mathcal{O}(f)$ the convex hull of the interior region bounded by $f(\mathbb{T})$. For any integer $m \geq 3$, we thus get a map $\Phi = \Phi(m)$:*

$$\Phi\colon \mathbf{Conv}^m \ni (f^{(i)})_{i=1,\cdots,m} \mapsto \mathcal{D} := \mathbb{R}^2 \setminus \cup_{i=1}^m \mathcal{O}(f^{(i)}).$$

Let $\mathbf{W}_{\mathrm{sym}}(m) := \Phi^{-1}(\mathbf{B}_{\mathrm{sym}}(m)) \subset \mathbf{Conv}^m$, and endow it with the topology induced by the product topology on $(C^\omega(\mathbb{T}, \mathbb{R}^2))^m$. Then we equip $\mathbf{B}_{\mathrm{sym}}(m)$ with the topology coinduced by the map Φ.

Our main result is the following.

Main Theorem. *For any $m \geq 3$, there exists an open and dense set of billiard tables $\mathbf{B}^*_{\mathrm{sym}}(m) \subset \mathbf{B}_{\mathrm{sym}}(m)$ so that if $\mathcal{D} \in \mathbf{B}^*_{\mathrm{sym}}(m)$, then the geometry of $\mathcal{D}$ is entirely determined (modulo isometries) by its Marked Length Spectrum $\mathcal{MLS}(\mathcal{D})$.*

Remark 1.5. *In fact, the open and dense condition is a* non-degeneracy condition*: it means that after a change of coordinates, the first coefficient in the expansion of the dynamics is non-zero (see Remark 1.9 and condition ($\star$) in Lemma 6.6).*

It is a standard observation that any continuous deformation of smooth domains which preserves the (unmarked) Length Spectrum $\mathcal{LS}(\mathcal{D})$ automatically preserves the Marked Length Spectrum (see e.g. [Sib, Proposition 3.2.2]); therefore our result could be also stated as the following spectral rigidity result. Let us first introduce a definition.

Definition 1.6. *A family $(\mathcal{D}_t)_{t \in (-1,1)}$ of billiards is called an iso-length-spectral deformation in $\mathbf{B}^*_{\mathrm{sym}}(m)$ if*

- *each $\mathcal{D}_t$ is in $\mathbf{B}^*_{\mathrm{sym}}(m)$, and the map $(-1,1) \ni t \mapsto \mathcal{D}_t \in \mathbf{B}^*_{\mathrm{sym}}(m)$ is continuous;*
- *$\mathcal{LS}(\mathcal{D}_t) = \mathcal{LS}(\mathcal{D}_0)$, for all $t \in (-1,1)$.*

Theorem. *Any iso-length-spectral deformation $(\mathcal{D}_t)_{t \in (-1,1)}$ in $\mathbf{B}^*_{\mathrm{sym}}(m)$ is isometric.*

Remark 1.7. *We have stated the above results in the case of m scatterers, with $m \geq 3$ but indeed it suffices to show that the result holds for $m = 3$. In fact, fix $m > 3$ and let*

$$\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^{m} \mathcal{O}_i \in \mathbf{B}_{\text{sym}}(m);$$

for $2 < i \leq m$, define

$$\mathcal{D}_i := \mathbb{R}^2 \setminus (\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_i).$$

It is immediate to show that $\mathcal{D}_i \in \mathbf{B}_{\text{sym}}(3)$ (since the non-eclipse condition holds automatically). Now let

$$\mathbf{B}^*_{\text{sym}}(m) := \{\mathcal{D} \in \mathbf{B}_{\text{sym}}(m) \text{ s.t. } \forall\, 2 < i \leq m,\, \mathcal{D}_i \in \mathbf{B}^*_{\text{sym}}(3)\}.$$

It is easy to check that $\mathbf{B}^*_{\text{sym}}(m)$ is open and dense. Since $\mathcal{MLS}(\mathcal{D}_i)$ is the restriction of $\mathcal{MLS}(\mathcal{D})$ to the periodic orbits that only collide with $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_i$, we can apply our Main Theorem for $m = 3$ to $\mathcal{D}_i$ and recover the geometry of $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_i$ for any i. Since i was arbitrary, we proved the Main Theorem for m.

The proof of the Main Theorem in the case $m = 3$ is given in Corollary 6.7 and Corollary 6.8 in Section 6, based on the constructions provided in detail in the preceding sections. From now on, we will consider the case of three scatterers. We will abbreviate $\mathbf{B} := \mathbf{B}(3)$ and $\mathbf{B}_{\text{sym}} := \mathbf{B}_{\text{sym}}(3)$.

Let $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^3 \mathcal{O}_i \in \mathbf{B}$ be a billiard table, and let $\mathcal{F} := \mathcal{F}(\mathcal{D})$. A key object in our study is the so-called *Birkhoff Normal Form* for saddle fixed points of symplectic local surface diffeomorphisms, whose definition we now recall. We introduce it for period two orbits since this is the case we will consider in the following, but the same can be done for any periodic orbit (given a periodic orbit of period $p \geq 2$, each point in the orbit is a saddle fixed point of $\mathcal{F}^p$). Let $j \neq k \in \{1, 2, 3\}$, and let $(s(j,k), 0)$ be the (s,r)-coordinates of the point of $\mathcal{O}_j$ in the orbit (jk). Recall that by [Mos, Ste], there exists an analytic symplectomorphism $R\colon \mathcal{U} \to \mathcal{V}$ from a neighborhood $\mathcal{U} \subset \mathcal{M}$ of $(s(j,k), 0)$ to a neighborhood $\mathcal{V} \subset \mathbb{R}^2$ of $(0,0)$ and a unique analytic map $\Delta = \Delta(\mathcal{D}, j, k) \in C^\omega(\mathbb{R}, \mathbb{R}^*)$, with $\Delta(z) = \lambda + \sum_{\ell \geq 1} a_\ell z^\ell$, s.t.

$$R \circ \mathcal{F}^2|_{\mathcal{U}} = N \circ R|_{\mathcal{U}},$$

where N is the *Birkhoff Normal Form* of $\mathcal{F}^2|_{\mathcal{U}}$:

$$N = N(\mathcal{D}, j, k)\colon (\xi, \eta) \mapsto (\Delta(\xi\eta)\xi, \Delta(\xi\eta)^{-1}\eta).$$

In the following, we refer to $(a_\ell)_\ell$ as the *Birkhoff invariants* or *coefficients* of N.

Remark 1.8. *The two symmetries described above are needed because of two different issues. Let us consider a billiard table $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^3 \mathcal{O}_i \in \mathbf{B}_{\text{sym}}$.*

- *The axial symmetry between $\mathcal{O}_1$ and $\mathcal{O}_2$ we ask for is similar to the one that appears for instance in the work of Zelditch. It is explained by the fact that in order to speak about Birkhoff Normal Forms, we need a fixed point. As the billiard map has no such fixed point, we need at least to consider its square. In the process, some information is lost, unless the pair $\{\mathcal{O}_1, \mathcal{O}_2\}$ has some $\mathbb{Z}_2$-symmetry; otherwise, we are a priori only able to recover some averaged information between $\mathcal{O}_1$ and $\mathcal{O}_2$. In [BDKL], we have started to analyze ways* *to avoid this additional symmetry requirement, and in a upcoming work we will carry on this approach further in the analytic case.*

- *The second symmetry we require is due to a well known observation made in [CdV]. Indeed, the reason why we ask the obstacles $\mathcal{O}_1, \mathcal{O}_2$ to be symmetric follows from the fact that the Birkhoff Normal Form has some intrinsic symmetries (it has two axes of symmetry), and only conveys partial information on the billiard dynamics, which has a priori only one natural symmetry, given by the time-reversal property. Roughly speaking, we lose half of the information on the billiard map, unless it itself has some additional symmetry, which can be indeed ensured provided that $\mathcal{O}_1, \mathcal{O}_2$ have some $\mathbb{Z}_2$-symmetry.*

Remark 1.9. *In the following, we show that the Lyapunov exponents of the orbits $(h_n)_n$ can be expanded as a series indexed by $\mathbb{Z}^2$ (see (1.6)), each of whose coefficients is a $\mathcal{MLS}$-invariant. The expression of these coefficients combines three different sets of geometric data, including the Birkhoff invariants we want to reconstruct. We show that under some open and dense condition, it is possible to extract enough information from the first three lines of the coefficients of the series in order to recover separately the three sets of data. More precisely, the condition we need is the non-vanishing of the first Birkhoff invariant (see condition $(\star)$ in Lemma 6.6), which can be seen dynamically as some twist condition. It guarantees that certain linear systems in the data we want to recover are invertible. Note that Lemma 4.10 gives an effective way of checking whether a given billiard table $\mathcal{D} \in \mathbf{B}_{\mathrm{sym}}$ satisfies this twist condition; in other words, the property "$\mathcal{D} \in \mathbf{B}^*_{\mathrm{sym}}$" itself is a $\mathcal{MLS}$-invariant. Besides, this condition comes from the particular subset of coefficients we consider (which is easiest to work with), and it is likely that considering other subsets of coefficients would produce another non-degeneracy condition involving different Birkhoff invariants. In particular, it seems reasonable to believe that as long as the Birkhoff Normal Form is not degenerate (i.e., is not linear), our construction can be adapted to produce invertible systems in the coefficients we want to reconstruct.*

Let us now give another way to state the above result. We denote by $\widehat{\mathbf{B}}$ the set of all connected hexagonal domains $\widehat{\mathcal{D}} \subset \mathbb{R}^2$ of the plane bounded by six arcs labeled by $1, 2, 3$, $(12), (13), (23)$, such that:

- $1, 2, 3$ are three arcs such that at each point in one of those arcs, the curvature keeps a constant sign, and the center of the osculating circle is in $\mathbb{R}^2 \setminus \widehat{\mathcal{D}}$;
- for $j < k \in \{1, 2, 3\}$, the arc labeled by (jk) is a line segment which meets each arc j and k perpendicularly;
- for $j < k \in \{1, 2, 3\}$, the convex hull of the arcs j and k is contained in $\widehat{\mathcal{D}}$.

The last condition is the same as our previous non-eclipse condition. For instance, the table $\widehat{\mathcal{D}}_1$ below is admissible, while $\widehat{\mathcal{D}}_2$ is not. By analyticity, there is a bijective correspondence between tables $\mathcal{D} \in \mathbf{B}$ defined previously and tables $\widehat{\mathcal{D}} \in \widehat{\mathbf{B}}$, where $\mathcal{D} \mapsto \widehat{\mathcal{D}}(\mathcal{D})$ is obtained by considering the domain bounded by 2-periodic orbits. Indeed, by Lemma 3.1 in [BDKL], any non-escaping trajectory stays in this domain. Besides, any periodic orbit of the billiard map of $\widehat{\mathcal{D}}(\mathcal{D})$ whose points all have images in the arcs labeled by $1, 2$ and 3 can be assigned a unique periodic trajectory of $\mathcal{D}$.

This correspondence is surjective, except for 2-periodic orbits of $\mathcal{D}$, since they now correspond to edges of $\widehat{\mathcal{D}}$. It is thus natural to define the Marked Length Spectrum of $\widehat{\mathcal{D}} = \widehat{\mathcal{D}}(\mathcal{D})$ as $\mathcal{MLS}(\widehat{\mathcal{D}}) := \mathcal{MLS}(\mathcal{D})$. It encodes only partially the information about periodic orbits of $\widehat{\mathcal{D}}$, which are much more numerous than those of $\mathcal{D}$.

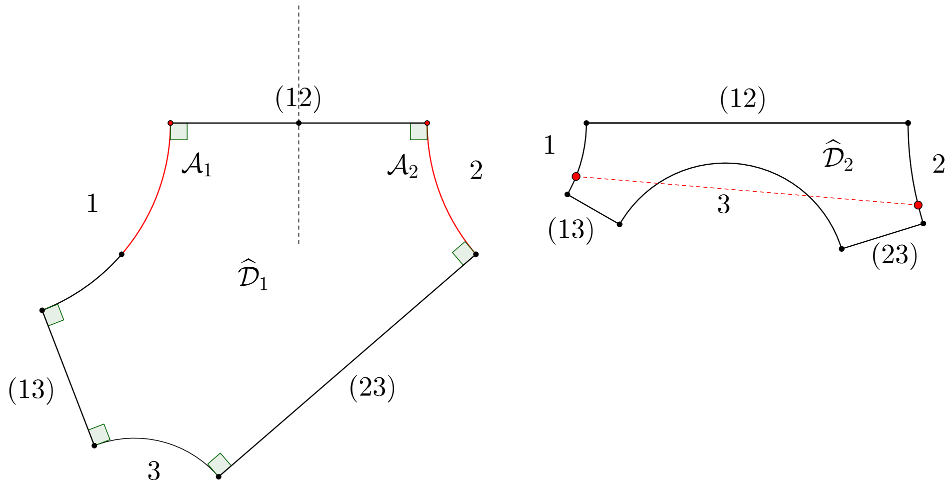


FIGURE 1. Hexagonal domains bounded by the trace of 2-periodic orbits.

Similarly, we denote by $\widehat{\mathbf{B}}_{\rm sym} \subset \widehat{\mathbf{B}}$ the subset of tables which have an additional symmetry as the above table $\widehat{\mathcal{D}}_1$, i.e., such that the jets of the curvature $\mathcal{K}$ at the endpoints of (12) are equal and even.

In this setting, our Main Theorem can be rephrased as follows:

Main Theorem (alternate version). *There exists an open and dense set of billiard tables $\widehat{\mathbf{B}}^*_{\rm sym} \subset \widehat{\mathbf{B}}_{\rm sym}$ so that if $\widehat{\mathcal{D}} \in \widehat{\mathbf{B}}^*_{\rm sym}$, then the geometry of $\widehat{\mathcal{D}}$ is entirely determined by the Marked Length Spectrum $\mathcal{MLS}(\widehat{\mathcal{D}})$.*

1.2. **Idea of the proof.** Let us give an idea of the proof of the above results. We fix a billiard table $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^3 \mathcal{O}_i \in \mathbf{B}_{\rm sym}$, and let $\mathcal{F} := \mathcal{F}(\mathcal{D})$. Note that it is natural to focus on 2-periodic orbits, since we may hope to determine $\mathcal{F}^2$ instead of $\mathcal{F}^p$ for some higher exponent $p \geq 3$, and also because of the additional symmetries of such orbits. As we shall see, the Birkhoff coefficients $(a_\ell)_{\ell\geq 1}$ above are directly related to the variation of the Lyapunov exponent for certain periodic orbits which spend a lot of time near the periodic orbit (12). As in [BDKL], we define a sequence of periodic orbits $(h_n)_n$ with a certain palindromic symmetry that accumulate some orbit h_∞ homoclinic to (12).

A key step in the construction is the extension of the coordinates given by the conjugacy R between $\mathcal{F}^2$ and its Birkhoff Normal Form, which is initially defined only in a neighborhood of the saddle fixed point. Indeed, in order to make the connection with the Lyapunov exponent of the orbits $(h_n)_n$, it is crucial to extend the conjugacy to describe them globally. The construction of the extension follows a classical procedure, by using the dynamics to propagate R along the separatrices, i.e., the stable and unstable manifolds of the origin. This is actually sufficient to describe all points in the orbits $(h_n)_n$, since for n large enough, these orbits stay in a small neighborhood of the separatrices. In this way, we produce convenient coordinates to describe the dynamics in a neighborhood of the separatrices, which can be seen as a hyperbolic analog of the coordinates provided by the Birkhoff Normal Form near the boundary of the billiard table, in the elliptic case, and which were used for instance in [CdV]. The problem is that we are extending our coordinates along two different directions, and at some point, since the trajectory is periodic, these two extensions will overlap in the collision space. In particular, we will need to perform a "gluing" of the two charts obtained in this way in a neighborhood of the homoclinic point on the third scatterer, and we will explain how to take care of this issue in the sequel.

By the palindromic property, we can write an equation for the images under the conjugacy map R of the points in the periodic orbits $(h_n)_n$ (see Lemma 4.6). This allows us to find an implicit expression of the parameters of those points in terms of the Birkhoff invariants and the coefficients of the arc of points where those orbits start (as we shall see, this arc is made of points on the second obstacle which bounce perpendicularly on the third obstacle after one iteration of the dynamics). As we have seen in [BDKL] (see Subsection 1.4), the Marked Lyapunov Spectrum is a $\mathcal{MLS}$-invariant. In Lemma 4.20, we show that for each integer $n \geq 0$, the Lyapunov exponent of h_n can be expanded as a series; more precisely, it holds

$$2\lambda^n \cosh(2(n+1)\mathrm{LE}(h_n)) = \sum_{p=0}^{+\infty} \sum_{q=0}^{p} L_{q,p} n^q \lambda^{np}, \tag{1.6}$$

for some sequence $(L_{q,p})_{\substack{p=0,\cdots,+\infty \\ q=0,\cdots,p}}$. In particular, each coefficient $L_{q,p}$ is a $\mathcal{MLS}$-invariant, and by restricting ourselves to $q = 0, 1, 2$, this gives enough information to recover the Birkhoff coefficients. Note that the expansion (1.6) obtained for the Lyapunov exponents of palindromic orbits in the horseshoe associated to the homoclinic orbit h_∞ can be seen as some "hyperbolic" analogue of the expansion obtained in [MM] (see for instance [MM, (1.11)] on p.3) for the maximum lengths of periodic orbits with a certain rotation number and period (the integer n being related to the period in either case). We also refer the reader to [FY] where similar expansions were studied for a different purpose.

One technical issue comes from the fact that the orbits $(h_n)_n$ bounce on the third obstacle, thus there are additional terms which come from a certain gluing map $\mathcal{G}$ taking this last bounce into account, and we need to find a way to recover this data as well. The idea is to leverage the "triangular" structure of the coefficients: at each step, there are certain additive constants associated with some terms that we already know, as well as new coefficients that we want to recover. Then, we derive a linear system in the new coefficients, and show that it is invertible under a suitable twist condition (non-vanishing of the first Birkhoff invariant). By induction, modulo some "homoclinic parameter" $\xi_\infty \in \mathbb{R}$, we can thus recover the Birkhoff invariants, as well as some information on the third obstacle associated to the differential of the gluing map $\mathcal{G}$.

More precisely, we consider some arc $\Gamma_\infty \ni (0, \xi_\infty)$ which is the image in Birkhoff coordinates of some small arc of points associated with a perpendicular bounce on the third scatterer (see also Figure 4.1). This arc is the graph of some analytic function γ that is determined by the gluing map $\mathcal{G} = (\mathcal{G}^+, \mathcal{G}^-)$, i.e., for ξ small, $\eta = \xi_\infty + \gamma(\xi)$ satisfies the implicit equation

$$\mathcal{G}^+(\xi, \eta)\mathcal{G}^-(\xi, \eta) = \xi\eta.$$

We show that from the sequence $(L_{q,p})_{\substack{p=0,\cdots,+\infty\\ q=0,1,2}}$, up to the parameter ξ_∞, it is possible to recover the value of the Birkhoff invariants, as well as the function γ and the differential $\mathcal{DG}|_{T\Gamma_\infty}$.

By looking only at the Marked Lyapunov Spectrum, we somehow forget the "scale" of the billiard table (note that homotheties preserve the Lyapunov exponents), and the missing parameter ξ_∞ can be seen as this "scaling factor". In Subsection 5.3, we prove that its value is a $\mathcal{MLS}$-invariant: we show (see Proposition 5.1 and Remark 5.4) that for some $\mathcal{L}^\infty \in \mathbb{R}$, the quantity

$$\mathcal{L}(h_n) - (n+1)\mathcal{L}(12) - \mathcal{L}^\infty \tag{1.7}$$

decays exponentially fast as n goes to infinity (see Section 2 for the notation), and can be expanded as a series of the same form as the one obtained in (1.6) – and also similar to the expansion obtained in [MM]. The first order term in this expansion is a $\mathcal{MLS}$-invariant and can be written in terms of ξ_∞^2 and of a certain quadratic form. By our previous results (see Subsection 1.4), the latter is a $\mathcal{MLS}$-invariant, thus ξ_∞ is a $\mathcal{MLS}$-invariant too.

Let us emphasize that all the results until Section 6 – in particular, the $\mathcal{MLS}$-determination of the Birkhoff Normal Form and of the gluing map $\mathcal{G}$ – do not require any symmetry assumption. Indeed, in our approach, axial symmetries are needed only to reconstruct the geometry from the Birkhoff Normal Form and the map $\mathcal{G}$.

Let us now consider the case of symmetric billiard tables $\mathcal{D} \subset \mathbf{B}_{\mathrm{sym}}$. In this case, we can introduce some flat wall between $\mathcal{O}_1$ and $\mathcal{O}_2$, and "fold" the table in order to virtually create a fixed point of the billiard dynamics. For some auxiliary billiard table $\mathcal{D}^*$ one of whose obstacles is now flat, we can extract enough information from $\mathcal{MLS}(\mathcal{D})$ to reconstruct the Birkhoff Normal Form N^* of the square $T^* := (\mathcal{F}^*)^2$ of the new billiard map $\mathcal{F}^* = \mathcal{F}^*(\mathcal{D}^*)$ near the new 2-periodic orbit. Besides, by the construction of N^* in [Mos], and *due to the symmetry of* $\mathcal{O}_1, \mathcal{O}_2$*, the jets of* N^* *and* T^* *are in one-to-one correspondence, by some invertible triangular system.* Furthermore, as Colin de Verdière [CdV] already observed in the elliptic setting, there is a bijective correspondence between the jet of T^* and the jet of the graphs of $\mathcal{O}_1, \mathcal{O}_2$, which can thus be reconstructed.

In order to recover the geometry of the last obstacle, we analyze the information that comes from the gluing map that we were mentioning previously. We can extract from this map some "averaged" information between the first two obstacles $\mathcal{O}_1, \mathcal{O}_2$ and the third obstacle $\mathcal{O}_3$, and since the geometry of $\mathcal{O}_1, \mathcal{O}_2$ is known, we can also reconstruct the local geometry of $\mathcal{O}_3$ near a certain homoclinic point. This determines the obstacle $\mathcal{O}_3$ entirely, by analyticity.

1.3. **Organization of the paper.** We use the notations introduced in Subsection 1.2. The proof of the Main Theorem follows different steps:

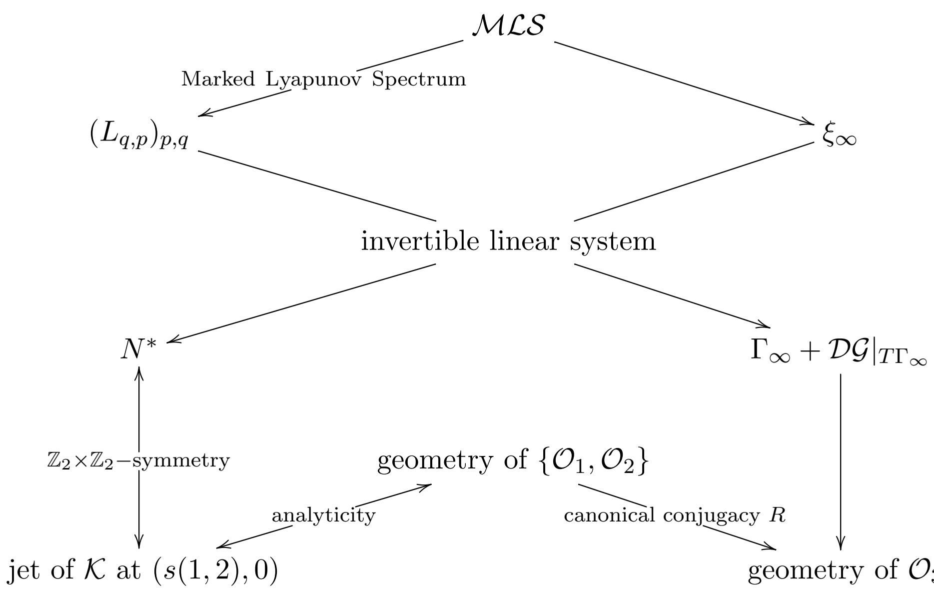
* **Step 1:** existence of a *canonical* (which respects the symmetries of the billiard dynamics) change of coordinates under some non-degeneracy condition;
* **Step 2:** extension of the system of coordinates and expression of the palindromic orbits $(h_n)_n$ in those coordinates;
* **Step 3:** definition of the gluing map $\mathcal{G}$;
* **Step 4:** asymptotic expansion (in n) of the Lyapunov exponent of h_n;
* **Step 5:** extracting a triangular system from the Lyapunov expansion in terms of "scaled" Birkhoff coefficients and gluing terms;

- $*$ **Step 6:** invertibility of this system under some non-degeneracy condition (proof by induction: compute the i^{th} order terms using the previous ones);
- $*$ **Step 7:** determination of the missing "scale" parameter through $\mathcal{MLS}$;
- $*$ **Step 8:** Step 5 + Step 6 $\Rightarrow$ the Birkhoff Normal Form and the differential of the gluing map are $\mathcal{MLS}$-invariants;
- $*$ **Step 9:** determination of the geometry from the Birkhoff invariants + the gluing map $\mathcal{G}$ in the case of symmetric billiard tables which satisfy a non-degeneracy condition $(\star)$.

Those steps are detailed respectively in:
(1) Section 2; **(2)** Section 3; **(3)** Subsection 4.1; **(4)**-**(5)** Subsection 4.2; **(6)** Subsection 4.3; **(7)** Subsections 5.1-5.2; **(8)** Subsection 5.3; **(9)** Section 6.

The central technical part of the proof is Steps 4-5; an outline of the computations carried out there is given after Remark 4.9 (see also Remark 4.16). Let us also emphasize formula (4.6) which follows from the palindromic symmetry of the orbits $(h_n)_n$ and on which the induction is based.

Moreover, the scheme of the proof can be summarized as follows:



1.4. **Previous results.** Let us also recall some results we obtained previously in [BDKL], joint with P. Bálint, and that will be needed in the following.

Theorem 1.10 ([BDKL, Theorem A])**.** *Consider a 2-periodic orbit encoded by a word $\sigma = (\sigma_1\sigma_0) \in \{1,2,3\}^2$, with $\sigma_0 \neq \sigma_1$. Let τ_1 be such that $\{\tau_1, \sigma_0, \sigma_1\} = \{1,2,3\}$, and set $\tau := (\tau_1\sigma_0)$. We denote by $R_0, R_1 > 0$ the respective radii of curvature at the points with symbols σ_0 and σ_1, and we let $\lambda = \lambda(\sigma) < 1$ be the smallest eigenvalue of $D\mathcal{F}^2$ at the points of σ.*

Then, for n sufficiently large, the following estimates hold:

$$\mathcal{L}(\tau\sigma^{2n}) - (2n+1)\mathcal{L}(\sigma) - \mathcal{L}^\infty = -C\cdot\mathcal{Q}\left(\frac{2R_0}{\mathcal{L}(\sigma)}, \frac{2R_1}{\mathcal{L}(\sigma)}\right)\lambda^{2n} + O(\lambda^{3n}),$$

$$\mathcal{L}(\tau\sigma^{2n+1}) - (2n+2)\mathcal{L}(\sigma) - \mathcal{L}^\infty = -C\cdot\mathcal{Q}\left(\frac{2R_1}{\mathcal{L}(\sigma)}, \frac{2R_0}{\mathcal{L}(\sigma)}\right)\lambda^{2n+1} + O(\lambda^{3n}),$$

for some real number $\mathcal{L}^\infty = \mathcal{L}^\infty(\sigma,\tau) \in \mathbb{R}$, some constant $C = C(\sigma,\tau) > 0$, and the quadratic form $\mathcal{Q}\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$:

$$\mathcal{Q}(X,Y) := (1+\lambda^2)(1+X)^2 - (1+\lambda)^2 XY + 2\lambda(1+Y)^2.$$

Corollary 1.11 ([BDKL, Corollary C])**.** *The radii of curvature R_0 and R_1 at the bouncing points of periodic orbits of period two are $\mathcal{MLS}$-invariants.*

By studying general periodic orbits, we have also obtained the following result.

Theorem 1.12 ([BDKL, Corollary E])**.** *The Marked Lyapunov Spectrum is determined by the Marked Length Spectrum (see* (1.3) *and* (1.5) *for the definitions).*

2. The Birkhoff Normal Form in a neighborhood of period two orbits

Let us fix a billiard table $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^3 \mathcal{O}_i \in \mathbf{B}$ and study the local dynamics near 2-periodic orbits. Without loss of generality, we focus on the 2-periodic orbit $\sigma = (12)$; it has two perpendicular bounces on the first and the second obstacles. Let us denote by $x(0) = (s(0), 0)$ and $x(1) = (s(1), 0)$ the coordinates of the points in this orbit, where $s(0)$, (resp. $s(1)$) is the position of the point on the first (resp. second) obstacle. We extend this notation by periodicity by setting $x(k) := x(k \bmod 2)$, for $k \in \mathbb{Z}$. We let $\tau := (32)$ and, given any integer $n \geq 1$, we set

$$h_n = h_n(\sigma,\tau) := (\tau\sigma^n) = (32\underbrace{1212\ldots12}_{2n}).$$

The word h_n encodes a periodic orbit of period $2n+2$.

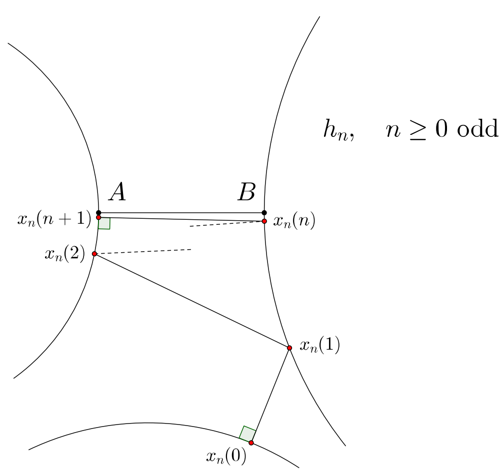


Figure 2. Trace of the orbits h_n on the billiard table when $n \geq 0$ is odd.

Let $\mathcal{F} = \mathcal{F}(\mathcal{D})\colon x_0 = (s_0, r_0) \mapsto x_1 = (s_1, r_1)$ be the billiard map. In such coordinates, $\mathcal{F}$ is exact symplectic, with generating function

$$h(s,s') := \|\Upsilon(s) - \Upsilon(s')\|, \tag{2.1}$$

where $\|\Upsilon(s)-\Upsilon(s')\|$ is the Euclidean length of the line segment between the two points identified by parameters s and s'. In other words, we have

$$dh(s_0,s_1) = -r_0 ds_0 + r_1 ds_1, \tag{2.2}$$

i.e.,

$$\partial_1 h(s_0,s_1) = -r_0, \quad \partial_2 h(s_0,s_1) = r_1.$$

Let us denote by $(x_n(k)=(s_n(k),r_n(k)))_{k=0,\cdots,2n+1}$ the coordinates of the points in the orbit h_n, where $x_n(0)=(s_n(0),0)$ is the only collision on the third obstacle, and $x_n(n+1)=(s_n(n+1),0)$ is the point of the orbit which is closest to the periodic orbit $\sigma=(12)$ (see Figure 2). Again, thanks to the $2n+2$-periodicity of h_n, we can extend those coordinates to any $k\in\mathbb{Z}$. As we have already observed in [BDKL] (see Lemma 3.2 in this paper), by the palindromic symmetry, for any $k\in\{0,\cdots,n+1\}$, it holds

$$x_n(2n+2-k) = \mathcal{I}(x_n(k)), \tag{2.3}$$

where $\mathcal{I}(s,r):=(s,-r)$. Recall that for a periodic orbit encoded by a finite word ς, we denote by $\mathcal{L}(\varsigma)$ its total perimeter. We have

$$\mathcal{L}(h_n)-(n+1)\mathcal{L}(\sigma) = 2\sum_{k=0}^{n}\Big(h(s_n(k),s_n(k+1))-h(s(k),s(k+1))\Big).$$

Let $h_\infty=h_\infty(\sigma,\tau)$ be the homoclinic trajectory encoded by the infinite word $(\sigma^\infty\tau\sigma^\infty)=(\dots 21212321212\dots)$. We denote by $(x_\infty(k))_{k\in\mathbb{Z}}$ its coordinates, with $x_\infty(k)=(s_\infty(k),r_\infty(k))$, for $k\in\mathbb{Z}$. We label them in such a way that $x_\infty(0)$ is associated with the unique bounce on the third obstacle, and $r_\infty(k)r_n(k)\geq 0$ for all $k\in\mathbb{Z}$. As we have shown in [BDKL], the terms $h(s_\infty(k),s_\infty(k+1))-h(s(k),s(k+1))$ decay exponentially fast as $k\to+\infty$, and the limit $\mathcal{L}^\infty=\mathcal{L}^\infty(\sigma):=\lim_{n\to\infty}(\mathcal{L}(h_n)-(n+1)\mathcal{L}(\sigma))$ is well defined:

$$\mathcal{L}^\infty = \lim_{n\to\infty}(\mathcal{L}(h_n)-(n+1)\mathcal{L}(\sigma)) = 2\sum_{k=0}^{+\infty}\Big(h(s_\infty(k),s_\infty(k+1))-h(s(k),s(k+1))\Big).$$

Then, we have

$$\mathcal{L}(h_n)-(n+1)\mathcal{L}(\sigma)-\mathcal{L}^\infty = \Sigma_n^1+\Sigma_n^2,$$

where

$$\Sigma_n^1 := 2\sum_{k=0}^{n}\Big(h(s_n(k),s_n(k+1))-h(s_\infty(k),s_\infty(k+1))\Big), \tag{2.4}$$

$$\Sigma_n^2 := 2\sum_{k=n+1}^{+\infty}\Big(h(s(k),s(k+1))-h(s_\infty(k),s_\infty(k+1))\Big). \tag{2.5}$$

The point $x(1)=(s(1),0)$ is a saddle fixed point of $\mathcal{F}^2$, with eigenvalues $\lambda<1$ and $\lambda^{-1}>1$.

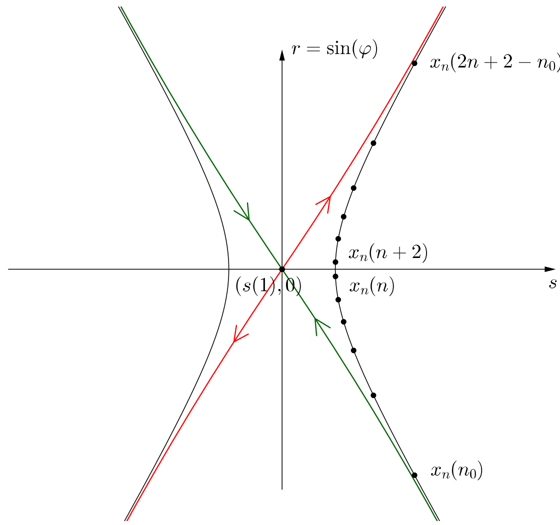


FIGURE 3. (s,r)-representation of the points in h_n near the orbit (12).

Let us recall the following result of [BDKL].

Proposition 2.1 ([BDKL, Proposition 4.1])**.** *There exists an integer $n_* \geq 1$ such that the following holds:*

$$\begin{aligned} \|x_\infty(k)-x(k)\| &= O(\lambda^{\frac{k}{2}}), && \textit{for all } k\in\mathbb{N},\\ \|x_n(k)-x_\infty(k)\| &= O(\lambda^{n-\frac{|k|}{2}}), && \textit{for all } n\geq n_* \textit{ and } k\in\{0,\cdots,n+1\}. \end{aligned}$$

The first estimate tells us that the points in the homoclinic orbit are on the stable manifold of the point $x(1)$. Moreover, as n goes to infinity, everything happens in a neighborhood of the unstable and stable manifolds of the periodic orbit σ. Indeed, for the first half of the orbit h_n, i.e., for $k\in\{0,\cdots,n+1\}$, the second estimate above tells us that the points $x_n(k)$ shadow closely the associated points $x_\infty(k)$ in the homoclinic orbit h_∞, and thus, stay close to the stable manifold of $x(1)$. On the other hand, for the second half of h_n, i.e., for $k\in\{n+1,\cdots,2n+2\}$, then by the palindromic symmetry (2.3), the points $x_n(k)$ shadow closely the points $\mathcal{I}(x_\infty(k))$, and thus, stay close to the unstable manifold of $x(1)$.

Let us consider the case where n is odd, i.e., $n=2m-1$ for some integer $m\geq 1$, and let us study the dynamics of $T:=\mathcal{F}^2$. Here, the period of $h_n=h_{2m-1}$ is equal to $2n+2=4m$. For simplicity, we assume in the following that $s(1)=0$.

By (2.2), the map $\mathcal{F}$ is symplectic for the form $ds\wedge dr$, where $r:=\sin(\varphi)$. It follows that T is symplectic too, i.e., $T^*(ds\wedge dr)=ds\wedge dr$. Then, by [Mos] (see also [Ste]), there exists a neighborhood $\mathcal{U}$ of $(0,0)$ in the (s,r)-plane, and an analytic symplectic change of coordinates

$$R\colon \begin{cases} \mathcal{U} & \to \quad \mathbb{R}^2,\\ (s,r) & \mapsto \quad (\xi,\eta) \end{cases}$$

with $d\xi \wedge d\eta = ds \wedge dr$, and which conjugates T to its Birkhoff Normal Form $N = R \circ T \circ R^{-1}$:

$$N = N_\Delta \colon (\xi, \eta) \mapsto (\Delta(\xi\eta) \cdot \xi, \Delta(\xi\eta)^{-1} \cdot \eta),$$

for some analytic function $\Delta \colon z \mapsto \lambda + \sum_{k=1}^{+\infty} a_k z^k$:

$$\Delta(\xi\eta) = \lambda + a_1 \xi\eta + a_2 (\xi\eta)^2 + \dots$$

The numbers $(a_k)_{k \geq 1}$ are called the *Birkhoff invariants* or *coefficients* of T at $(0,0)$.

In the following, we will denote by $\mathrm{Sympl}^\omega(\mathbb{R}^2, \mathbb{R}^2)$ the set of real analytic symplectomorphisms of $\mathbb{R}^2$ which preserve the form $d\xi \wedge d\eta$.

2.1. **Canonical choice of the conjugacy map R.** The following lemma ensures that the map N is well defined in a neighborhood of the vertical and horizontal axes.

Lemma 2.2. *For all $(\xi, \eta) \in R(\mathcal{U})$, and for each $k \in \mathbb{Z}$, the point $(\Delta(\xi\eta)^{-k}\xi, \Delta(\xi\eta)^k \eta)$ is in the domain of definition of N. In particular, N is well defined at each point in the orbit of (ξ, η).*

Proof. Let $k \in \mathbb{Z}$. Clearly, N is well defined at a point $(\Delta(\xi\eta)^{-k}\xi, \Delta(\xi\eta)^k\eta)$ if and only if Δ is well defined at the point $(\Delta(\xi\eta)^{-k}\xi) \cdot (\Delta(\xi\eta)^k \eta) = \xi\eta$, which is true, provided that $(\xi, \eta) \in R(\mathcal{U})$. □

Lemma 2.3. *Let $\mathcal{D} \in \mathbf{B}$ be such that the Birkhoff invariants $(a_k)_{k \geq 1}$ are not all equal to zero. Then there exists a neighborhood $\mathcal{V}$ of $(0,0)$ in $\mathbb{R}^2$ such that the centralizer of $N|_\mathcal{V}$, which is defined as*

$$\mathcal{C}_N^\mathcal{V} := \{F \in \mathrm{Sympl}^\omega(\mathbb{R}^2, \mathbb{R}^2) : F \circ N|_\mathcal{V} = N \circ F|_\mathcal{V}\},$$

is reduced to the set of maps of the same form as N: for any $F \in \mathcal{C}_N^\mathcal{V}$, we have

$$F|_\mathcal{V} = N_{\widetilde{\Delta}}|_\mathcal{V} \colon (\xi, \eta) \mapsto (\widetilde{\Delta}(\xi\eta)\xi, \widetilde{\Delta}(\xi\eta)^{-1}\eta), \text{ for some } \widetilde{\Delta} \in C^\omega(\mathbb{R}, \mathbb{R}^*). \tag{2.6}$$

Proof. Clearly, any map of the above form commutes with N. The equality of the two sets follows from the fact that any such map has to respect the symmetries of N; in particular, it has to map hyperbolas to hyperbolas, and preserve the rate of contraction/expansion along each of them.

More precisely, let us take $F \in \mathcal{C}_N^\mathcal{V}$ for some neighborhood $\mathcal{V}$ of $(0,0)$ to be chosen later, with $F \colon (\xi, \eta) \mapsto (u(\xi, \eta), v(\xi, \eta))$. For any $j \geq 0$, we have $F \circ N^j = N^j \circ F$, and then, by projection on the two coordinates, for each $(\xi, \eta) \in \mathcal{V}$, we get

$$u(\Delta^j(\xi\eta)\xi, \Delta^{-j}(\xi\eta)\eta) = \Delta^j(u(\xi,\eta)v(\xi,\eta))u(\xi,\eta),$$
$$v(\Delta^j(\xi\eta)\xi, \Delta^{-j}(\xi\eta)\eta) = \Delta^{-j}(u(\xi,\eta)v(\xi,\eta))v(\xi,\eta).$$

For $\eta = 0$, we have $\Delta(\xi\eta) = \lambda$, so that

$$\Delta^j(u(\xi,0)v(\xi,0))v(\lambda^j\xi, 0) = v(\xi, 0).$$

By letting $j \to +\infty$, we deduce that $v(\xi, 0) = 0$. Similarly, we have $u(0, \eta) = 0$ for all $\eta \in \mathbb{R}$. In particular, $F(0,0) = (0,0)$. Let us write

$$u(\xi, \eta) = \sum_{k,\ell \geq 0} u_{k,\ell}\xi^k\eta^\ell.$$

The above equation yields:

$$\begin{aligned}\Delta^j(u(\xi,\eta)v(\xi,\eta))u(\xi,\eta) &= u(\Delta^j(\xi\eta)\xi, \Delta^{-j}(\xi\eta)\eta)\\ &= \sum_{k,\ell\geq 0} u_{k,\ell}\Delta^{j(k-\ell)}(\xi\eta)\xi^k\eta^\ell.\end{aligned}$$

The left hand side goes to zero as j goes to infinity, hence $u_{k,\ell}=0$ for $\ell\geq k$. We deduce that for $j\gg 1$,[2]

$$\Delta^j(u(\xi,\eta)v(\xi,\eta))u(\xi,\eta)\sim \Delta^j(\xi\eta)\cdot\eta^{-1}\sum_{k\geq 1}u_{k,k-1}(\xi\eta)^k,$$

and thus,

$$\left(\frac{\Delta(u(\xi,\eta)v(\xi,\eta))}{\Delta(\xi\eta)}\right)^j\sim\frac{\eta^{-1}\sum_{k\geq 1}u_{k,k-1}(\xi\eta)^k}{u(\xi,\eta)}.$$

Since the right hand side does not depend on j, we obtain

$$\Delta(u(\xi,\eta)v(\xi,\eta))=\Delta(\xi\eta), \tag{2.7}$$

and also

$$u(\xi,\eta)=\eta^{-1}\sum_{k\geq 1}u_{k,k-1}(\xi\eta)^k=\sum_{k\geq 1}u_{k,k-1}\xi^k\eta^{k-1}. \tag{2.8}$$

Let c_1 be the analytic function $z\mapsto\sum_{k\geq 1}u_{k,k-1}z^k$, so that $u(\xi,\eta)\eta=c_1(\xi\eta)$. Similarly, we have $v(\xi,\eta)\xi=: c_2(\xi\eta)$ for some analytic function c_2. For any $z\neq 0$, we set $c(z):=\frac{c_1(z)c_2(z)}{z}$. Then, for each $(\xi,\eta)\in\mathcal{V}\backslash\{(0,0)\}$, it holds

$$u(\xi,\eta)v(\xi,\eta)=\frac{c_1(\xi\eta)c_2(\xi\eta)}{\xi\eta}=c(\xi\eta).$$

Therefore, F maps the hyperbola $\mathcal{H}_\flat=\{\xi\eta=c_\flat\}$ to the hyperbola $\mathcal{H}_\sharp=\{uv=c_\sharp\}$, where $c_\sharp:=c(c_\flat)$. By (2.7), we also have $\Delta(uv)=\Delta(\xi\eta)$, i.e., $\Delta(c(\xi\eta))=\Delta(\xi\eta)$. Note that $\lim_{\xi\eta\to 0}c(\xi\eta)=\lim_{\xi\eta\to 0}u(\xi,\eta)v(\xi,\eta)=0$. By assumption, the Birkhoff invariants $(a_k)_{k\geq 1}$ are not all equal to zero. Let $k_0\geq 1$ be the smallest positive integer such that $a_{k_0}\neq 0$. Then, $\Delta(z)-\lambda\sim_0 a_{k_0}z^{k_0}$ near the origin, and thus, there exists a neighborhood $\mathcal{N}_0$ of 0 such that $\Delta|_{\mathbb{R}_+\cap\mathcal{N}_0}$ is strictly monotonic. Let us assume that the neighborhood $\mathcal{V}$ was chosen in such a way that $\xi\eta\in\mathcal{N}_0$ for all $(\xi,\eta)\in\mathcal{V}$. It follows from the previous discussion that for all $(\xi,\eta)\in\mathcal{V}$, we have

$$\Delta(c(\xi\eta))-\lambda=\sum_{k\geq k_0}a_k(c(\xi\eta))^k=\sum_{k\geq k_0}a_k(\xi\eta)^k=\Delta(\xi\eta)-\lambda,$$

and then, $c(\xi\eta)=\xi\eta$, by the strict monotonicity of $\Delta|_{\mathbb{R}_+\cap\mathcal{N}_0}$. In other words, since F maps hyperbolas to hyperbolas, the local non-degeneracy of Δ together with (2.7) compel F to fix each hyperbola near the origin, i.e.,

$$c(\xi\eta)=u(\xi,\eta)v(\xi,\eta)=\xi\eta.$$

For any $(\xi,\eta)\in\mathcal{V}$ such that $\xi\eta\neq 0$, let us set

$$\widetilde{\Delta}(\xi\eta):=\frac{u(\xi,\eta)}{\xi}=\frac{c(\xi\eta)}{v(\xi,\eta)\xi}=\frac{\xi\eta}{v(\xi,\eta)\xi}=\frac{\eta}{v(\xi,\eta)}.$$

[2] By symplecticity, $u_{1,0}v_{0,1}=1$ hence the right hand side is different from zero for $\xi\eta\ll 1$.

For any $(\xi,\eta)\neq(0,0)$, we also have $\widetilde{\Delta}(\xi\eta)=\frac{u(\xi,\eta)}{\xi}=\sum_{j\geq 0}u_{j+1,j}(\xi\eta)^j$, thus we set $\widetilde{\Delta}(0):=\lim_{(\xi,\eta)\to(0,0)}\frac{u(\xi,\eta)}{\xi}=u_{1,0}=v_{0,1}^{-1}$. We conclude that $\widetilde{\Delta}\in C^\omega(\mathbb{R},\mathbb{R}^*)$, and

$$F(\xi,\eta)=(u(\xi,\eta),v(\xi,\eta))=(\widetilde{\Delta}(\xi\eta)\xi,\widetilde{\Delta}(\xi\eta)^{-1}\eta)=N_{\widetilde{\Delta}}(\xi,\eta),$$

as desired. □

Remark 2.4. *In (s,r)-coordinates, the horizontal axis $\{r=0\}=\{\varphi=0\}$ plays a special role, because of the reflection symmetry of the billiard map:*

$$\mathcal{F}(s,r)=(s',r')\quad\Longleftrightarrow\quad\mathcal{F}(s',-r')=(s,-r).$$

This time-reserval symmetry also exchanges the stable and unstable spaces. In (ξ,η)-coordinates, the stable space is the horizontal axis $\{\eta=0\}$, while the unstable space is the vertical axis $\{\xi=0\}$. Moreover, 2-periodic points are on the axis of symmetry $\{r=0\}$ – and more generally, all the points associated to perpendicular bounces in palindromic orbits – hence their stable and unstable manifolds are symmetric with respect to $\{r=0\}$. It is thus natural to require the new axis of symmetry to be $\{\xi=\eta\}$. By the previous study, under some non-degeneracy condition, maps in the centralizer of N translate points along hyperbolas $\{\xi\eta=Const\}$, hence typically, they do not preserve the axis $\{\xi=\eta\}$. As a consequence, there is a canonical choice for the conjugacy map R defined above, which preserves this symmetry.

In the following, we assume that the Birkhoff invariants $(a_k)_{k\geq 1}$ are not all equal to zero, and that the neighborhood $\mathcal{U}$ in the definition of the change of coordinates R introduced at the beginning of Section 2 is sufficiently small such that the neighborhood $\mathcal{V}:=R(\mathcal{U})\subset\mathbb{R}^2$ of $(0,0)$ satisfies the conclusion of Lemma 2.3.

Corollary 2.5. *Assume that the Birkhoff invariants $(a_k)_{k\geq 1}$ are not all equal to zero. Then, there exists a unique map $R_0\in\mathrm{Sympl}^\omega(\mathbb{R}^2,\mathbb{R}^2)$ such that*

$$R_0\circ\mathcal{F}^2|_{\mathcal{U}}=N\circ R_0|_{\mathcal{U}},\qquad\textit{and}\qquad R_0(\{(s,0)\ s\geq 0\})=\{(\xi,\xi),\ \xi\geq 0\}.$$

Proof. Let us start by showing the uniqueness of R_0. Let $R,\widetilde{R}$ be two such maps. Then $R^{-1}\circ N\circ R=\widetilde{R}^{-1}\circ N\circ\widetilde{R}=\mathcal{F}^2$, so that

$$(\widetilde{R}R^{-1})\circ N\circ(\widetilde{R}R^{-1})^{-1}=N,$$

and $\widetilde{R}R^{-1}\in\mathcal{C}_N^{\mathcal{V}}$. By Lemma 2.3, the centralizer $\mathcal{C}_N^{\mathcal{V}}$ is reduced to the set of maps which translate points along hyperbolas $\{\xi\eta=\mathrm{Const}\}$, and then

$$\widetilde{R}R^{-1}(\xi,\eta)=N_{\widetilde{\Delta}}(\xi,\eta)=(\widetilde{\Delta}(\xi\eta)\xi,\widetilde{\Delta}(\xi\eta)^{-1}\eta),$$

for some real analytic map $\widetilde{\Delta}\in C^\omega(\mathbb{R},\mathbb{R}^*)$. Since both R and $\widetilde{R}$ fix the positive axis $\{(\xi,\xi),\ \xi\geq 0\}$, so does $N_{\widetilde{\Delta}}$, and then,

$$(\widetilde{\Delta}(\xi^2)\xi,\widetilde{\Delta}(\xi^2)^{-1}\xi)=(\tilde{\xi}(\xi),\tilde{\xi}(\xi)),\qquad\forall\,\xi\in\mathbb{R},$$

for some function $\tilde{\xi}\colon\mathbb{R}\to\mathbb{R}$. By taking the product of the two coordinates, we deduce that $\xi^2=(\tilde{\xi}(\xi))^2$, and then $\tilde{\xi}(\xi)=\xi$, since $\xi,\tilde{\xi}\geq 0$. We deduce that $\widetilde{\Delta}\equiv 1$, and then $\widetilde{R}R^{-1}=\mathrm{id}$, which concludes the proof of uniqueness..

To show the existence of such a map R_0, let us fix an analytic symplectomorphism $R(s,r)=(\xi(s,r),\eta(s,r))$ such that $R\circ T\circ R^{-1}=N$.

After possibly composing R with $-\mathrm{id}$,[3] we may assume that $\xi(s,0),\eta(s,0)\geq 0$ for all $s\geq 0$. Let $R^{-1}\colon(\xi,\eta)\mapsto(\mathcal{S}(\xi,\eta),\mathcal{R}(\xi,\eta))$, and let $\pi\colon(\xi,\eta)\mapsto(\pi_1(\xi\eta),\pi_2(\xi\eta))$ be the projection along hyperbolas $\{\xi\eta=\mathrm{Const}\}$ onto the set $\{\mathcal{R}(\xi,\eta)=0\}$. We denote by $\theta\in[0,\frac{\pi}{2}]$ the angle between the positive parts of the horizontal axis and of the unstable space of T. Since the coordinate η vanishes only on the stable space $\{r=\tan(\theta)s\}$, we may define $\delta(\xi\eta):=\sqrt{\frac{\pi_2(\xi\eta)}{\pi_1(\xi\eta)}}$, and we set $N_\delta\colon(\xi,\eta)\mapsto(\delta(\xi\eta)\xi,\delta^{-1}(\xi\eta)\eta)$. Clearly, $N_\delta\in\mathrm{Sympl}^\omega(\mathbb{R}^2,\mathbb{R}^2)$, and $N_\delta(\{\mathcal{R}(\xi,\eta)=0,\ \xi,\eta\geq 0\})=\{(\tilde{\xi},\tilde{\xi}),\ \tilde{\xi}\geq 0\}$. Then, the map $R_0\colon(s,r)\mapsto N_\delta\circ R(s,r)$ satisfies the required conditions:

$$R_0(s,0)=(\delta(\xi(s,0)\eta(s,0))\xi(s,0),\delta(\xi(s,0)\eta(s,0))^{-1}\eta(s,0))\in\{(\tilde{\xi},\tilde{\xi}),\ \tilde{\xi}\geq 0\},$$

and $R_0TR_0^{-1}=N_\delta RTR^{-1}N_\delta^{-1}=N_\delta NN_\delta^{-1}=N$. □

We call *Birkhoff Coordinates* the coordinates (ξ,η) obtained via the change of coordinates R_0.

2.2. **The time reversal involution in Birkhoff coordinates.** By the time-reversal property, the map $\mathcal{I}\colon(s,r)\mapsto(s,-r)$ conjugates the billard map $\mathcal{F}$ to its inverse $\mathcal{F}^{-1}$, and thus, $\mathcal{I}\circ T\circ\mathcal{I}=T^{-1}$. Assume that the Birkhoff invariants $(a_k)_{k\geq 1}$ are not all equal to zero, and let $R_0\colon\mathcal{U}\to\mathbb{R}^2$ be the canonical symplectic change of coordinates given by Lemma 2.5. Since $R_0\circ T\circ R_0^{-1}=N$, we get

$$(R_0\circ\mathcal{I}\circ R_0^{-1})\circ N\circ(R_0\circ\mathcal{I}\circ R_0^{-1})=R_0\circ T^{-1}\circ R_0^{-1}=N^{-1}.$$

Set $\mathcal{I}^*:=R_0\circ\mathcal{I}\circ R_0^{-1}$. We thus have

$$\mathcal{I}^*\circ N\circ\mathcal{I}^*=N^{-1}. \tag{2.9}$$

Lemma 2.6. *The map $\mathcal{I}^*$ is the reflection along the bisectrix $\{\xi=\eta\}$:*

$$\mathcal{I}^*=\mathcal{I}_0\colon(\xi,\eta)\mapsto(\eta,\xi). \tag{2.10}$$

Proof. Let us write $\mathcal{I}^*(\xi,\eta)=(u,v)$, with $u=u(\xi,\eta)$ and $v=v(\xi,\eta)$. For every $(\xi,\eta)\in\mathbb{R}^2$, we have

$$u(\xi,0)=0,\qquad v(0,\eta)=0. \tag{2.11}$$

In other words, $\mathcal{I}^*$ maps the horizontal axis $\{\eta=0\}=\{(\xi,0):\xi\in\mathbb{R}\}$ to the vertical axis $\{\xi=0\}=\{(0,\eta):\eta\in\mathbb{R}\}$, and vice versa. Indeed it follows fom the definition of the map N that $\{\eta=0\}$ is the stable manifold of $(0,0)$, since $N^j(\xi,0)=(\lambda^j\xi,0)$, for $j\geq 0$, and similarly, $\{\xi=0\}$ is the unstable manifold of the origin. Moreover, (2.9) implies that N exchanges the stable manifold with the unstable manifold: given $p\in\mathbb{R}^2$ such that $\lim_{j\to+\infty}N^j(p)=(0,0)$, then its image $p^*:=\mathcal{I}^*(p)$ satisfies

$$\lim_{j\to+\infty}N^{-j}(p^*)=\mathcal{I}^*(\lim_{j\to+\infty}N^j(p))=\mathcal{I}^*(0,0)=(0,0).$$

Here, we have used that $\mathcal{I}^*(0,0)=(0,0)$ (by (2.11)).

Moreover, by (2.9), we know that $\mathcal{I}^*\circ N^{-1}=N\circ\mathcal{I}^*$. Therefore, given any $(\xi,\eta)\in\mathbb{R}^2$, we obtain

$$\mathcal{I}^*(\Delta(\xi\eta)^{-1}\xi,\Delta(\xi\eta)\eta)=(\Delta(u(\xi,\eta)v(\xi,\eta))u(\xi,\eta),\Delta(u(\xi,\eta)v(\xi,\eta))^{-1}v(\xi,\eta)).$$

[3] By symplecticity, we do not need to consider the reflections $(\xi,\eta)\mapsto(\xi,-\eta)$ or $(\xi,\eta)\mapsto(-\xi,\eta)$.

In particular, by considering the projection on the first coordinate, we get

$$u(\Delta(\xi\eta)^{-1}\xi, \Delta(\xi\eta)\eta) = \Delta(u(\xi,\eta)v(\xi,\eta))u(\xi,\eta). \tag{2.12}$$

For $\xi = 0$, by the power series expansion of Δ, and by (2.11), we have $\Delta(\xi\eta) = \Delta(0\cdot\eta) = \Delta(0) = \lambda$ and $\Delta(u(\xi,\eta)v(\xi,\eta)) = \Delta(u(0,\eta)v(0,\eta)) = \Delta(u(0,\eta)\cdot 0) = \Delta(0) = \lambda$. We deduce from (2.12) that for any $\eta \in \mathbb{R}$,

$$u(0,\lambda\eta) = \lambda u(0,\eta).$$

By considering the power series expansion $u(\xi,\eta) = \sum_{k,\ell\geq 0} u_{k,\ell}\xi^k\eta^\ell$, this relation implies that $u_{0,\ell} = 0$ for all $\ell \neq 1$, and then,

$$u(0,\eta) = u_{0,1}\eta.$$

Besides, for any $(\xi,\eta) \in \mathbb{R}^2$, and any $j \geq 0$, we have $\mathcal{I}^* \circ N^j = N^{-j} \circ \mathcal{I}^*$. Similarly, by projecting on the first coordinate, we obtain

$$u(\Delta(\xi\eta)^j\xi, \Delta(\xi\eta)^{-j}\eta) = \Delta(u(\xi,\eta)v(\xi,\eta))^{-j}u(\xi,\eta). \tag{2.13}$$

For any $j \geq 0$, we have

$$\begin{aligned} u(\xi,\eta) &= \Delta(u(\xi,\eta)v(\xi,\eta))^j \cdot u(\Delta(\xi\eta)^j\xi, \Delta(\xi\eta)^{-j}\eta) \\ &= \left(\frac{\Delta(u(\xi,\eta)v(\xi,\eta))}{\Delta(\xi\eta)}\right)^j \cdot \left(u_{0,1}\eta + \Delta(\xi\eta)^j \sum_{k=1}^{+\infty}\sum_{\ell=0}^{+\infty} u_{k,\ell}\Delta(\xi\eta)^{j(k-\ell)} \cdot \xi^k\eta^\ell\right), \end{aligned}$$

where we have used that $u(0,\Delta(\xi\eta)^{-j}\eta) = u_{0,1}\Delta(\xi\eta)^{-j}\eta$. Since the left hand side is bounded independently of j, then, arguing as in Lemma 2.3, we get $\Delta(u(\xi,\eta)v(\xi,\eta)) = \Delta(\xi\eta)$, and

$$u(\xi,\eta) = u_{0,1}\eta.$$

Similarly, there exists $v_{1,0} \in \mathbb{R}$ such that $v(\xi,\eta) = v_{1,0}\xi$. Since $\mathcal{I}^*$ is anti-symplectic (R_0 is symplectic and $\mathcal{I}$ is anti-symplectic), we have

$$d\xi \wedge d\eta = dv \wedge du = (u_{0,1}v_{1,0})d\xi \wedge d\eta,$$

and then $u_{0,1}, v_{1,0} \in \mathbb{R}^*$, and $v_{1,0} = u_{0,1}^{-1}$. Besides, $R_0^{-1} = (S,\Phi)$ maps $\{(\xi,\xi),\ \xi \geq 0\}$ to $\{(s,0),\ s \geq 0\}$, hence for any $\xi \geq 0$, we have

$$(u_{0,1}\xi, u_{0,1}^{-1}\xi) = \mathcal{I}^*(\xi,\xi) = R_0 \circ \mathcal{I}(S(\xi,\xi),0) = R_0(S(\xi,\xi),0) \in \{(\xi,\xi),\ \xi \geq 0\},$$

and then $u_{0,1} = v_{1,0} = 1$. We conclude that

$$\mathcal{I}^*(\xi,\eta) = (\eta,\xi). \qquad \square$$

Remark 2.7. *Note that (2.10) can also be obtained as follows: by (2.9), both $\mathcal{I}^* = R_0\mathcal{I}R_0^{-1}$ and $\mathcal{I}_0$ conjugate N with N^{-1}, hence $\mathcal{I}^* \circ \mathcal{I}_0^{-1}$ is in the centralizer of N. By Lemma 2.3 and since $\mathcal{I}^*, \mathcal{I}_0$ preserve the bisectrix $\{\xi = \eta\}$ (as R_0 does), we conclude that $\mathcal{I}^* = \mathcal{I}_0$.*

3. Extension of the Birkhoff coordinates along the separatrices and symmetries of the billiard problem

Let us fix a billiard table $\mathcal{D} \in \mathbf{B}$. In this section, we consider the Birkhoff Normal Form N introduced above for the 2-periodic (12) and we assume that the Birkhoff invariants $(a_k)_{k\geq 1}$ are not all equal to zero. We denote by $R_0 \colon \mathcal{U} \to \mathbb{R}^2$ the canonical symplectic change of coordinates given by Lemma 2.5 and we set $\mathcal{V} := R_0(\mathcal{U})$. We will also use the notation introduced at the beginning of Section 2.

Up to this point, the model for the dynamics of T given by its Birkhoff Normal Form N only accounts for the dynamics in a neighborhood of the 2-periodic orbit. In this section, we explain how to extend this model in such a way that it also describes the global dynamics of the palindromic orbits $(h_n)_{n\geq 1}$ introduced earlier. In the (ξ, η)-coordinates, the only non-wandering point of the map N is the origin $(0, 0)$; to describe recurrence properties of the dynamics of T, we explain a *gluing* construction for some points in this model, for which we have more information due to additional symmetries. This is, in particular, the case for the palindromic orbits $(h_n)_{n\geq 1}$, which have two symmetries, and for which we have a good control on the gluing map. Moreover, for n large enough, those orbits always stay in a neighborhood of the separatrices, and the local dynamics of N near the fixed point is sufficient to describe them, based on the relation $NR_0 = R_0T$ which can be used to extend the system of coordinates by the dynamics. Although this relation is only true locally (some points escape in the billiard dynamics, so the map T is not everywhere defined), it is sufficient for our purpose, which consists in determining explicitly a link between the Birkhoff invariants and the Lyapunov exponents of the palindromic orbits. The extension of the coordinates to a neighborhood of the separatrices that we describe in the following can be seen as a hyperbolic analog of the local coordinates in a neighborhood of the boundary given by the Birkhoff Normal Form in the elliptic setting, which was used, for instance, in [CdV].

After possibly replacing $\mathcal{U}$ with $\mathcal{U} \cap \mathcal{I}(\mathcal{U})$, where $\mathcal{I} \colon (s, r) \mapsto (s, -r)$, we can assume that the neighborhood $\mathcal{U}$ is symmetric with respect to the axis $\{r = 0\}$. For any sufficiently large odd integer $n = 2m - 1$, and after a certain time, Proposition 2.1 implies that the iterates under $T = \mathcal{F}^2$ of the point $x_n(1)$ in the palindromic orbit h_n are contained in the neighborhood $\mathcal{U}$. More precisely, there exists $m_0 \geq 0$ such that if $n = 2m - 1 \geq n_0 := 2m_0 - 1$, we have $x_n(2k + 1) \in \mathcal{U}$, for all $k \in \{m_0, m_0 + 1, \cdots, 2m - m_0 - 1\}$. We denote by $(\xi_n(2k + 1), \eta_n(2k + 1))$ the coordinates of the point $R_0(x_n(2k + 1))$. The contraction rate Δ is constant along this orbit segment: for any integer $k \in \{m_0, m_0 + 1, \cdots, 2m - m_0 - 1\}$, we have

$$\Delta(\xi_n(2k + 1)\eta_n(2k + 1)) = \Delta(\xi_n(2m - 1)\eta_n(2m - 1)) =: \Delta_n.$$

By Lemma 2.2, the map N is well defined in a neighborhood of the coordinate axes. In particular, due to the relations $R_0 = NR_0T^{-1}$ and $R_0 = N^{-1}R_0T$, it is possible to extend the system of coordinates given by R_0 to a neighborhood of the separatrices as follows.

Let $\mathcal{NE}^{-1}$ be the set of all parameters $(s, r) \in \mathcal{M}$ in the collision space such that $T^{-1}(s, r)$ is well defined, i.e., such that both $\mathcal{F}^{-1}(s, r)$ and $\mathcal{F}^{-2}(s, r)$ are well defined. For any $(s, r) \in T^{-1}(\mathcal{U} \cap \mathcal{NE}^{-1})$, we have $T(s, r) \in \mathcal{U}$, thus we can set

$R_{-1}(s,r) := N^{-1}R_0T(s,r)$. By induction, for each integer $\ell \geq 2$, we define

$$\mathcal{NE}^{-\ell} := \{(s,r) \in \mathcal{NE}^{-1} : T^{-1}(s,r) \in \mathcal{NE}^{-(\ell-1)}\},$$

and for each $(s,r) \in \mathcal{U}^{-\ell} := T^{-\ell}(\mathcal{U} \cap \mathcal{NE}^{-\ell})$, we set

$$R_{-\ell}(s,r) := N^{-1}R_{-(\ell-1)}T(s,r) = N^{-\ell}R_0T^{\ell}(s,r).$$

On $\mathcal{U}^{-(\ell-1)} \cap \mathcal{U}^{-\ell}$, it holds $N^{-1}R_{-(\ell-1)}T = R_{-(\ell-1)}$, hence $R_{-\ell}$ coincides with $R_{-(\ell-1)}$ on this set. We define R_- as the map obtained in this way by extending the conjugacy R_0 to a neighborhood of the arc of the stable manifold between the points $(0,0)$ and $x_\infty(1)$. More precisely, R_- is defined on $\mathcal{U}^- := \bigcup_{\ell=0}^{m_0} \mathcal{U}^{-\ell}$ as follows: for any $\ell \in \{0, \cdots, m_0\}$ and $(s,r) \in \mathcal{U}^{-\ell} \setminus \bigcup_{k=0}^{\ell-1} \mathcal{U}^{-k}$, we set $R_-(s,r) := R_{-\ell}(s,r)$. In a symmetric way, we define $\mathcal{U}^+$ and we extend R_0 to a map R_+ defined on a neighborhood of the arc of the unstable manifold between the points $(0,0)$ and $x_\infty(-1)$.

By the above remark, for any integer $n = 2m-1 \geq n_0$, every point $h_n(k)$ labeled with some odd integer k belongs to the set $\mathcal{U}^+ \cup \mathcal{U}^-$, thus it has an image either by R_+ or R_-. We let R be the map defined on $\mathcal{U}^+ \cup \mathcal{U}^-$ by $R|_{\mathcal{U}^\pm} := R_\pm$. For each $k \in \{0, \cdots, m-m_0-1\}$, we have $R(x_n(2m \pm (2k+1))) = R_0(x_n(2m \pm (2k+1)))$, while for $k \in \{m-m_0, \cdots, m-1\}$, the point $R(x_n(2m \pm (2k+1))) = R_{\pm(m-k)}(x_n(2m \pm (2k+1)))$ is well defined. Moreover, for some neighborhood $\mathcal{U}_n \subset \mathcal{U}$ of the point $x_n(2m) = x_n(n+1)$, it follows from the above definitions that

$$R \circ T^{\pm k}|_{\mathcal{U}_n} = N^{\pm k} \circ R|_{\mathcal{U}_n} = N^{\pm k} \circ R_0|_{\mathcal{U}_n}, \qquad \forall\, k \in \{0, \cdots, m-1\}. \tag{3.1}$$

More generally, for each $x \in \mathcal{U}^+ \cup \mathcal{U}^-$ and each integer k such that $x, \cdots, T^k(x) \in \mathcal{U}^+ \cup \mathcal{U}^-$, we have

$$R \circ T^k(x) = N^k \circ R(x). \tag{3.2}$$

In particular, for each $n = 2m-1$ with $m \geq m_0$, and for $k \in \{0, \cdots, n\}$, it holds

$$R(x_n(2k+1)) = (\xi_n(2k+1), \eta_n(2k+1)) := ((\Delta_n^{m-1-k})^{-1}\xi_n(n), \Delta_n^{m-1-k}\eta_n(n)).$$

Let us abbreviate

$$R(x_n(1)) = (\xi_n, \eta_n) := ((\Delta_n^{m-1})^{-1}\xi_n(n), \Delta_n^{m-1}\eta_n(n)).$$

Then, we have

$$(\xi_n(2k+1), \eta_n(2k+1)) = (\Delta_n^k \xi_n, \Delta_n^{-k}\eta_n), \quad \forall\, k \in \{0, \cdots, n\}. \tag{3.3}$$

In the same way, we can extend our system of coordinates such that the images of the forward iterates of the point $x_\infty(1)$ in the homoclinic orbit h_∞ are

$$R(x_\infty(2k+1)) = R_-(x_\infty(2k+1)) = (\xi_\infty(2k+1), 0) = (\lambda^k \xi_\infty, 0), \quad \forall\, k = 0, 1, \ldots$$

for some nonzero real number $\xi_\infty \in \mathbb{R}$ (the second coordinate has to vanish since we are on the stable manifold $\{\eta = 0\}$ of the origin). Recall that $\mathcal{I}\colon (s,r) \mapsto (s,-r)$. By Proposition 2.1, for all $n = 2m-1 \geq n_0$ and $k \geq 0$, we have $x_n(-2k-1) = x_n(2(2m-1-k)+1) = \mathcal{I}(x_n(2k+1))$. Thus, we extend analogously the coordinates in the past, such that the preimages of $x_\infty(-1)$ have coordinates $R(x_\infty(-2k-1)) = R_+(x_\infty(-2k-1))$, i.e.,

$$R(x_\infty(-2k-1)) = (0, \xi_\infty(-2k-1)) = (0, \lambda^k \xi_\infty), \quad \forall\, k = 1, 2, \ldots$$

Remark 3.1. *In the previous construction, we stop the extension after the time $\pm m_0$ where we reach a neighborhood of the point $x_\infty(\pm 1)$. Indeed, after that time, in the initial (s,r)-collision space, the neighborhoods of the separatrices start to overlap; in particular, they both contain a neighborhood of the point $x_\infty(0)$ on the third obstacle. Besides, the point of this construction is to study the dynamics of the map T through its Birkhoff Normal Form N. Note that the latter only depends on the obstacles $\mathcal{O}_1, \mathcal{O}_2$. By analyticity, as long as the points bounce between the first two obstacles, it is legitimate to replace the billiard dynamics with that of N, but it does not carry any meaningful information once the points reach the third obstacle.*

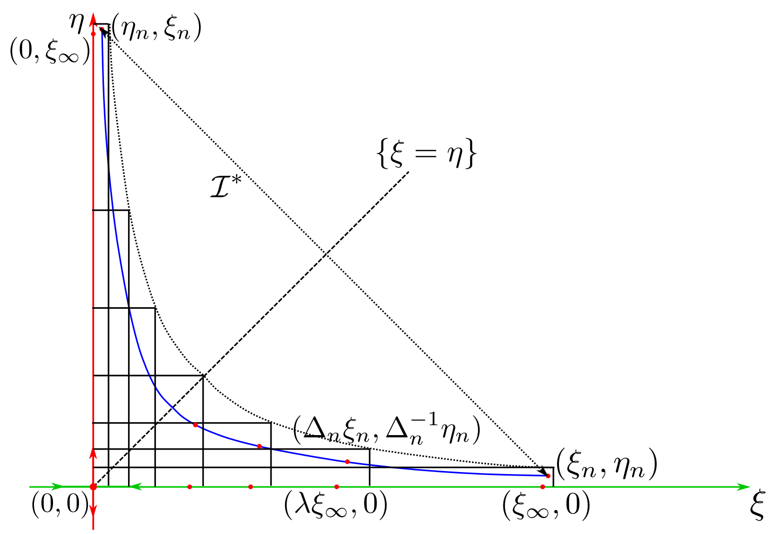


FIGURE 4. (ξ,η)-representation of the points in the palindromic orbit h_n.

The next lemma says that after the extension, the image of the time reversal involution is still given by the map $\mathcal{I}_0 \colon (\xi,\eta) \mapsto (\eta,\xi)$:

Lemma 3.2. *The extended system of coordinates R satisfies*

$$R \circ \mathcal{I} \circ R^{-1} = \mathcal{I}_0. \tag{3.4}$$

Proof. It follows directly from (3.2) and Lemma 2.6. □

4. Marked Lyapunov Spectrum and Birkhoff invariants

4.1. **Preliminary estimates on the parameters.** Recall that $T = \mathcal{F}^2$ where $\mathcal{F}$ is the billiard map and that $\mathcal{M}_i$ denotes the set of (s,r)-coordinates of collisions emanating from the i^{th} scatterer.

We let $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^2$ be small neighborhoods of $(0,0)$ as defined in Section 2.

Lemma 4.1. *For any $x = (s,0) \in \mathcal{U} \cap \{r = 0\}$, we have $R_0(x) \in \mathcal{V} \cap \{\xi = \eta\}$.*

Proof. The lemma follows directly from the definition of R_0. □

Let $\mathscr{O}_\infty \subset \mathcal{M}_2$ be a small neighborhood of the point $x_\infty(-1)$. We denote by $\Omega_\infty := R(\mathscr{O}_\infty)$ the image of $\mathscr{O}_\infty$ in Birkhoff coordinates, and we let

$$\mathcal{G} := R \circ T \circ R^{-1}|_{\Omega_\infty} = R_- \circ T \circ R_+^{-1}|_{\Omega_\infty}$$

be the gluing map between R_+ and R_-. It satisfies the time-reversal property

$$\mathcal{G}^{-1} = \mathcal{I}_0 \circ \mathcal{G} \circ \mathcal{I}_0, \qquad \mathcal{I}_0 \colon (\xi, \eta) \mapsto (\eta, \xi). \tag{4.1}$$

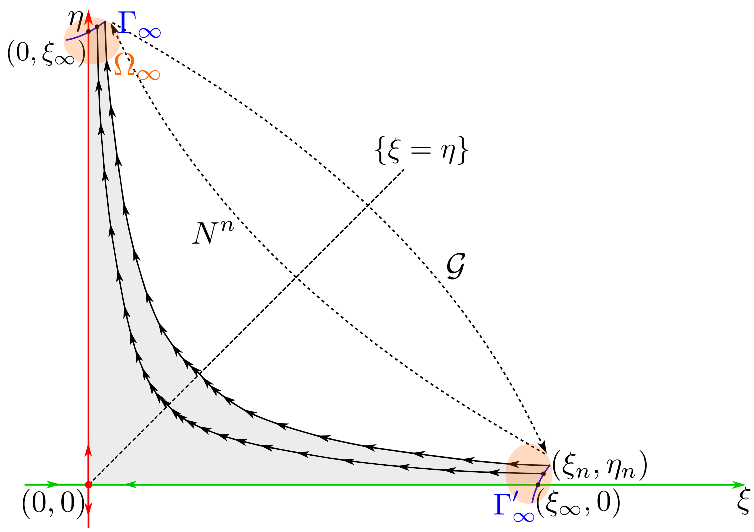


FIGURE 5. Gluing map between the extensions R_+, R_-.

Let $\mathscr{A}_\infty := \mathscr{O}_\infty \cap \mathcal{F}^{-1}(\{r = 0\})$ be the curve in $\mathscr{O}_\infty$ containing $x_\infty(-1)$ and made of points whose image under the billiard map $\mathcal{F}$ is associated to an orthogonal collision on $\mathcal{O}_3$; let $\Gamma_\infty := R(\mathscr{A}_\infty) \subset \Omega_\infty$ be the image of $\mathscr{A}_\infty$ in Birkhoff coordinates, and set $\Gamma'_\infty := \mathcal{G}(\Gamma_\infty) \subset \mathcal{G}(\Omega_\infty)$.

Lemma 4.2. *For any $x = (s, r) \in \mathscr{A}_\infty$, we have $T(s, r) = \mathcal{I}(x) = (s, -r)$, and $D_{(s,r)}T \in \mathrm{SL}(2, \mathbb{R})$. Analogously, for any $(\xi, \eta) \in \Gamma_\infty$, it holds $\mathcal{G}(\xi, \eta) = \mathcal{I}_0(\xi, \eta) = (\eta, \xi)$, and $D_{(\xi,\eta)}\mathcal{G} \in \mathrm{SL}(2, \mathbb{R})$. In particular, we have $\Gamma'_\infty = \mathcal{I}_0(\Gamma_\infty)$.*

Proof. Let $x = (s, r) \in \mathscr{A}_\infty$. The point $\mathcal{F}(x) \in \{r = 0\}$ is invariant under the involution $\mathcal{I} \colon (s, r) \mapsto (s, -r)$, hence

$$T(x) = \mathcal{F}(\mathcal{F}(x)) = \mathcal{F} \circ \mathcal{I}(\mathcal{F}(x)) = \mathcal{I} \circ \mathcal{F}^{-1}(\mathcal{F}(x)) = \mathcal{I}(x).$$

Moreover, by (1.2), we have $\det D_x T = \det D_{(s,r)}\mathcal{F}^2 = 1$. By (3.4), $R \circ \mathcal{I} = \mathcal{I}_0 \circ R$, hence for any $(\xi, \eta) = R(x) \in \Gamma_\infty$, we also have

$$\mathcal{G}(\xi, \eta) = RTR^{-1}(R(x)) = RT(x) = R \circ \mathcal{I}(x) = \mathcal{I}_0 \circ R(x) = \mathcal{I}_0(\xi, \eta),$$

and $\det D_{(\xi,\eta)}\mathcal{G} = \det D_x T = 1$. $\square$

As in Section 3, for any large integer $n \geq n_0$, we let $(\xi_n, \eta_n) := R_-(x_n(1))$, $\Delta_n := \Delta(\xi_n \eta_n)$, and we let $(\xi_\infty, 0) := R_-(x_\infty(1)) = \lim_{n \to +\infty}(\xi_n, \eta_n)$.

We let $\gamma \colon \xi \mapsto \sum_{j=1}^{\infty} \gamma_j \xi^j$ be the analytic function such that Γ_∞ is the graph of $\xi_\infty + \gamma(\cdot)$, i.e., for any $(\xi, \eta) \in \Gamma_\infty$, we have $\eta = \xi_\infty + \gamma(\xi)$. As we have seen in Lemma 4.2, $\Gamma'_\infty = \mathcal{I}_0(\Gamma_\infty)$, hence for any $(\xi, \eta) \in \Gamma'_\infty$, we also have $\xi = \xi_\infty + \gamma(\eta)$.

Lemma 4.3. *For each integer $n \geq n_0$, it holds*

$$R(x_n(-1)) = (\eta_n, \xi_n), \qquad R(x_\infty(-1)) = (0, \xi_\infty), \qquad \xi_n = \xi_\infty + \gamma(\eta_n).$$

In particular, $(\eta_n, \xi_n), (0, \xi_\infty) \in \Gamma_\infty$, while $(\xi_n, \eta_n), (\xi_\infty, 0) \in \Gamma'_\infty$.

Proof. Let $n \geq n_0$. We have $x_n(0) = \mathcal{F}(x_n(-1)) \in \{r = 0\}$, i.e., $x_n(-1) \in \mathscr{A}_\infty$, and $R(x_n(-1)) \in \Gamma_\infty$. It follows from Lemma 4.2 that $(\xi_n, \eta_n) = R(T(x_n(-1))) = \mathcal{G}(R(x_n(-1))) = \mathcal{I}_0(R(x_n(-1)))$, which gives the first identity. Besides, $(\eta_n, \xi_n) \in \Gamma_\infty$, and $(\xi_n, \eta_n) = R(x_n(1)) \in \Gamma'_\infty$, so that $\xi_n = \xi_\infty + \gamma(\eta_n)$.

Similarly, $x_\infty(-1) \in \mathscr{A}_\infty$, $R(x_\infty(-1)) \in \Gamma_\infty$, and by Lemma 4.2, we have

$$R(x_\infty(-1)) = \mathcal{G}^{-1}(\xi_\infty, 0) = \mathcal{I}_0(\xi_\infty, 0) = (0, \xi_\infty). \qquad \square$$

Let us denote by $\mathcal{G}^\pm$ the coordinate functions of $\mathcal{G}$, i.e., $\mathcal{G}\colon (\xi, \eta) \mapsto (\mathcal{G}^+(\xi, \eta), \mathcal{G}^-(\xi, \eta))$. Since $\mathcal{G}(0, \xi_\infty) = (\xi_\infty, 0)$, for any $(\xi, \eta) \in \Omega_\infty$, we may write

$$\mathcal{G}(\xi, \eta) = (\mathcal{G}^+(\xi, \eta), \mathcal{G}^-(\xi, \eta)) = (\xi_\infty + G^+(\xi, \eta - \xi_\infty), G^-(\xi, \eta - \xi_\infty)),$$

for two analytic functions $G^\pm\colon (\xi, \eta) \mapsto \sum_{j+k \geq 1} G^\pm_{j,k} \xi^j \eta^k$. Note that by the time-reversal property (4.1), for any $(\xi, \eta) \in \mathcal{G}(\Omega_\infty)$, we have

$$\mathcal{G}^{-1}(\xi, \eta) = (G^-(\eta, \xi - \xi_\infty), \xi_\infty + G^+(\eta, \xi - \xi_\infty)). \tag{4.2}$$

As a consequence of Lemma 4.2, we get, for $|\eta|$ sufficiently small:

$$G^-(\eta, \gamma(\eta)) = \eta, \qquad\qquad G^+(\eta, \gamma(\eta)) = \gamma(\eta). \tag{4.3}$$

For $i = 1, 2$, we set $G^\pm_i\colon \eta \mapsto \partial_i G^\pm(\eta, \gamma(\eta))$.

Lemma 4.4. *The following relations hold:*

$$\begin{aligned} G_1^- &= -G_2^+ = 1 - \gamma' G_2^-, \\ G_1^+ &= \gamma'(2 - \gamma' G_2^-). \end{aligned}$$

Proof. By differentiating (4.3), for $|\eta|$ sufficiently small, we obtain

$$\begin{aligned} \partial_1 G^-(\eta, \gamma(\eta)) + \gamma'(\eta)\partial_2 G^-(\eta, \gamma(\eta)) &= G_1^-(\eta) + \gamma'(\eta) G_2^-(\eta) = 1, \\ \partial_1 G^+(\eta, \gamma(\eta)) + \gamma'(\eta)\partial_2 G^+(\eta, \gamma(\eta)) &= G_1^+(\eta) + \gamma'(\eta) G_2^+(\eta) = \gamma'(\eta). \end{aligned}$$

Now, by Lemma 4.2, the differential $D_{(\eta, \xi_\infty + \gamma(\eta))}\mathcal{G}$ of the gluing map is in $\mathrm{SL}(2, \mathbb{R})$. We have

$$D_{(\eta, \xi_\infty + \gamma(\eta))}\mathcal{G} = \begin{pmatrix} \partial_1 G^+(\eta, \gamma(\eta)) & \partial_2 G^+(\eta, \gamma(\eta)) \\ \partial_1 G^-(\eta, \gamma(\eta)) & \partial_2 G^-(\eta, \gamma(\eta)) \end{pmatrix} = \begin{pmatrix} G_1^+(\eta) & G_2^+(\eta) \\ G_1^-(\eta) & G_2^-(\eta) \end{pmatrix},$$

and thus,

$$G_1^-(\eta) G_2^+(\eta) = G_1^+(\eta) G_2^-(\eta) - 1.$$

We deduce from the relations obtained previously that

$$\begin{aligned} (G_1^+(\eta) - \gamma'(\eta))(\gamma'(\eta) G_2^-(\eta) - 1) &= \gamma'(\eta) G_1^-(\eta) G_2^+(\eta) \\ &= \gamma'(\eta)(G_1^+(\eta) G_2^-(\eta) - 1), \end{aligned}$$

which yields

$$G_1^+ = \gamma'(2 - \gamma' G_2^-).$$

Combining this with the relations obtained above, we conclude that $G_1^- = 1 - \gamma' G_2^-$ and $G_2^+ = 1 - (\gamma')^{-1} G_1^+ = \gamma' G_2^- - 1 = -G_1^-$. $\square$

The above lemma leads us to define the analytic function

$$g\colon \eta \mapsto \sum_{k=0}^{+\infty} g_k \eta^k := G_2^-(\eta) = \partial_2 G^-(\eta, \gamma(\eta)),$$

so that for $|\eta|$ sufficiently small, we have

$$D_{(\eta,\xi_\infty+\gamma(\eta))}\mathcal{G} = \begin{pmatrix} \gamma'(\eta)(2-\gamma'(\eta)g(\eta)) & \gamma'(\eta)g(\eta)-1 \\ 1-\gamma'(\eta)g(\eta) & g(\eta) \end{pmatrix}. \tag{4.4}$$

Remark 4.5. *Let $\mathcal{W}_\infty^+ = \{(\eta, \xi_\infty + w(\eta)) : |\eta| \text{ small}\} \subset \Omega_\infty$ be the arc of the stable manifold of $(0,0)$ containing the homoclinic point $(0,\xi_\infty)$, for some analytic function $w\colon \xi \mapsto \sum_{k\geq 1} w_k \xi^k$, and let $\mathcal{W}_\infty^- \subset \mathcal{G}(\Omega_\infty)$ be the arc of the unstable manifold of $(0,0)$ containing $(\xi_\infty, 0)$. By the time-reversal symmetry, we have $T \circ R_+^{-1}(\mathcal{W}_\infty^+) = \mathcal{I}(T^{-1} \circ R_-^{-1}(\mathcal{W}_\infty^-))$, and then,*

$$\mathcal{W}_\infty^- = \mathcal{I}_0(\mathcal{W}_\infty^+),$$

i.e., $\mathcal{W}_\infty^- = \{(\xi_\infty + w(\eta), \eta) : |\eta| \text{ small}\} \subset \mathcal{G}(\Omega_\infty)$. By (4.2) and the fact that the gluing map $\mathcal{G}$ preserves the invariant subspaces of the saddle fixed point $(0,0)$, we can write analogous relations between G^+ and G^-, but involving the function w instead of γ, i.e., for $|\eta|$ sufficiently small, it holds

$$G^-(\eta, w(\eta)) = 0, \qquad G^+(0,\eta) = w(G^-(0,\eta)).$$

Differentiating $G^-(\eta, w(\eta)) = 0$ and evaluating at 0, we get $G_1^-(0) = -w_1 G_2^-(0)$. On the other hand, the previous identities yield $G_1^-(0) = 1 - \gamma_1 G_2^-(0)$. In particular, it follows that $g_0 = G_2^-(0) = (\gamma_1 - w_1)^{-1}$.

The arc $\mathcal{W}_\infty^+$ of the stable manifold is transverse to the unstable manifold of N at the homoclinic point $(0,\xi_\infty)$, which is vertical in those coordinates, hence $w_1 \neq \infty$. Besides, $\mathscr{A}_\infty \subset T^{-1}(\{r=0\})$ is the image under T^{-1} of some arc on the third scatterer, and then, its image Γ_∞ under R is also transverse to the unstable manifold of N at $(0,\xi_\infty)$, i.e., $\gamma_1 \neq \infty$. Since the gluing map $\mathcal{G} = R \circ T \circ R^{-1}|_{\Omega_\infty}$ is defined dynamically, we deduce that

$$\Gamma'_\infty = \mathcal{G}(\Gamma_\infty) = \mathcal{I}_0(\Gamma_\infty) = \{(\xi_\infty + \gamma(\eta), \eta) : |\eta| \text{ small}\}$$
$$\text{and } \mathcal{W}_\infty^- = \mathcal{G}(\{\xi = 0\}) = \mathcal{I}_0(\mathcal{W}_\infty^+) = \{(\xi_\infty + w(\eta), \eta) : |\eta| \text{ small}\}$$

are still transverse at $(\xi_\infty, 0)$, and then, $\gamma_1 \neq w_1$, which implies

$$|g_0| = |G_2^-(0)| = |\gamma_1 - w_1|^{-1} \in (0, +\infty). \tag{4.5}$$

Let us recall that for any integer $n \geq n_0$, we let $\Delta_n := \Delta(\zeta_n)$, with $\zeta_n := \xi_n \eta_n$.

Lemma 4.6. *For any integer $n \geq n_0$, it holds*

$$\eta_n = \Delta_n^n \xi_n,$$

which can also be rewritten as

$$\eta_n = \Delta(\eta_n(\xi_\infty + \gamma(\eta_n)))^n (\xi_\infty + \gamma(\eta_n)). \tag{4.6}$$

Proof. Let $n \geq n_0$. We note that

$$(\eta_n, \xi_n) = R(x_n(-1)) = R(x_n(2n+1)) = R \circ T^n(x_n(1)) = N^n(\xi_n, \eta_n),$$

which yields the first identity. On the other hand, by definition, $\Delta_n = \Delta(\zeta_n) = \Delta(\xi_n \eta_n)$ and by Lemma 4.3, we have $\xi_n = \xi_\infty + \gamma(\eta_n)$, which concludes the proof. $\square$

In other words, Lemma 4.6 tells us that for each integer $n \geq n_0$, the coordinates $(\xi_n, \eta_n) = (\xi_\infty + \gamma(\eta_n), \eta_n)$ of the image under R of the periodic point $x_n(1)$ are defined implicitely in terms of the coefficients of Δ and γ, according to the previous equation.

4.2. **Lyapunov exponents and asymptotic expansions of the parameters.** In this part, we use the same notation as in the previous subsection, and show the relation between the above formulas and the Marked Lyapunov Spectrum of the billiard table.

Remark 4.7. *Let us make a few comments and introduce some notation.*

(1) *Given $\xi_\infty \in \mathbb{R}$ and the pair of functions (γ, g), then by (4.4), it is possible to reconstruct the restriction of the gluing map $\mathcal{G}|_{\Gamma_\infty}$. Conversely, given $\xi_\infty \in \mathbb{R}$ and the coordinate functions $(\mathcal{G}^+, \mathcal{G}^-)$ of $\mathcal{G}$, then the function γ can be recovered. Indeed, the gluing map $\mathcal{G}$ is dynamically defined, hence it maps some unstable cone at $(0, \xi_\infty)$ into some unstable cone at $(\xi_\infty, 0)$. In particular, for ξ small, $\eta = \xi_\infty + \gamma(\xi)$ is determined by the implicit equation*

$$\mathcal{G}^+(\xi, \eta)\mathcal{G}^-(\xi, \eta) = \xi\eta.$$

(2) *The homoclinic parameter $\xi_\infty \in \mathbb{R}$ can be regarded as a scaling factor: we will show in Subsection 5.2 how its value is determined by the Marked Length Spectrum. For any integer $j \geq 0$, we introduce scaled coefficients*

$$\bar{a}_j := \lambda^{-1} a_i \xi_\infty^{2j}, \quad \bar{\gamma}_j := \gamma_j \xi_\infty^{j-1}, \quad \textit{and} \quad \bar{g}_j := g_j \xi_\infty^{j}, \tag{4.7}$$

with $a_0 := \lambda$ and $\gamma_0 := \xi_\infty$. Note that $\bar{a}_0 = \bar{\gamma}_0 = 1$ and $\bar{g}_0 = g_0$.

In the following, for any integer $n \geq n_0$, we also let

$$\bar{\eta}_n := (\xi_\infty \lambda^n)^{-1} \eta_n, \quad \textit{and} \quad \bar{\zeta}_n := (\xi_\infty^2 \lambda^n)^{-1} \zeta_n = (\xi_\infty^2 \lambda^n)^{-1} \xi_n \eta_n. \tag{4.8}$$

(3) *Given the homoclinic parameter $\xi_\infty \in \mathbb{R}$, the Birkhoff Normal Form N and the gluing map $\mathcal{G}$, we will find an explicit expression for the parameters $(\xi_n, \eta_n)_n$ of the periodic orbits $(h_n)_{n \geq 0}$ and of their Lyapunov exponent. More precisely, under the assumption that the first Birkhoff invariant a_1 does not vanish, we show in Lemma 4.20 and Corollary 4.21 that there is a one-to-one correspondence between the sequence of Lyapunov exponents $(\mathrm{LE}(h_n))_{n \geq 0}$ and the coefficients $(\bar{a}_j, \bar{\gamma}_j, \bar{g}_j)_{j=0}^{\infty}$.*

By the above remark, if we know the value of ξ_∞ and of the scaled coefficients $\{\bar{a}_j\}_j$, $\{\bar{\gamma}_j\}_j$ and $\{\bar{g}_j\}_j$, then it is possible to reconstruct $\{a_j\}_j$, $\{\gamma_j\}_j$ and $\{g_j\}_j$. In order to ease our notation, we henceforth assume that $\xi_\infty = 1$ in the rest of this section.

The next lemma tells us how the Lyapunov exponent of the associated orbit can be expressed in terms of the new coordinates.

Lemma 4.8. *For each integer $n \geq n_0$, we let*

$$\Delta'_n := \Delta'(\zeta_n)\zeta_n = \sum_{k=1}^{+\infty} k a_k \zeta_n^k.$$

Then, the Lyapunov exponent of the periodic orbit h_n satisfies

$$2\cosh(2(n+1)\mathrm{LE}(h_n)) = \lambda^{-n}\mathrm{I}_n + \mathrm{II}_n + \lambda^n \mathrm{III}_n,$$

where

$$\begin{aligned} \mathrm{I}_n &:= \lambda^n \Delta_n^{-n}\big(1 - n\Delta_n'\Delta_n^{-1}\big)g(\eta_n),\\ \mathrm{II}_n &:= 2n\Delta_n'\Delta_n^{-1}\big(1-\gamma'(\eta_n)g(\eta_n)\big),\\ \mathrm{III}_n &:= \lambda^{-n}\Delta_n^n\big(1+n\Delta_n'\Delta_n^{-1}\big)\gamma'(\eta_n)\big(2-\gamma'(\eta_n)g(\eta_n)\big). \end{aligned}$$

Proof. By the $(2n+2)$-periodicity of h_n, we have $T^{n+1}(x_n(-1)) = x_n(-1)$, and since $D_{x_n(-1)}T^{n+1} \in \mathrm{SL}(2,\mathbb{R})$, we obtain

$$\begin{aligned} 2\cosh(2(n+1)\mathrm{LE}(h_n)) &= \mathrm{tr}(D_{x_n(-1)}T^{n+1}) = \mathrm{tr}(D_{x_n(-1)}(T^nR^{-1}\circ RTR^{-1}\circ R))\\ &= \mathrm{tr}(D_{(\xi_n,\eta_n)}(RT^nR^{-1})\cdot D_{(\eta_n,\xi_n)}(RTR^{-1})) = \mathrm{tr}(D_{(\xi_n,\eta_n)}N^n\cdot D_{(\eta_n,\xi_n)}\mathcal{G}). \end{aligned}$$

By Lemma 4.6, we have

$$D_{(\xi_n,\eta_n)}N^n = \begin{pmatrix}\Delta_n^n & 0\\ 0 & \Delta_n^{-n}\end{pmatrix} + n\Delta_n'\Delta_n^{-1}\begin{pmatrix}\Delta_n^n & 1\\ -1 & -\Delta_n^{-n}\end{pmatrix},$$

with $\Delta_n' := \Delta'(\zeta_n)\zeta_n$, and then, it follows from (4.4) that

$$\begin{aligned} &2\cosh(2(n+1)\mathrm{LE}(h_n))\\ &= \mathrm{tr}\left(\begin{pmatrix}\Delta_n^n & 0\\ 0 & \Delta_n^{-n}\end{pmatrix}\cdot\begin{pmatrix}G_1^+(\eta_n) & G_2^+(\eta_n)\\ G_1^-(\eta_n) & G_2^-(\eta_n)\end{pmatrix}\right)\\ &+ n\Delta_n'\Delta_n^{-1}\mathrm{tr}\left(\begin{pmatrix}\Delta_n^n & 1\\ -1 & -\Delta^{-n}\end{pmatrix}\cdot\begin{pmatrix}G_1^+(\eta_n) & G_2^+(\eta_n)\\ G_1^-(\eta_n) & G_2^-(\eta_n)\end{pmatrix}\right)\\ &= \Delta_n^{-n}g(\eta_n) + \Delta_n^n\gamma'(\eta_n)(2-\gamma'(\eta_n)g(\eta_n))\\ &- n\Delta_n'\Delta_n^{-1}\big[\Delta_n^{-n}g(\eta_n) + 2(\gamma'(\eta_n)g(\eta_n)-1) - \Delta_n^n\gamma'(\eta_n)(2-\gamma'(\eta_n)g(\eta_n))\big]\\ &= \Delta_n^{-n}\big(1-n\Delta_n'\Delta_n^{-1}\big)g(\eta_n) + 2n\Delta_n'\Delta_n^{-1}\big(1-\gamma'(\eta_n)g(\eta_n)\big)\\ &+ \Delta_n^n\big(1+n\Delta_n'\Delta_n^{-1}\big)\gamma'(\eta_n)(2-\gamma'(\eta_n)g(\eta_n)). \end{aligned}$$

□

Remark 4.9. *Our choice for the definitions of* I_n, II_n, III_n *will become clearer in the following. Roughly speaking, we write them in this way so that their expansions begin with the "same weight", i.e., are* 0*-triangular in the sense of Definition 4.17.*

In the following, as explained in Remark 4.7, we derive asymptotic expansions with respect to n of the parameters η_n and of the other symbols which appear in the expression of the Lyapunov exponent $\mathrm{LE}(h_n)$ obtained in Lemma 4.8. In Lemma 4.10, we compute the first terms in these expansions. In Lemma 4.11, we study their general structure, and show that they can be expressed as certain series mixing polynomials and exponentials in n, and whose coefficients are "homogeneous" combinations of the gluing terms and of Birkhoff coefficients. In Lemma 4.15, we compute the value of the coefficients of the different terms in the expansions of the parameters; each time, we focus on the terms with the largest index, as we see them for the first time, while the previous terms appear as additive constants.

Due to the different roles that the various coefficients play, we expect to be able to distinguish between them in the estimates; in particular, Birkhoff coefficients have a larger weight than the gluing terms, since the periodic orbits h_n spend much more time in a neighborhood of the saddle than in the gluing region. In a first time, we compute inductively the expansion of η_n in terms of the coefficients of Δ and γ thanks to the formula given by Lemma 4.6. Next, we compute the expansions of the other expressions which appear in the formula given by Lemma 4.8.

Lemma 4.10. *With the notation introduced in (4.7)–(4.8), it holds:*

$$\bar{\eta}_n = 1 + [n\bar{a}_1 + \bar{\gamma}_1]\lambda^n + \Big[n^2\frac{3\bar{a}_1^2}{2} + n\big(-\frac{\bar{a}_1^2}{2} + 4\bar{a}_1\bar{\gamma}_1 + \bar{a}_2\big) + (\bar{\gamma}_1^2 + \bar{\gamma}_2)\Big]\lambda^{2n} + O(n^3\lambda^{3n}).$$

By the fact that $\Delta_n = \Delta(\zeta_n)$ *and* $\Delta_n' := \Delta'(\zeta_n)\zeta_n$*, the previous estimates give*

$$\lambda^n\Delta_n^{-n} = 1 - n\bar{a}_1\lambda^n - \Big[n^2\frac{\bar{a}_1^2}{2} + n\big(2\bar{a}_1\bar{\gamma}_1 + \bar{a}_2 - \frac{\bar{a}_1^2}{2}\big)\Big]\lambda^{2n} + O(n^3\lambda^{3n}),$$

$$1 - n\Delta_n'\Delta_n^{-1} = 1 - n\bar{a}_1\lambda^n - [n^2\bar{a}_1^2 + n(2\bar{a}_1\bar{\gamma}_1 + 2\bar{a}_2 - \bar{a}_1^2)]\lambda^{2n} + O(n^3\lambda^{3n}),$$

and

$$\lambda^n\Delta_n^{-n}\big(1 - n\Delta_n'\Delta_n^{-1}\big) = 1 - 2n\bar{a}_1\lambda^n - \Big[n^2\frac{\bar{a}_1^2}{2} + n\big(4\bar{a}_1\bar{\gamma}_1 + 3\bar{a}_2 - \frac{3\bar{a}_1^2}{2}\big)\Big]\lambda^{2n} + O(n^3\lambda^{3n}).$$

In particular,

$$2\cosh(2(n+1)\mathrm{LE}(h_n)) = \lambda^{-n}g_0 - 2ng_0\bar{a}_1 + O(1),$$

hence the coefficients g_0 *and* $\bar{a}_1$ *are determined by the Marked Length Spectrum.*

Proof. Let $n \geq n_0$. By Lemma 4.6, and since we assume that $\xi_\infty = 1$, we have

$$\begin{aligned}\eta_n &= \Delta(\eta_n(1+\gamma(\eta_n)))^n(1+\gamma(\eta_n))\\ &= \Big(\sum_{j=0}^3 a_j\eta_n^j \cdot \Big(\sum_{k=0}^3 \gamma_k\eta_n^k\Big)^j + O(\eta_n^4)\Big)^n \cdot \Big(\sum_{\ell=0}^3 \gamma_\ell\eta_n^\ell + O(\eta_n^4)\Big),\end{aligned}$$

which yields the expansion

$$\begin{aligned}\eta_n = \lambda^n\Big(1 + [n\lambda^{-1}a_1 + \gamma_1]\eta_n + \Big[n^2\frac{(\lambda^{-1}a_1)^2}{2}\\ + n\Big(-\frac{(\lambda^{-1}a_1)^2}{2} + 2\lambda^{-1}a_1\gamma_1 + \lambda^{-1}a_2\Big) + \gamma_2\Big]\eta_n^2 + O(n^3\eta_n^3)\Big).\end{aligned}$$

By considering first order terms, we obtain $\eta_n = \lambda^n + O(n\lambda^{2n})$. Plugging this back into the previous equation, we deduce that

$$\eta_n = \lambda^n + [n\bar{a}_1 + \bar{\gamma}_1]\lambda^{2n} + O(n^2\lambda^{3n}).$$

We thus obtain

$$\begin{aligned}\bar{\eta}_n &= \lambda^{-n}\Big(\lambda^n + [n\bar{a}_1 + \bar{\gamma}_1]\lambda^{2n} + \Big[(n\bar{a}_1 + \bar{\gamma}_1)^2 + n^2\frac{\bar{a}_1^2}{2} +\\ &+ n\Big(-\frac{\bar{a}_1^2}{2} + 2\bar{a}_1\bar{\gamma}_1 + \bar{a}_2\Big) + \bar{\gamma}_2\Big]\lambda^{3n} + O(n^3\lambda^{4n})\Big)\\ &= 1 + [n\bar{a}_1 + \bar{\gamma}_1]\lambda^n + \Big[n^2\frac{3\bar{a}_1^2}{2} + n\Big(-\frac{\bar{a}_1^2}{2} + 4\bar{a}_1\bar{\gamma}_1 + \bar{a}_2\Big) + (\bar{\gamma}_1^2 + \bar{\gamma}_2)\Big]\lambda^{2n} + O(n^3\lambda^{3n}).\end{aligned}$$

To obtain the expansions of $\Delta_n^{\pm n}$, we argue as follows: by definition, we have $\Delta_n = \Delta(\zeta_n)$, with $\zeta_n = \eta_n(1+\gamma(\eta_n)) = \lambda^n\bar{\zeta}_n$ as in (4.8), so that

$$\begin{aligned}\bar{\zeta}_n &= \bar{\eta}_n + \bar{\gamma}_1\lambda^n\bar{\eta}_n^2 + \bar{\gamma}_2\lambda^{2n}\bar{\eta}_n^3 + O(n^3\lambda^{3n})\\ &= 1 + [n\bar{a}_1 + 2\bar{\gamma}_1]\lambda^n + \Big[n^2\frac{3\bar{a}_1^2}{2} + n\Big(-\frac{\bar{a}_1^2}{2} + 6\bar{a}_1\bar{\gamma}_1 + \bar{a}_2\Big) + (3\bar{\gamma}_1^2 + 2\bar{\gamma}_2)\Big]\lambda^{2n} + O(n^3\lambda^{3n}).\end{aligned}$$

To conclude, it suffices to expand the following expressions:

$$\Delta_n^{-n} = (\Delta^{-1}(\zeta_n))^n = \lambda^{-n}\big(1 - \bar{a}_1\lambda^n\bar{\zeta}_n + (\bar{a}_1^2 - \bar{a}_2)\lambda^{2n}\bar{\zeta}_n^2\big)^n + O(n^3\lambda^{2n}),$$

and

$$\begin{aligned} 1 - n\Delta_n'\Delta_n^{-1} &= 1 - n\Delta'(\zeta_n)\zeta_n \cdot \Delta^{-1}(\zeta_n) \\ &= 1 - n\lambda^n\bar{\zeta}_n\big(\bar{a}_1 + 2\bar{a}_2\lambda^n\bar{\zeta}_n\big) \cdot \big(1 - \bar{a}_1\lambda^n\bar{\zeta}_n\big) + O(n^3\lambda^{3n}) \\ &= 1 - n\lambda^n\big(\bar{a}_1\bar{\zeta}_n + (2\bar{a}_2 - \bar{a}_1^2)\lambda^n\bar{\zeta}_n^2\big) + O(n^3\lambda^{3n}). \end{aligned}$$

The previous estimates and the expression obtained in Lemma 4.8 yield

$$2\cosh(2(n+1)\mathrm{LE}(h_n)) = \lambda^{-n}g_0 - 2ng_0\bar{a}_1 + O(1).$$

Indeed, the other expressions in the formula given by Lemma 4.8 are bounded, since $\Delta_n^{\pm n} = O(\lambda^{\pm n})$, while $\eta_n = O(\lambda^n)$.

By Theorem 1.12, the quantities on the left hand side can be computed, thus we can recover the value of g_0 and $\bar{a}_1$ by separating terms growing at different speeds:

$$\begin{aligned} g_0 &= \lim_{n\to+\infty} 2\cosh(2(n+1)\mathrm{LE}(h_n))\lambda^n, \\ \bar{a}_1 &= \lim_{n\to+\infty} \frac{1}{2n}(\lambda^{-n} - 2g_0^{-1}\cosh(2(n+1)\mathrm{LE}(h_n))). \end{aligned}$$

Indeed, recall that by Remark 4.5, the coefficient g_0 does not vanish. □

More generally, we prove the following lemma.

Lemma 4.11. *There exists $n_0 > 0$ so that for any integer $i \geq 1$, there exists a sequence $\left(P_k^{(i)}\right)_{k\geq 1}$ of polynomials such that for any integer $n \geq n_0$:*

$$\bar{\eta}_n^i = 1 + \sum_{k=1}^{+\infty} P_k^{(i)}(n)\lambda^{nk},$$

where for each $k \geq 0$, the polynomial $P_k^{(i)}(X) = \sum_{j=0}^k \mu_{j,k}^{(i)} X^j$ has degree k. For simplicity, we abbreviate $P_k^{(1)} = P_k$ and $\mu_{j,k}^{(1)} = \mu_{j,k}$ in the following.[4]

Similarly, there exist three sequences $(Q_k^{\pm})_{k\geq 0}$, $(R_k)_{k\geq 0}$ of polynomials such that

$$\lambda^{\mp n}\Delta_n^{\pm n} = 1 + \sum_{k=1}^{+\infty} Q_k^{\pm}(n)\lambda^{nk},$$

$$1 - n\Delta_n'\Delta_n^{-1} = 1 + \sum_{k=1}^{+\infty} R_k(n)\lambda^{nk},$$

where for each $k \geq 0$, the polynomials $Q_k^{\pm}(X) = \sum_{j=0}^k \nu_{j,k}^{\pm} X^j$ and $R_k(X) = \sum_{j=0}^k \rho_{j,k}^{\pm} X^j$ have degree k.

In particular, by Lemma 4.10, it holds

$$\begin{cases} \mu_{0,0}^{(i)} = 1, & \mu_{1,1}^{(i)} = i\bar{a}_1, & \mu_{2,2}^{(i)} = \frac{i(i+2)}{2}\bar{a}_1^2, \\ \nu_{0,0}^{-} = 1, & \nu_{1,1}^{-} = -\bar{a}_1, & \nu_{2,2}^{-} = -\frac{\bar{a}_1^2}{2}, \\ \rho_{0,0} = 1, & \rho_{1,1} = -\bar{a}_1, & \rho_{2,2} = -\bar{a}_1^2. \end{cases} \tag{4.9}$$

[4]Note that it is sufficient to show the result for $i = 1$, as such expansions are stable by taking powers. This is what we are going to do in the following proof.

Proof. Let us first consider $\bar{\eta}_n$, $n \geq n_0$. We will prove by induction on $\ell \geq 0$ that $\bar{\eta}_n = \bar{\eta}_{n,\ell} + O\big(n^{\ell+1}\lambda^{n(\ell+1)}\big)$, where

$$\bar{\eta}_{n,\ell} = 1 + \sum_{k=1}^{\ell} P_k(n)\lambda^{nk}, \tag{4.10}$$

for certain polynomials $P_1, P_2, \cdots, P_\ell$ satisfying the above properties.

It is clear for $\ell = 0$. Assume that it holds for $\ell - 1 \geq 0$. To show the result for ℓ, we use the formula given by Lemma 4.6:

$$\begin{aligned}
&\bar{\eta}_n = \lambda^{-n}\Delta(\eta_n(1+\gamma(\eta_n)))^n(1+\gamma(\eta_n))\\
&\quad= \Big(1 + \sum_{p=1}^{\ell} \lambda^{-1}a_p\eta_n^p(1+\gamma(\eta_n))^p + O(\eta_n^{\ell+1})\Big)^n(1+\gamma(\eta_n))\\
&\quad= \sum_{r=0}^{n}\binom{n}{r}\Big(\sum_{p=1}^{\ell}\lambda^{-1}a_p\eta_n^p\Big(1+\sum_{q=1}^{\ell}\gamma_q\eta_n^q\Big)^p + O(\eta_n^{\ell+1})\Big)^r\Big(1+\sum_{s=1}^{\ell}\gamma_s\eta_n^s + O(\eta_n^{\ell+1})\Big)\\
&\quad= \sum_{r=0}^{\ell}\binom{n}{r}\Big(\sum_{p=1}^{\ell}\bar{a}_p\lambda^{np}\bar{\eta}_{n,\ell-1}^p\Big(1+\sum_{q=1}^{\ell}\bar{\gamma}_q\lambda^{nq}\bar{\eta}_{n,\ell-1}^q\Big)^p\Big)^r\Big(1+\sum_{s=1}^{\ell}\bar{\gamma}_s\lambda^{ns}\bar{\eta}_{n,\ell-1}^s\Big)\\
&\quad+ O\big(n^{\ell+1}\lambda^{n(\ell+1)}\big).
\end{aligned}$$

Indeed, it is sufficient to consider the $(\ell-1)$-expansion $\bar{\eta}_{n,\ell-1} = 1 + \sum_{k=1}^{\ell-1}P_k(n)\lambda^{nk}$ of $\bar{\eta}_n$ obtained previously to go from $\ell - 1$ to ℓ, as the summations indices p, q, s are all at least equal to one, and hence each term $\bar{\eta}_n^*$ in the above expression is multiplied by a factor λ^{nk}, with $k \geq 1$. Moreover, we can restrict ourselves to indices $r, p, q, s \in \{0, \cdots, \ell\}$, since for $r, p, q, s \geq \ell + 1$, the associated terms are of order $O\big(n^{\ell+1}\lambda^{n(\ell+1)}\big)$.

We claim that the degree of the polynomial in n associated to the factor $\lambda^{n\ell}$ is at most ℓ. Indeed, the expansion of the previous expression is a combination of powers of $\bar{\eta}_{n,\ell-1}$ (which are themselves combinations of polynomials in n multiplied by powers of λ^n, where the degree of the polynomial is at most equal to the exponent of λ^n) multiplied by binomial coefficients $\binom{n}{*}$ and powers of $\bar{a}_*\lambda^{n*}$ or $\bar{\gamma}_*\lambda^{n*}$. Besides the degree of the polynomial in n associated to binomial coefficients is always less than or equal to the exponent of λ^n $\big(r$ in $\binom{n}{r}$ versus at least $rp \geq r$ because of the factor $\lambda^{nrp}\big)$.

We conclude that the new expansion will be of the same form as before, i.e., for some polynomial P_ℓ of degree at most ℓ, we have

$$\bar{\eta}_n = 1 + \sum_{k=1}^{\ell}P_k(n)\lambda^{nk} + O\big(n^{\ell+1}\lambda^{n(\ell+1)}\big).$$

Let us now consider the expansion of $\Delta_n^{\pm n}$. We first remark that $\Delta(z)^{\pm} = \lambda^{\pm} + \sum_{k=1}^{+\infty}a_k^{\pm}z^k$, where for each $k \geq 1$, $a_k^+ = a_k$, and

$$a_k^- = -\lambda^{-2}a_k - \lambda^{-1}(a_1a_{k-1}^- + \ldots + a_{k-1}a_1^-).$$

Similarly, for $i \geq 1$, let $\bar{a}_i^\pm := \lambda^\mp a_i^\pm \xi_\infty^{2i} = \lambda^\mp a_i^\pm$. As a result, for each $k \geq 0$, it holds

$$\bar{a}_k^- = -\bar{a}_k - (\bar{a}_1\bar{a}_{k-1}^- + \ldots + \bar{a}_{k-1}\bar{a}_1^-). \tag{4.11}$$

Thus, for any integer $\ell \geq 0$, we obtain

$$\begin{aligned}
&\Delta_n^{\pm n} = (\Delta(\eta_n(1+\gamma(\eta_n)))^{\pm 1})^n \\
&= \lambda^{\pm n}\Big(1 + \sum_{p=1}^{\ell} \lambda^{\mp 1} a_p^\pm \eta_n^p (1+\gamma(\eta_n))^p + O(\eta_n^{\ell+1})\Big)^n \\
&= \lambda^{\pm n}\left(1 + \sum_{r=1}^{\ell} \binom{n}{r} \left(\sum_{p=1}^{\ell} \bar{a}_p^\pm \lambda^{np} \bar{\eta}_n^p \Big(1 + \sum_{q=1}^{\ell} \bar{\gamma}_q \lambda^{nq} \bar{\eta}_n^q\Big)^p\right)^r\right) + O(n^{\ell+1}\lambda^{n(\ell+1\pm 1)}).
\end{aligned}$$

The form of the expansions of $\Delta_n^{\pm n}$ and $1 - n\Delta_n'\Delta_n^{-1}$ follows from the expression of $\bar{\eta}_n$ obtained previously, since $\Delta_n = \Delta(\zeta_n)$ and $\Delta_n' = \Delta'(\zeta_n)\zeta_n$, with $\zeta_n = \lambda^n\bar{\eta}_n(1 + \gamma(\lambda^n\bar{\eta}_n))$. □

Remark 4.12. *On a formal level, we see that $\bar{\eta}_n$, $\lambda^{\mp n}\Delta_n^{\pm n}$ and $1 - n\Delta_n'\Delta_n^{-1}$ can be expressed as (formal) series in λ^n with coefficients in the ring of polynomials in n. Moreover the coefficient of order k is a polynomial of degree k. Let us call* balanced *those formal series with coefficients in the ring of polynomials in n with the property that the coefficient of order k is a polynomial of degree* at most k. *Observe that such series are closed under sum and product; moreover they are also closed under composition with an analytic function. We conclude that I_n, II_n and III_n are also balanced series. Let us also note that expansions of a similar type were studied earlier in the paper [FY] for a different purpose.*

Remark 4.13. *For any integers $k \geq 1$, $j \in \{0, \cdots, k\}$, and $\ell \geq 1$, the coefficients $\mu_{j,k}^{(\ell)}, \nu_{j,k}^\pm, \rho_{j,k}$ are "homogeneous" expressions in the parameters $\{\bar{a}_i\}_i$, $\{\bar{\gamma}_i\}_i$:*

$$*_{j,k} = *_{j,k}(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_{k-j+1}, \bar{\gamma}_1, \bar{\gamma}_2, \cdots, \bar{\gamma}_{k-j}), \qquad *_{j,k} = \mu_{j,k}^{(\ell)}, \nu_{j,k}^\pm, \rho_{j,k},$$

where $*_{j,k}$ is a linear combination of terms of the form

$$\bar{a}_1^{p_1}\bar{a}_2^{p_2}\cdots\bar{a}_{k-j+1}^{p_{k-j+1}}\bar{\gamma}_1^{q_1}\bar{\gamma}_2^{q_2}\cdots\bar{\gamma}_{k-j}^{q_{k-j}}, \tag{4.12}$$

with

$$\text{i)}\ \sum_{i=1}^{k-j} p_i \geq j, \qquad \text{ii)}\ \sum_{i=1}^{k-j+1} ip_i + \sum_{i=1}^{k-j} iq_i = k.$$

Proof. Let us study how the coefficients $\{\mu_{j,k}\}_{j,k}$ in the expansion of $\bar{\eta}_n$ depend on the parameters $\{\bar{a}_i\}_i$, $\{\bar{\gamma}_i\}_i$. The "homogeneous" structure of the expansion of the coefficients $\{\mu_{j,k}^{(\ell)}\}_{j,k}$, $\{\nu_{j,k}^\pm\}_{j,k}$ and $\{\rho_{j,k}\}_{j,k}$ is shown in the same way as for the coefficients $\{\mu_{j,k}\}_{j,k}$.

For any integers $\ell \geq 1$ and $n \geq n_0$, recall the equation for $\bar{\eta}_n$ obtained in the proof of Lemma 4.11:

$$\bar{\eta}_n = \sum_{r=0}^{\ell} \binom{n}{r} \left(\sum_{p=1}^{\ell} \bar{a}_p \lambda^{np}\bar{\eta}_n^p \Big(1 + \sum_{q=1}^{\ell} \bar{\gamma}_q \lambda^{nq} \bar{\eta}_n^q\Big)^p\right)^r \Big(1 + \sum_{s=1}^{\ell} \bar{\gamma}_s \lambda^{ns} \bar{\eta}_n^s\Big) + O\big(n^{\ell+1}\lambda^{n(\ell+2)}\big).$$

The expansion of this expression is a combination of terms as in (4.12). In particular, for any integer $\ell \geq 1$, we see that $\bar{a}_\ell$ first appears with the weight $n\lambda^{n\ell}$ (for $r = 1$

and $p=\ell$ with the above notation), while $\bar\gamma_\ell$ first appears with the weight $\lambda^{n\ell}$ (for $r=0$ and $s=\ell$ with the above notation).

More generally, the "homogeneity" property ii) above is essentially due to the fact that in the previous expansion, $\bar a_p$ always comes together with the weight λ^{np}, while $\bar\gamma_q$ always comes together with the weight λ^{nq}.

Note that increasing the exponent of n in the expansion of $\bar\eta_n$ corresponds to taking derivatives of the function Δ in formula (4.6). In terms of the above expansion, those derivatives are associated to certain binomial coefficients, and each time we increase the exponent of n by one, we increase the exponent of λ^n by at least one too, depending on the weight of the coefficient $\bar a_p$ associated to this derivative. Together with the previous remark on the first appearance of $\bar a_\ell$, $\bar\gamma_\ell$, this explains the constraint in (4.12) and ii) on the coefficients which can enter the expression associated to a specific weight, and why they depend on the difference $k-j$ between the exponent j of n^j and the exponent k of λ^{nk} (the coefficients $\bar a_{k-j+1}^{p_{k-j+1}}$ and $\bar\gamma_{k-j}^{q_{k-j}}$ are obtained when all the derivatives we take are associated to $\bar a_1$).

Besides, the reason why we have an inequality and not an equality in point i) is because the binomial coefficients $\begin{pmatrix} n \\ r\end{pmatrix}$ are not homogeneous polynomials in n. $\square$

Remark 4.14. *The reason why the respective weights of $\bar a_\ell$ and $\bar\gamma_\ell$ on their first appearance differ by a factor n is due to the fact that any orbit under consideration spends much more time (n steps) in the region where we have Birkhoff coordinates, while the gluing term associated to the coefficient $\bar\gamma_\ell$ accounts for a bounded number of steps in the orbit.*

Lemma 4.15. *With the notation introduced in Lemma 4.11, for every $k\geq 1$, there exist constants $c_{0,k}, c_{j,k+j}, c^{\pm}_{j,k+j}, c'_{j,k+j}\in\mathbb{R}$, $j=1,2$, with*

$$c_{0,k} = c_{0,k}(\bar\gamma_1,\bar\gamma_2,\cdots,\bar\gamma_{k-1}),$$
$$*_{j,k+j} = *_{j,k+j}(\bar a_1,\bar a_2,\cdots,\bar a_k,\bar\gamma_1,\bar\gamma_2,\cdots,\bar\gamma_{k-1}),\qquad\qquad * = c, c^{\pm}, c',$$

such that

$$\left\{\begin{array}{rcl}\mu_{0,k} &=& \bar\gamma_k + c_{0,k},\\ \mu_{1,k+1} &=& (k+3)\bar a_1\cdot\bar\gamma_k + \bar a_{k+1} + c_{1,k+1},\\ \mu_{2,k+2} &=& \frac{(k+3)(k+5)}{2}\bar a_1^2\cdot\bar\gamma_k + (k+3)\bar a_1\cdot\bar a_{k+1} + c_{2,k+2};\end{array}\right. \tag{4.13a}$$

$$\left\{\begin{array}{rcl}\nu^{\pm}_{0,k} &=& 0,\\ \nu^{\pm}_{1,k+1} &=& \pm(2\bar a_1\cdot\bar\gamma_k + \bar a_{k+1}) + c^{\pm}_{1,k+1},\\ \nu^{\pm}_{2,k+2} &=& \pm(k+2\pm 1)\bar a_1(2\bar a_1\cdot\bar\gamma_k+\bar a_{k+1}) + c^{\pm}_{2,k+2};\end{array}\right. \tag{4.13b}$$

$$\left\{\begin{array}{rcl}\rho_{0,k} &=& 0,\\ \rho_{1,k+1} &=& -2\bar a_1\cdot\bar\gamma_k - (k+1)\bar a_{k+1} + c'_{1,k+1},\\ \rho_{2,k+2} &=& -2(k+2)\bar a_1^2\cdot\bar\gamma_k - ((k+1)^2+1)\bar a_1\cdot\bar a_{k+1} + c'_{2,k+2}.\end{array}\right. \tag{4.13c}$$

Remark 4.16. *Before giving the details of the proof, let us explain how the computations are carried out. We first focus on $\bar\eta_n$ and study the coefficients of the expansion given by Lemma 4.11; the expressions of $\lambda^{\mp n}\Delta_n^{\pm n}$ and $1-n\Delta_n'\Delta_n^{-1}$ follow from that of $\bar\eta_n$, as they are obtained by evaluating the functions $\Delta,\Delta',\gamma,\dots$ at the point $\bar\eta_n$. Note that the equation in Lemma 4.6 can be rewritten as*

$$\bar\eta_n = \bar\Delta(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))^n\bar\gamma(\lambda^n\bar\eta_n), \tag{4.14}$$

with

$$\bar{\Delta}\colon z \mapsto 1 + \sum_{j=1}^{+\infty} \bar{a}_j z^j, \quad \text{and} \quad \bar{\gamma}\colon z \mapsto 1 + \gamma(z) = 1 + \sum_{j=1}^{+\infty} \bar{\gamma}_j z^j.$$

Fix an integer $k \geq 1$. Based on the implicit equation (4.14) satisfied by $\bar{\eta}_n$, we determine inductively the coefficients of $\bar{a}_{k+1}$ and $\bar{\gamma}_k$ in the expression of $\bar{\eta}_n$. More precisely, in order to explicit the dependence of $\bar{\eta}_n$ on $\bar{a}_{k+1}$, resp. $\bar{\gamma}_k$, we differentiate (4.14) with respect to $\bar{a}_{k+1}$, resp. $\bar{\gamma}_k$, and plug the expansion we already have in the right hand side. The presence of the extra factor λ^n acts as a shift, i.e., it "propagates" the information we have one step further. Besides, in order to avoid seeing new unknown quantities $\bar{a}_{k+2}, \bar{\gamma}_{k+1}, \dots$, we only consider the terms in the expansion of $\bar{\eta}_n$ which are aligned along the line of slope 1 based at the points where $\bar{a}_{k+1}$ and $\bar{\gamma}_k$ first appear (see Fig. 6). At each step, the expressions we obtain in the derivative of (4.14) are combinations of terms of two kinds:

- *terms of the form $n^A \bar{a}_1^B \bar{a}_{k+1} \lambda^{nC} \bar{\eta}_n^D$ or $n^A \bar{a}_1^B \bar{\gamma}_k \lambda^{nC} \bar{\eta}_n^D$ with C large; in this case, we use the expressions of the first coefficients of $\bar{\eta}_n^D$ given by (4.9);*
- *terms of the form $n^A \bar{a}_1^B \lambda^{nC} \bar{\eta}_n^D$ with C small; in this case, based on the expansion computed previously, we identify the coefficients of $\bar{a}_{k+1}$ and $\bar{\gamma}_k$ in the expression of $\bar{\eta}_n^D$ in order to go one step further in the expansion.*

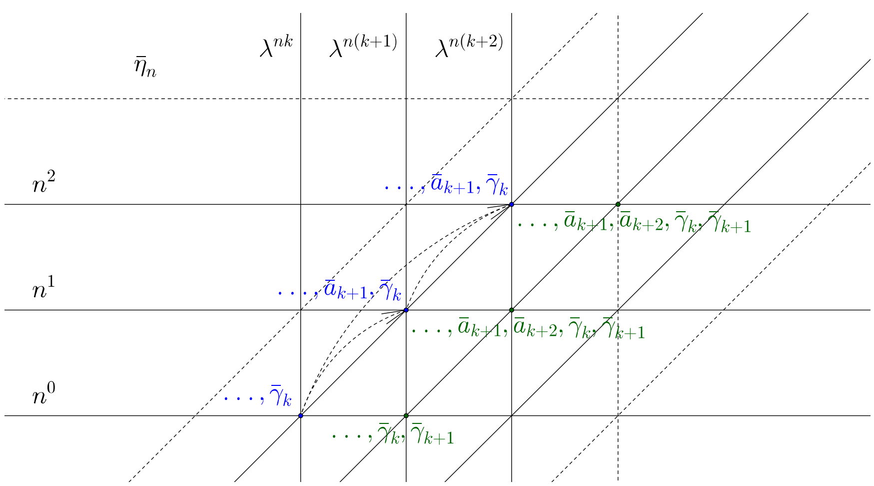


FIGURE 6. Coefficients in the series expansion of $\bar{\eta}_n$.

Definition 4.17. *For any integer $k \geq 0$, a formal series $\mathcal{S}$ is called k-triangular if*

(1) *it is of the form $\mathcal{S} = \sum_{p-q \geq k, q \geq 0} s_{q,p} n^q \lambda^{np}$;*
(2) *its principal part $\mathbb{P}(\mathcal{S}) := \sum_{p-q=k} s_{q,p} n^q \lambda^{np}$ is non-zero, i.e., $\mathbb{P}(\mathcal{S}) \neq 0$.*

In other words, $\mathcal{S}$ is the product of λ^{nk} and of a balanced series with non-zero principal part.

Let us state some basic properties of triangular series which will be useful in the following.

Remark 4.18. *Let $\mathcal{S}_0$ be a k-triangular series, for some integer $k \geq 0$.*

- *For any k-triangular series $\mathcal{S}_1$ such that $\mathbb{P}(\mathcal{S}_0) + \mathbb{P}(\mathcal{S}_1) \neq 0$, the series $\mathcal{S}_0 + \mathcal{S}_1$ is k-triangular, and $\mathbb{P}(\mathcal{S}_0 + \mathcal{S}_1) = \mathbb{P}(\mathcal{S}_0) + \mathbb{P}(\mathcal{S}_1)$.*
- *For any integer $\ell \geq 0$ and any ℓ-triangular series $\mathcal{S}_2$ such that $\mathbb{P}(\mathcal{S}_0)\mathbb{P}(\mathcal{S}_2) \neq 0$, the series $\mathcal{S}_0\mathcal{S}_2$ is $(k+\ell)$-triangular and it holds*

$$\mathbb{P}(\mathcal{S}_0\mathcal{S}_2) = \mathbb{P}(\mathcal{S}_0)\mathbb{P}(\mathcal{S}_2).$$

In particular, if $\mathcal{S}$ is a balanced series and if $\omega\colon z \mapsto \sum_{j=0}^{+\infty} \omega_j z^j$ is an analytic function with $\omega_0 \neq 0$, then $\omega(\lambda^n \mathcal{S})$ is a 0-triangular series, and $\mathbb{P}(\omega(\mathcal{S})) = \omega_0$.

Proof of Lemma 4.15. Let $k \geq 1$.

- *Proof of* (4.13a). Since $\partial_{\bar{a}_{k+1}}\bar{\Delta}\colon z \mapsto z^{k+1}$, we thus get

$$\partial_{\bar{a}_{k+1}}\bar{\eta}_n = n\bar{\Delta}(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))^{n-1}\Big[\lambda^{n(k+1)}\bar{\eta}_n^{k+1}(\bar{\gamma}(\lambda^n\bar{\eta}_n))^{k+2} + \lambda^n\partial_{\bar{a}_{k+1}}\bar{\eta}_n[\bar{\gamma}(\lambda^n\bar{\eta}_n)+$$
$$+\lambda^n\bar{\eta}_n\gamma'(\lambda^n\bar{\eta}_n)]\bar{\Delta}'(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))\bar{\gamma}(\lambda^n\bar{\eta}_n)\Big] + \partial_{\bar{a}_{k+1}}\bar{\eta}_n \cdot \bar{\Delta}(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))^n\lambda^n\gamma'(\lambda^n\bar{\eta}_n).$$

By the fact that $\bar{\eta}_n$ is a 0-triangular series, it follows from this expression that $\partial_{\bar{a}_{k+1}}\bar{\eta}_n$ is a k-triangular series with leading term

$$\partial_{\bar{a}_{k+1}}\bar{\eta}_n = n\lambda^{n(k+1)} + \dots \tag{4.15}$$

Besides, the terms $\gamma(\lambda^n\bar{\eta}_n)$ and $\lambda^n\gamma'(\lambda^n\bar{\eta}_n)$ are 1-triangular and are mutiplied by k-triangular terms in the previous expression, hence by Remark 4.18 they do not contribute to the principal part of $\partial_{\bar{a}_{k+1}}\bar{\eta}_n$. For the same reason, the only term in the expansion of $\bar{\Delta}'(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))$ which contributes to the principal part of $\partial_{\bar{a}_{k+1}}\bar{\eta}_n$ is the constant term $\bar{a}_1$. By (4.9) and (4.15), we thus get

$$\begin{aligned}\mathbb{P}(\partial_{\bar{a}_{k+1}}\bar{\eta}_n) &= \mathbb{P}\Big(\bar{\Delta}(\lambda^n\bar{\eta}_n)^{n-1}\Big[n\lambda^{n(k+1)}\bar{\eta}_n^{k+1} + n\lambda^n\partial_{\bar{a}_{k+1}}\bar{\eta}_n\bar{a}_1\Big]\Big)\\ &= (1+n\bar{a}_1\lambda^n)\Big[n\lambda^{n(k+1)}(1+(k+1)n\bar{a}_1\lambda^n) + n^2\bar{a}_1\lambda^{n(k+2)}\Big] + O(n^3\lambda^{n(k+3)})\\ &= n\lambda^{n(k+1)} + (k+3)n^2\bar{a}_1\lambda^{n(k+2)} + O(n^3\lambda^{n(k+3)}).\end{aligned}$$

Similarly, $\partial_{\bar{\gamma}_k}\bar{\gamma} = \partial_{\bar{\gamma}_k}\gamma\colon z \mapsto z^k$, hence

$$\partial_{\bar{\gamma}_k}\bar{\eta}_n = \bar{\Delta}(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))^n\Big[\lambda^{nk}\bar{\eta}_n^k + \lambda^n\partial_{\bar{\gamma}_k}\bar{\eta}_n\gamma'(\lambda^n\bar{\eta}_n)\Big] + n\bar{\Delta}'(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))\cdot$$
$$\bar{\Delta}(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))^{n-1}\Big[\lambda^n\partial_{\bar{\gamma}_k}\bar{\eta}_n[\bar{\gamma}(\lambda^n\bar{\eta}_n) + \lambda^n\bar{\eta}_n\gamma'(\lambda^n\bar{\eta}_n)] + \lambda^{n(k+1)}\bar{\eta}_n^{k+1}\Big]\bar{\gamma}(\lambda^n\bar{\eta}_n).$$

By the fact that $\bar{\eta}_n$ is a balanced series, it follows from this expression that $\partial_{\bar{\gamma}_k}\bar{\eta}_n$ is a k-triangular series with leading term

$$\partial_{\bar{\gamma}_k}\bar{\eta}_n = \lambda^{nk} + \dots$$

As previously, the terms $\gamma(\lambda^n\bar{\eta}_n)$ and $\lambda^n\gamma'(\lambda^n\bar{\eta}_n)$ do not contribute to the principal part of $\partial_{\bar{\gamma}_k}\bar{\eta}_n$, as they are 1-triangular and are mutiplied by k-triangular terms in the previous expression, and $\bar{\Delta}'(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))$ can be replaced with $\bar{a}_1$. Together with (4.9), the previous expansion thus yields

$$\begin{aligned}\mathbb{P}(\partial_{\bar\gamma_k}\bar\eta_n) &= \mathbb{P}\Big(\bar\Delta(\lambda^n\bar\eta_n)^n\lambda^{nk}\bar\eta_n^k + n\bar a_1\lambda^n\bar\Delta(\lambda^n\bar\eta_n)^{n-1}\Big[\partial_{\bar\gamma_k}\bar\eta_n + \lambda^{nk}\bar\eta_n^{k+1}\Big]\Big)\\ &= (1+n\bar a_1\lambda^n)\Big[\lambda^{nk}(1+kn\bar a_1\lambda^n) + n\bar a_1\lambda^n\cdot 2\lambda^{nk}\Big] + O(n^2\lambda^{n(k+2)})\\ &= \lambda^{nk} + (k+3)n\bar a_1\lambda^{n(k+1)} + O(n^2\lambda^{n(k+2)}).\end{aligned}$$

Plugging this back in the expression of $\mathbb{P}(\partial_{\bar\gamma_k}\bar\eta_n)$, and going one step further in the expansion of $\bar\Delta(\lambda^n\bar\eta_n)^n$, we can compute the third term in the principal part of $\partial_{\bar\gamma_k}\bar\eta_n$ (we do not give the details here as it will not be needed in the following).

• *Proof of* (4.13b). The derivation of the coefficients $\{\nu^\pm_{j,k}\}_{j,k}$ is done in a similar way. Yet, unlike $\bar\eta_n$, now there is no implicit equation anymore, thus we can differentiate $\lambda^{\mp n}\Delta_n^{\pm n}$ directly and use the expressions of $\{\mu_{j,k}\}_{j,k}$ obtained above. We note that

$$\lambda^{\mp n}\Delta_n^{\pm n} = \bar\Delta(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))^{\pm n}.$$

We will detail the calculations only for $\pm = -$ which is the case we will need in the following. The case where $\pm = +$ is analogous. By the above formula, we thus get

$$\begin{aligned}\partial_{\bar a_{k+1}}\lambda^n\Delta_n^{-n} = -n\bar\Delta(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))^{-(n+1)}\Big[&\lambda^{n(k+1)}\bar\eta_n^{k+1}(\bar\gamma(\lambda^n\bar\eta_n))^{k+1}+\\ &+\lambda^n\partial_{\bar a_{k+1}}\bar\eta_n[\bar\gamma(\lambda^n\bar\eta_n)+\lambda^n\bar\eta_n\gamma'(\lambda^n\bar\eta_n)]\bar\Delta'(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))\Big].\end{aligned}$$

We see that the associated series is also k-triangular. For the same reason as before, the terms $\gamma(\lambda^n\bar\eta_n)$ and $\lambda^n\gamma'(\lambda^n\bar\eta_n)$ need not be considered, and the only term of $\bar\Delta'(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))$ which contributes to the principal part of $\partial_{\bar a_{k+1}}\lambda^n\Delta_n^{-n}$ is $\bar a_1$. Replacing $\partial_{\bar a_{k+1}}\bar\eta_n$ with the value computed previously, and by (4.9), we thus obtain

$$\begin{aligned}&\mathbb{P}(\partial_{\bar a_{k+1}}\lambda^n\Delta_n^{-n}) = -\mathbb{P}\Big(\bar\Delta(\lambda^n\bar\eta_n)^{-(n+1)}\Big[n\lambda^{n(k+1)}\bar\eta_n^{k+1} + n\bar a_1\lambda^n\partial_{\bar a_{k+1}}\bar\eta_n\Big]\Big)\\ &= -(1-n\bar a_1\lambda^n)\Big[n\lambda^{n(k+1)}(1+(k+1)n\bar a_1\lambda^n) + n^2\bar a_1\lambda^{n(k+2)}\Big] + O(n^3\lambda^{n(k+3)})\\ &= -n\lambda^{n(k+1)} - (k+1)n\bar a_1\lambda^{n(k+2)} + O(n^3\lambda^{n(k+3)}).\end{aligned}$$

Similarly,

$$\begin{aligned}\partial_{\bar\gamma_k}\lambda^n\Delta_n^{-n} = -n\bar\Delta'(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))\bar\Delta(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))^{-(n+1)}\Big[&\lambda^{n(k+1)}\bar\eta_n^{k+1}+\\ &+\lambda^n\partial_{\bar\gamma_k}\bar\eta_n[\bar\gamma(\lambda^n\bar\eta_n)+\lambda^n\bar\eta_n\gamma'(\lambda^n\bar\eta_n)]\Big].\end{aligned}$$

Arguing as before, and replacing $\partial_{\bar\gamma_k}\bar\eta_n$ with the value computed previously, we get

$$\begin{aligned}&\mathbb{P}(\partial_{\bar\gamma_k}\lambda^n\Delta_n^{-n}) = -\mathbb{P}\Big(n\bar a_1\bar\Delta(\lambda^n\bar\eta_n)^{-(n+1)}\Big[\lambda^{n(k+1)}\bar\eta_n^{k+1} + \lambda^n\partial_{\bar\gamma_k}\bar\eta_n\Big]\Big)\\ &= -n\bar a_1\lambda^n(1-n\bar a_1\lambda^n)\Big[\lambda^{nk}(1+(k+1)n\bar a_1\lambda^n) + \lambda^{nk} + (k+3)n\bar a_1\lambda^{n(k+1)}\Big] + O(n^3\lambda^{n(k+3)})\\ &= -2n\lambda^{n(k+1)} - 2(k+1)n^2\bar a_1\lambda^{n(k+2)} + O(n^3\lambda^{n(k+3)}).\end{aligned}$$

• *Proof of* (4.13c). Let us now deal with the coefficients $\{\rho_{j,k}\}_{j,k}$. The computations are carried out in the same way as for the coefficients $\{\nu^-_{j,k}\}_{j,k}$. Set $\bar D\colon z\mapsto\sum_{j=1}^{+\infty} j\bar a_j z^j$. It holds

$$1 - n\Delta_n'\Delta_n^{-1} = 1 - n\bar D(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))\bar\Delta(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))^{-1}.$$

We have $\partial_{\bar a_{k+1}}\bar D\colon z\mapsto (k+1)z^{k+1}$. It follows that

$$\partial_{\bar a_{k+1}}(1-n\Delta_n'\Delta_n^{-1}) = -n\lambda^{n(k+1)}\bar\eta_n^{k+1}(\bar\gamma(\lambda^n\bar\eta_n))^{k+1}\bar\Delta(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))^{-1}\Big[(k+1)-$$
$$-\Delta_n'\Delta_n^{-1}\Big] - n\lambda^n\partial_{\bar a_{k+1}}\bar\eta_n[\bar\gamma(\lambda^n\bar\eta_n)+\lambda^n\bar\eta_n\gamma'(\lambda^n\bar\eta_n)]\bar\Delta(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))^{-1}\cdot$$
$$\cdot\Big[\bar D'(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n)) - \bar\Delta'(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))\Delta_n'\Delta_n^{-1}\Big].$$

In the previous expression, $[(k+1)-\Delta_n'\Delta_n^{-1}]$ is a 0-triangular series whose principal part is reduced to $k+1$, as $\Delta_n'\Delta_n^{-1}$ is 1-triangular. Similarly, $\bar D'(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n)) - \bar\Delta'(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))\Delta_n'\Delta_n^{-1}$ is 0-triangular, and its principal part is equal to $\bar a_1$. We see that $\partial_{\bar a_{k+1}}(1-n\Delta_n'\Delta_n^{-1})$ is k-triangular, and as before, the terms $\gamma(\lambda^n\bar\eta_n)$ and $\lambda^n\gamma'(\lambda^n\bar\eta_n)$ do not contribute to the principal part of $\partial_{\bar a_{k+1}}(1-n\Delta_n'\Delta_n^{-1})$. We thus obtain

$$\begin{aligned}
&\mathbb{P}\Big(\partial_{\bar a_{k+1}}(1-n\Delta_n'\Delta_n^{-1})\Big)\\
&= -\mathbb{P}\Big(\bar\Delta(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))^{-1}\Big[n\lambda^{n(k+1)}\bar\eta_n^{k+1}(k+1)+n\bar a_1\lambda^n\partial_{\bar a_{k+1}}\bar\eta_n\Big]\Big)\\
&= -n\lambda^{n(k+1)}(1+(k+1)n\bar a_1\lambda^n)(k+1) - n\bar a_1\lambda^n\cdot n\lambda^{n(k+1)} + O(n^3\lambda^{n(k+3)})\\
&= -(k+1)n\lambda^{n(k+1)} - [(k+1)^2+1]n^2\bar a_1\lambda^{n(k+2)} + O(n^3\lambda^{n(k+3)}).
\end{aligned}$$

Finally, we have

$$\partial_{\bar\gamma_k}(1-n\Delta_n'\Delta_n^{-1}) = -n\Big[\lambda^n\partial_{\bar\gamma_k}\bar\eta_n[\bar\gamma(\lambda^n\bar\eta_n)+\lambda^n\bar\eta_n\gamma'(\lambda^n\bar\eta_n)] + \lambda^{n(k+1)}\bar\eta_n^{k+1}\Big]\cdot$$
$$\cdot\,\bar\Delta(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))^{-1}\cdot\Big[\bar D'(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n)) - \bar\Delta'(\lambda^n\bar\eta_n\bar\gamma(\lambda^n\bar\eta_n))\Delta_n'\Delta_n^{-1}\Big].$$

Arguing as above, we deduce that

$$\begin{aligned}
&\mathbb{P}\Big(\partial_{\bar\gamma_k}(1-n\Delta_n'\Delta_n^{-1})\Big) = -\mathbb{P}\Big(n\bar a_1(\lambda^n\partial_{\bar\gamma_k}\bar\eta_n + \lambda^{n(k+1)}\bar\eta_n^{k+1})\Big)\\
&= -n\bar a_1\lambda^{n(k+1)} - (k+3)n^2\bar a_1^2\lambda^{n(k+2)} - n\bar a_1\lambda^{n(k+1)} - (k+1)n^2\bar a_1^2\lambda^{n(k+2)} + O(n^3\lambda^{n(k+3)})\\
&= -2n\bar a_1\lambda^{n(k+1)} - 2(k+2)n^2\bar a_1^2\lambda^{n(k+2)} + O(n^3\lambda^{n(k+3)}).
\end{aligned}$$

□

We reported the above computations since they could be useful in some further developments of this work. In the current section we will in fact only rely upon some specific combinations, which occur in the term denoted with I_n and we collect in the following corollary.

Corollary 4.19. *The following holds:*

$$\lambda^n\Delta_n^{-n}(1-n\Delta_n'\Delta_n^{-1}) = \sum_{p=0}^{\infty}\sum_{q=0}^{p} L^*_{q,p}n^q\lambda^{np},$$

and we have:

$$L^*_{0,p} = c^*_{0,p}, \tag{4.16a}$$
$$L^*_{1,p+1} = -4\bar a_1\bar\gamma_p - (p+2)\bar a_{p+1} + c^*_{1,p+1}, \tag{4.16b}$$
$$L^*_{2,p+2} = -2(2p+1)\bar a_1^2\bar\gamma_p - (p+1)^2\bar a_1\bar a_{p+1} + c^*_{2,p+2}, \tag{4.16c}$$

where $c^*_{i,p+1}$ *depend only on the coefficients* $\{\bar\gamma_\ell, \bar a_{\ell+1}\}_{0\le\ell<p}$. *Moreover*

$$L^*_{0,0} = 1, \qquad L^*_{1,1} = -2\bar a_1, \qquad L^*_{2,2} = -\frac{1}{2}\bar a_1^2. \tag{4.17}$$

Proof. By definition of $L^*_{q,p}$, we gather that for any $p \ge 0$ and $0 \le q \le p$:

$$L^*_{q,p} = \sum_{\substack{p'+p''=p\\ q'+q''=q}} \nu^-_{q',p'}\rho_{q'',p''}.$$

Observe that the contribution of terms for which both $p'-q' < p$ and $p''-q'' < p$ can be absorbed in the terms c, since they do not depend on either $\bar a_{p+1}$ nor $\bar\gamma_p$ by Remark 4.13.

In particular, if $q = 0$, then necessarily $q' = q'' = 0$, hence:

$$L^*_{0,p} = \nu^-_{0,p}\rho_{0,0} + \nu^-_{0,0}\rho_{0,p} + c_{0,p};$$

and using (4.9) and Lemma 4.15 we obtain (4.16a). If $q = 1$ then either $q' = 1$ and $q'' = 0$ or $q' = 0$ and $q'' = 1$. Observe that by Lemma 4.15, the coefficients $\nu^-_{0,k}$ and $\rho_{0,k}$ are 0 unless $k = 0$; we conclude that:

$$L^*_{1,p+1} = \nu^-_{1,p+1}\rho_{0,0} + \nu^-_{0,0}\rho_{1,p+1} + c_{1,p+1},$$

which yields (4.16b). Finally, we consider the case $q = 2$; in this case one could have $q' = 0, 1, 2$ and correspondingly $q'' = 2 - q'$. This leads to:

$$L^*_{2,p+2} = \nu^-_{2,p+2}\rho_{0,0} + \nu^-_{1,p+1}\rho_{1,1} + \nu^-_{1,1}\rho_{1,p+1} + \nu^-_{0,0}\rho_{2,p+2},$$

which yields (4.16c). Equations (4.17) then follow from similar arguments, or directly from Lemma 4.10. □

4.3. **Determination of the scaled coefficients** $\{\bar g_\ell, \bar\gamma_\ell, \bar a_\ell\}_{\ell\ge0}$. In this part, we keep the same notation and show how the above estimates can be employed to show that the scaled coefficients $\{\bar g_\ell, \bar\gamma_\ell, \bar a_\ell\}_{\ell\ge0}$ introduced in (4.7) are $\mathcal{MLS}$-invariants.

Lemma 4.20. *There exists a sequence of real numbers*

$$(L_{q,p})_{\substack{p=0,\cdots,+\infty\\ q=0,\cdots,p}}$$

such that for any integer $n \ge n_0$, *we have the following expansion:*

$$2\lambda^n\cosh(2(n+1)\mathrm{LE}(h_n)) = \sum_{p=0}^{+\infty}\sum_{q=0}^{p} L_{q,p}n^q\lambda^{np}. \tag{4.18}$$

Moreover, for any $p \ge 1$, *the following linear relation holds:*

$$W_p = A_pV_p + C_p, \tag{4.19}$$

where $V_p, W_p, C_p \in \mathbb{R}^3$ *are defined as:*

$$V_p := \begin{pmatrix}\bar g_p\\ \bar\gamma_p\\ \bar a_{p+1}\end{pmatrix}, \qquad W_p := \begin{pmatrix}L_{0,p}\\ L_{1,p+1}\\ L_{2,p+2}\end{pmatrix}, \qquad C_p := \begin{pmatrix}C_{0,p}\\ C_{1,p+1}\\ C_{2,p+2}\end{pmatrix}, \tag{4.20}$$

$A_p \in \mathfrak{M}_3(\mathbb{R})$ *is given by:*

$$A_p := \begin{pmatrix} 1 & 0 & 0 \\ (p-2)\bar{a}_1 & -4\bar{a}_1 g_0 & -(p+2)g_0 \\ \frac{p^2-2p-1}{2}\bar{a}_1^2 & -2(2p+1)\bar{a}_1^2 g_0 & -(p+1)^2\bar{a}_1 g_0 \end{pmatrix},$$

and for $i \in \{0,1,2\}$, the constants $C_{i,p+i} \in \mathbb{R}$ only depend on the coefficients[5] $\{\bar{g}_\ell, \bar{\gamma}_\ell, \bar{a}_{\ell+1}\}_{0 \le \ell < p}$.

Proof. Let $n \ge n_0$. By Lemma 4.8, we have

$$2\lambda^n \cosh(2(n+1)\mathrm{LE}(h_n)) = \mathrm{I}_n + \lambda^n \mathrm{II}_n + \lambda^{2n}\mathrm{III}_n.$$

By Remark 4.12, I_n, II_n and III_n are balanced series, i.e.:

$$\mathrm{I}_n = \sum_{p=0}^{+\infty}\sum_{q=0}^{p} L^{\mathrm{I}}_{q,p} n^q \lambda^{np}, \quad \mathrm{II}_n = \sum_{p=0}^{+\infty}\sum_{q=0}^{p} L^{\mathrm{II}}_{q,p} n^q \lambda^{np}, \quad \mathrm{III}_n = \sum_{p=0}^{+\infty}\sum_{q=0}^{p} L^{\mathrm{III}}_{q,p} n^q \lambda^{np},$$

and therefore also the left hand side of (4.18) is a balanced series, i.e. (4.18) holds. We thus need to show (4.19). Let us fix an integer $p \ge 1$. Observe that

$$L_{q,p} = L^{\mathrm{I}}_{q,p} + L^{\mathrm{II}}_{q,p-1} + L^{\mathrm{III}}_{q,p-2}; \tag{4.21}$$

moreover, by construction (see Remark 4.13), we can also conclude that $L^*_{q,p}$ only depend on $\{\bar{a}_{i+1}, \bar{\gamma}_i, \bar{g}_i\}_{i=0,\cdots,p-q}$ for $* = \mathrm{I}_n$, II_n and III_n. Hence, the contributions to $L_{0,p}$, (resp. $L_{1,p+1}$, $L_{2,p+2}$) of the last two terms in (4.21) do contain no $\bar{g}_p$, $\bar{\gamma}_p$ or $\bar{a}_{p+1}$, and can thus be absorbed in $C_{0,p}$ (resp. $C_{1,p+1}$, $C_{2,p+2}$). In order to show (4.19) it thus suffices to study the coefficients $L^{\mathrm{I}}_{q,p}$ of the balanced series:

$$\mathrm{I}_n = \lambda^n \Delta_n^{-n}\big(1 - n\Delta_n'\Delta_n^{-1}\big) g(\lambda^n \bar{\eta}_n).$$

We begin to study the dependence of I_n on $\bar{g}_p$ (i.e. the first column of A_p). We write:

$$g(\eta_n) = \sum_{\ell=0}^{+\infty} \bar{g}_\ell \lambda^{n\ell}\bar{\eta}_n^\ell = \sum_{\ell=0}^{+\infty} \bar{g}_\ell \lambda^{n\ell} \sum_{k=0}^{\infty}\sum_{j=0}^{k} \mu^{(\ell)}_{j,k} n^j \lambda^{nk}.$$

Observe that in the expansion of g, the coefficient $\bar{g}_\ell$ is multiplied by $\lambda^\ell \bar{\eta}_n^\ell$; hence:

$$\begin{aligned} \mathrm{I}_n &= \left[\sum_{p'=0}^{\infty}\sum_{q'=0}^{p'} L^*_{q',p'} n^{q'}\lambda^{np'}\right] \cdot \left[\sum_{\ell=0}^{+\infty} \bar{g}_\ell \lambda^{n\ell} \sum_{k=0}^{\infty}\sum_{j=0}^{k} \mu^{(\ell)}_{j,k} n^j \lambda^{nk}\right] \\ &= \sum_{p=0}^{\infty} \sum_{p'+p''+p'''=p} \sum_{q=0}^{p'+p'''} \sum_{q'+q'''=q} L^*_{q',p'} \bar{g}_{p''} \mu^{(p'')}_{q''',p'''} n^q \lambda^{np}, \end{aligned}$$

which yields:

$$L^{\mathrm{I}}_{q,p} = \sum_{p'+p''+p'''=p} \sum_{\substack{q'+q'''=q \\ 0 \le q' \le p' \\ 0 \le q''' \le p'''}} L^*_{q',p'} \bar{g}_{p''} \mu^{(p'')}_{q''',p'''}.$$

[5] Recall the notation introduced in (4.7).

In order to extract the contribution of $\bar{g}_p$ we thus need to set $p'' = p$; we conclude that, for $i \in \{0,1,2\}$, the coefficient $\bar{g}_p$ appears in $L_{i,p+i}$ multiplied by a factor $K_i^p = \sum_{r+s=i} \mu_{r,r}^{(p)} L_{s,s}^*$ and by (4.9) and Corollary 4.19 we can at last conclude:

$$K_0^p = 1, \qquad K_1^p = (p-2)\bar{a}_1, \qquad K_2^p = \frac{p^2-2p-1}{2}\bar{a}_1^2.$$

We now proceed to study the second and third columns of A_p, which amounts to study the dependence on $\bar{\gamma}$ and $\bar{a}$. This, in principle, entails more work than the previous task, since the coefficients $\bar{\gamma}$ and $\bar{a}$ show up in the expansions of each of the terms in I_n, and not just the last term. As a matter of fact, the last term does not contribute at all; in fact notice that, as before, we can write:

$$g(\eta_n) = \sum_{\ell=0}^{+\infty} \bar{g}_\ell \lambda^{n\ell} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \mu_{j,k}^{(\ell)} n^j \lambda^{nk}.$$

As noted earlier, the coefficients $\bar{a}_{p+1}$ and $\bar{\gamma}_p$ would only occur in the expression for $\mu_{j,k}^{(\ell)}$ with $k - j \geq p$. If we consider $L_{0,p}$ (resp. $L_{1,p+1}$, $L_{2,p+2}$), we thus must set $k = p$ (resp. $p+1$, $p+2$); in turn this implies that $\ell = 0$ (since $\ell + k = p + i$). But then $\mu_{j,k}^{(0)} = 0$ for any j, k. Thus it suffices to consider the expansion of

$$\tilde{I}_n = g_0 \cdot \lambda^n \Delta_n^{-n} \big(1 - n\Delta_n' \Delta_n^{-1}\big)$$

and the statement follows from Corollary 4.19, □

We have seen in Lemma 4.10 that the values of g_0 and $\bar{a}_1$ are $\mathcal{MLS}$-invariant; moreover by Remark 4.5, $g_0 \neq 0$.

Corollary 4.21. *Under the assumption that the first Birkhoff coefficient does not vanish, i.e., $\bar{a}_1 \neq 0$, then the coefficients $\{\bar{g}_\ell, \bar{\gamma}_\ell, \bar{a}_\ell\}_{\ell \geq 0}$ are $\mathcal{MLS}$-invariants.*

Proof. By Theorem 1.12, the Marked Lyapunov Spectrum is a $\mathcal{MLS}$-invariant; in particular, for each $n \geq n_0$, $\mathrm{LE}(h_n)$ is a $\mathcal{MLS}$-invariant. By Lemma 4.20, we also have

$$2\lambda^n \cosh(2(n+1)\mathrm{LE}(h_n)) = \sum_{p=0}^{+\infty} \sum_{q=0}^{p} L_{q,p} n^q \lambda^{np}.$$

Notice that each term $\{n^q \lambda^{np}\}_{\substack{p=0,\cdots,+\infty,\\ q=0,\cdots,p}}$ grows at a different rate as $n \to +\infty$; hence, their associated weights can be determined inductively, i.e., each coefficient $L_{q,p}$ is a $\mathcal{MLS}$-invariant. It thus suffices to prove that the coefficients $L_{q,p}$ determine the coefficients $\{\bar{g}_p, \bar{\gamma}_p, \bar{a}_{p+1}\}_{p \geq 0}$.

As recalled above, g_0 and $\bar{a}_1$ are $\mathcal{MLS}$-invariants, and $g_0 \neq 0$, by transversality. We proceed by induction on p; by Corollary 4.10, $(\bar{g}_0, \bar{\gamma}_0, \bar{a}_1) = (g_0, 1, \bar{a}_1)$ is spectrally determined. Given $p \geq 1$, let us assume that the coefficients $\{\bar{g}_\ell, \bar{\gamma}_\ell, \bar{a}_{\ell+1}\}_{0 \leq \ell < p}$ are known; we want to compute V_p (recall (4.20)). By Lemma 4.20, we have $W_p = A_p V_p + C_p$ for some matrix $A_p \in \mathfrak{M}_3(\mathbb{R})$, which only depends on p, $\bar{a}_1$ and g_0 and hence it is spectrally determined; moreover:

$$\det A_p = -2p\bar{a}_1^2 g_0^2.$$

In particular, under the assumption that $\bar{a}_1 \neq 0$, we have $\det A_p \neq 0$, since $g_0 \neq 0$, and $p \geq 1$. Therefore:

$$V_p = A_p^{-1}(W_p - C_p).$$

By Lemma 4.20, the vector C_p is determined by inductive hypothesis; W_p is obtained from the coefficients $L_{q,p}$, which are $\mathcal{MLS}$-invariants; we conclude that the vector V_p is a $\mathcal{MLS}$-invariant. □

5. Further estimates on the Marked Length Spectrum

5.1. **Basic facts about twist maps and generating functions.** Let us define the one-form $\omega_1 := -r ds$, with $r := \sin(\varphi)$. Recall that the billiard map $\mathcal{F}\colon (s,r) \mapsto (s',r')$ is exact symplectic: for some generating function $\bar{S}_1 = \bar{S}_1(s,r)$, we have

$$\mathcal{F}^*\omega_1 - \omega_1 = d\bar{S}_1.$$

Actually, by the twist property, this can also be rewritten in terms of the generating function $h(s,s') := \|\Upsilon(s) - \Upsilon(s')\|$, where $\Upsilon(s)$ is the point on the boundary $\partial\mathcal{D}$ of the billiard table associated with the parameter s:

$$\mathcal{F}^*\omega_1 - \omega_1 = dh(s,s').$$

For $T := \mathcal{F}^2\colon (s,r) \mapsto (s'',r'')$, and since the pull-back commutes with the operator d, we get

$$\begin{aligned} T^*\omega_1 - \omega_1 &= dh(s,s') + dh(s',s'') \\ &= d\mathcal{F}^*\bar{S}_1 + d\bar{S}_1 = d\bar{S}, \end{aligned}$$

where $\bar{S} := \bar{S}_1 + \bar{S}_1 \circ \mathcal{F}$. In the neighborhood $\mathscr{O}_\infty$ of $x_\infty(-1)$, we have $\mathcal{G}R = RT$, so that $\mathcal{G}^* = (R^{-1})^* T^* R^*$, while in a neighborhood of any other points of the homoclinic orbit h_∞, we have $NR = RT$, so that $N^* = (R^{-1})^* T^* R^*$. Let us set $\omega := (R^{-1})^*\omega_1$. In either case, we obtain

$$(R^{-1})^* T^* R^* (R^{-1})^*\omega_1 - (R^{-1})^*\omega_1 = (R^{-1})^*(T^*\omega_1 - \omega_1) = (R^{-1})^* d\bar{S} = dS,$$

where we have set

$$S := (R^{-1})^*\bar{S} = (R^{-1})^*(\mathcal{F}^*\bar{S}_1 + \bar{S}_1) = (\mathcal{F}R^{-1})^*\bar{S}_1 + (R^{-1})^*\bar{S}_1 = S^1 + S^2,$$

and

$$S^1 := (R^{-1})^*\bar{S}_1, \qquad\qquad S^2 := (\mathcal{F}R^{-1})^*\bar{S}_1 = (R^{-1})^*(\mathcal{F}^*\bar{S}_1).$$

5.2. **Estimates on the Marked Length Spectrum.** In this part, we assume that the integer n in the definition of the orbits h_n is even, i.e., $n = 2m$ for some integer $m \geq m_0$. Note that the orbits $(h_{2m})_{m \geq m_0}$ still approximate the same homoclinic orbit h_∞. Recall the notation $(\xi_n, \eta_n) := R_-(x_n(1))$ and $\Delta_n := \Delta(\xi_n \eta_n)$. Recall also that $(\xi_\infty, 0) = R_-(x_\infty(1))$. Moreover, the period of $h_n = h_{2m}$ now also equals $2n + 2 = 4m + 2$.

As in Lemma 4.6 and in (3.3), $\eta_n = \Delta_n^n \xi_n$, and for any $k \in \{0, \cdots, n\}$, it holds

$$R(x_n(2k+1)) = (\xi_n(2k+1), \eta_n(2k+1)) = \xi_n(\Delta_n^k, \Delta_n^{n-k}). \tag{5.1}$$

Note that by the fact that $n = 2m$ is even, for $k = m$, we have

$$R(x_n(2m+1)) = R(x_n(n+1)) = (\xi_n(n+1), \eta_n(n+1)) = \xi_n \Delta_n^m (1,1).$$

Proposition 5.1. *Let Σ_n^1 and Σ_n^2 be as in (2.4) and (2.5). In (ξ,η)-coordinates, we get the following expression for $\Sigma_n^1+\Sigma_n^2$:*

$$\begin{aligned}\Sigma_n^1+\Sigma_n^2=&-\sum_\beta\frac{1}{\beta!}\Big[2\sum_{k=0}^{m-1}\partial^\beta S_{\xi_n(\Delta_n^k,\Delta_n^{n-k})}\cdot\Big(\lambda^k\xi_\infty-\Delta_n^k\xi_n,-\Delta_n^{n-k}\xi_n\Big)^\beta\\&\qquad-\partial^\beta S^1_{(0,0)}\cdot(\Delta_n^m\xi_n,\Delta_n^m\xi_n)^\beta+\partial^\beta S^2_{\xi_n(\Delta_n^n,1)}\cdot\big(-\Delta_n^n\xi_n,\xi_\infty-\xi_n\big)^\beta\Big]\\&-2\sum_{\ell=1}^{+\infty}\frac{\xi_\infty^\ell}{\ell!}\sum_{k=m}^{+\infty}\lambda^{k\ell}\partial_1^\ell S_{(0,0)}.\end{aligned}$$

Moreover, we have[6]

$$\mathcal{L}(h_n)-(n+1)\mathcal{L}(\sigma)-\mathcal{L}^\infty\sim-\frac{\xi_\infty^2}{2}\left(\frac{1+\lambda^2}{1-\lambda^2}\mathrm{tr}(d^2S_{(0,0)})-\mathrm{tr}(d^2S^1_{(0,0)})-2\partial_{12}S^1_{(0,0)}\right)\lambda^n,$$

with $\mathrm{tr}(d^2S^i_{(0,0)}):=\partial_{11}S^i_{(0,0)}+\partial_{22}S^i_{(0,0)}$, *for* $i=1,2$.

Proof. The value of the sum $\Sigma_n^1+\Sigma_n^2$ does not depend on the choice of symplectic coordinates (see [DRR] for more details in this direction), thus, we may rewrite it in terms of the new coordinates (ξ,η) and of the generating function $S=S^1+S^2$ introduced above. Recall that by the palindromic symmetry of the orbit h_n, we have $x_n(4m+2-k)=\mathcal{I}(x_n(k))$ for $k=1,\cdots,2m+1$. Besides, we have $\bar S\circ\mathcal{I}=\bar S$ and $S\circ\mathcal{I}^*=S$, and we note that ξ,η play symmetric roles in the following computations (see Remark 5.2). We thus obtain:

$$\begin{aligned}\Sigma_n^1+\Sigma_n^2&=2\sum_{k=0}^{n}\Big(h(s_n(k),s_n(k+1))-h(s_\infty(k),s_\infty(k+1))\Big)\\&\quad+2\sum_{k=n+1}^{+\infty}\Big(h(s(k),s(k+1))-h(s_\infty(k),s_\infty(k+1))\Big)\\&=2\sum_{k=0}^{m-1}\Big(\bar S(x_n(2k+1))-\bar S(x_\infty(2k+1))\Big)\\&\quad+\big(\bar S_1(x_n(2m+1))-\bar S_1(x(2m+1))\big)+\big(\bar S_1(x_n(0))-\bar S_1(x_\infty(0))\big)\\&\quad+2\sum_{k=m}^{+\infty}\Big(\bar S(x(2k+1))-\bar S(x_\infty(2k+1))\Big),\end{aligned}$$

i.e.,

$$\begin{aligned}\Sigma_n^1+\Sigma_n^2=&\,2\sum_{k=0}^{m-1}\big(S(\xi_n(2k+1),\eta_n(2k+1))-S(\xi_\infty(2k+1),0)\big)\\&+\big(S^1(\xi_n(n+1),\eta_n(n+1))-S^1(0,0)\big)+\big(S^2(\xi_n(-1),\eta_n(-1))-S^2(0,\xi_\infty)\big)\\&+2\sum_{k=m}^{+\infty}\big(S(0,0)-S(\xi_\infty(2k+1),0)\big),\end{aligned}$$

[6]Recall that $\mathcal{L}(h_n)-(n+1)\mathcal{L}(\sigma)-\mathcal{L}^\infty=\Sigma_n^1+\Sigma_n^2$.

and then,

$$
\begin{aligned}
\Sigma_n^1+\Sigma_n^2 = &-2\sum_{k=0}^{m-1}\big(S(\lambda^k\xi_\infty,0)-S(\Delta_n^k\xi_n,\Delta_n^{n-k}\xi_n)\big)\\
&+\big(S^1(\Delta_n^m\xi_n,\Delta_n^m\xi_n)-S^1(0,0)\big)-\big(S^2(0,\xi_\infty)-S^2(\Delta_n^n\xi_n,\xi_n)\big)\\
&-2\sum_{k=m}^{+\infty}\big(S(\lambda^k\xi_\infty,0)-S(0,0)\big)\\
=&-\sum_\beta\frac{1}{\beta!}\Big[2\sum_{k=0}^{m-1}\partial^\beta S_{\xi_n(\Delta_n^k,\Delta_n^{n-k})}\cdot\Big(\lambda^k\xi_\infty-\Delta_n^k\xi_n,-\Delta_n^{n-k}\xi_n\Big)^\beta\\
&-\partial^\beta S^1_{(0,0)}\cdot(\Delta_n^m\xi_n,\Delta_n^m\xi_n)^\beta+\partial^\beta S^2_{\xi_n(\Delta_n^n,1)}\cdot\big(-\Delta_n^n\xi_n,\xi_\infty-\xi_n\big)^\beta\Big]\\
&-2\sum_{\ell=1}^{+\infty}\frac{\xi_\infty^\ell}{\ell!}\sum_{k=m}^{+\infty}\lambda^{k\ell}\partial_1^\ell S_{(0,0)}.
\end{aligned}
$$

By Lemma 4.10, we have

$$|\xi_n-\xi_\infty|=O(\lambda^n),\qquad |\Delta_n-\lambda|=O(\lambda^n),$$

thus for any $0\le k\le m$, $|\Delta_n^k\xi_n-\lambda^k\xi_\infty|=O(\lambda^n)$, while $|\Delta_n^{n-k}\xi_n|\asymp O(\lambda^m)$ when k is close to m, hence the contribution of the second term overcomes that of the first term in $(\Delta_n^k\xi_n-\lambda^k\xi_\infty,\Delta_n^{n-k}\xi_n)$. For $k\sim m$, we have $\|\xi_n(\Delta_n^k,\Delta_n^{n-k})\|=O(\lambda^m)$ too, thus in order to estimate the leading term in $\Sigma_n^1+\Sigma_n^2$ we may consider partial derivatives $\partial^\beta_{(0,0)}$ instead of $\partial^\beta_{\xi_n(\Delta_n^k,\Delta_n^{n-k})}$.

As we have seen in [BDKL], first order terms (for $|\beta|=1$) vanish in the expansion of $\Sigma_n^1+\Sigma_n^2$, since the periodic orbits h_n and (12) are critical points of the length functional. Then, by considering the terms with $|\beta|=2$, we get the estimate

$$
\begin{aligned}
&\Sigma_n^1+\Sigma_n^2\\
&\sim-\frac{\xi_\infty^2}{2}\Bigg(2\sum_{k=0}^{m-1}\partial_{22}S_{(0,0)}\cdot\lambda^{2(n-k)}-2(\partial_{11}S^1_{(0,0)}+\partial_{12}S^1_{(0,0)})\cdot\lambda^{2m}+2\sum_{k=m}^{+\infty}\partial_{11}S_{(0,0)}\cdot\lambda^{2k}\Bigg)\\
&\sim-\frac{\xi_\infty^2}{2}\Bigg(\mathrm{tr}(d^2S_{(0,0)})\Big(\sum_{k=0}^{m-1}\lambda^{2(n-k)}+\sum_{k=m}^{+\infty}\lambda^{2k}\Big)-\big(\mathrm{tr}(d^2S^1_{(0,0)})+2\partial_{12}S^1_{(0,0)}\big)\lambda^n\Bigg)\\
&\sim-\frac{\xi_\infty^2}{2}\Bigg(\frac{1+\lambda^2}{1-\lambda^2}\mathrm{tr}(d^2S_{(0,0)})-\mathrm{tr}(d^2S^1_{(0,0)})-2\partial_{12}S^1_{(0,0)}\Bigg)\lambda^n. \qquad\square
\end{aligned}
$$

Remark 5.2. *The roles of ξ and η are symmetric in the previous calculations. Indeed, by the time reversal symmetry $\mathcal{I}\circ\mathcal{F}\circ\mathcal{I}=\mathcal{F}^{-1}$, with $\mathcal{I}(s,r)=(s,-r)$, the relation $(s',r')=\mathcal{F}(s,r)$ can be rewritten as $(s,-r)=\mathcal{F}(s',-r')$. The generating function (see (2.1)) $h(s,s')=\|\Upsilon(s)-\Upsilon(s')\|$ satisfies $h(s,s')=h(s',s)$, and*

$$\partial_1h(s,s')=-r,\qquad \partial_2h(s,s')=r'.$$

In (ξ,η)-coordinates, the time reversal symmetry becomes $\mathcal{I}^*\circ N\circ\mathcal{I}^* = N^{-1}$, with $\mathcal{I}^*(\xi,\eta)=(\eta,\xi)$. We have $S^i\circ\mathcal{I}=S^i$, for $i=1,2$, which implies

$$\partial_{11}S^i_{(0,0)} = \partial_{22}S^i_{(0,0)} = \frac{1}{2}\mathrm{tr}(d^2S^i_{(0,0)}).$$

Remark 5.3. *Note that the parameter ξ_∞ in the present paper is different from – although related to – the analogous quantity introduced in* [BDKL]. *The corresponding formulae involving ξ_∞ (see e.g.* [BDKL, (31)]*) therefore differ from the ones obtained in this paper (see e.g. the estimates in Proposition 5.1).*

Remark 5.4. *In Proposition 5.1, we have considered the case where n is even. But by Theorem 1.10, the analogue estimate when n is odd is obtained by swapping the roles of the first and the second scatterers. We deduce that for $n=2m-1\gg 1$,*

$$\mathcal{L}(h_n)-(n+1)\mathcal{L}(\sigma)-\mathcal{L}^\infty \sim -\frac{\xi_\infty^2}{2}\left(\frac{1+\lambda^2}{1-\lambda^2}\mathrm{tr}(d^2S_{(0,0)})-\mathrm{tr}(d^2S^2_{(0,0)})-2\partial_{12}S^2_{(0,0)}\right)\lambda^n.$$

5.3. **$\mathcal{MLS}$-determination of the Birkhoff data.** As a consequence of the above estimates on the Marked Length Spectrum, we can conclude the following result.

Corollary 5.5. *Let $\mathcal{D}\in\mathbf{B}$. With the same notations as above, for any 2-periodic orbit (jk), with $j<k\in\{1,2,3\}$, the value of the parameter $\xi_\infty=\xi_\infty(\mathcal{D},j,k)$ associated to the homoclinic orbit $h_\infty=h_\infty(\mathcal{D},j,k)$*[7] *is a $\mathcal{MLS}$-invariant.*

Proof. Without loss of generality, we assume that $(jk)=(12)$. By the estimates obtained in Proposition 5.1 and Remark 5.4, we see that the quantities $\xi_\infty^2\big(\frac{1+\lambda^2}{1-\lambda^2}\mathrm{tr}(d^2S_{(0,0)})-\mathrm{tr}(d^2S^i_{(0,0)})-2\partial_{12}S^i_{(0,0)}\big)$, $i=1,2$, are $\mathcal{MLS}$-invariants. Therefore, their sum $\xi_\infty^2\big(\frac{1+3\lambda^2}{1-\lambda^2}\mathrm{tr}(d^2S_{(0,0)})-2\partial_{12}S_{(0,0)}\big)$ is also a $\mathcal{MLS}$-invariant. We claim that in fact, the three quantities $\mathrm{tr}(d^2S_{(0,0)})$, $\partial_{12}S_{(0,0)}$ and ξ_∞ are $\mathcal{MLS}$-invariants.

Indeed, $\mathrm{tr}(d^2S_{(0,0)})$ only depends on the local geometry near the 2-periodic orbit (12) and is a symplectic invariant, hence, with the notation of Subsection 5.1, it equals the trace $\mathrm{tr}(d^2\bar{S})$ at $(s(2,1),0)$ in (s,r)-coordinates, assuming that $(s(i,j),0)$ are the coordinates of the point in the 2-periodic orbit (12) which belongs to $\mathcal{M}_i$, for $\{i,j\}=\{1,2\}$. Equivalently, this trace can be expressed in terms of the generating function $h(s,s')$ for the dynamics of $\mathcal{F}$. Moreover, the trace $\mathrm{tr}(d^2h_{(s(1,2),0)}+d^2h_{(s(2,1),0)})$ only depends on the length $\mathcal{L}(12)$ and the radii of curvature R_1,R_2 at the two bouncing points in the periodic orbit (12) (see Lemma 5.3 in [BDKL]). By the results of [BDKL] recalled in Theorem 1.10 and Corollary 1.11, R_1,R_2 are determined by $\mathcal{MLS}$ and hence, the value of $\mathrm{tr}(d^2S_{(0,0)})=\mathrm{tr}(d^2\bar{S}_{(s(2,1),0)})$ is a $\mathcal{MLS}$-invariant.

Note that the Hessian matrix of S at the point $(0,0)$ has the form $d^2S_{(0,0)} = \begin{pmatrix} a & b\\ b & a\end{pmatrix}$ with $a:=\partial_{11}S_{(0,0)}=\partial_{22}S_{(0,0)}$ and $b:=\partial_{12}S_{(0,0)}=\partial_{21}S_{(0,0)}$. By the above discussion, the coefficient a is a $\mathcal{MLS}$-invariant. Besides, the determinant $\det\, d^2S_{(0,0)}=a^2-b^2$ of this matrix is a conjugacy invariant, thus it also equals $\det\, d^2\bar{S}_{(s(2,1),0)}$. As above, this quantity only depends on $\mathcal{L}(12)$ and R_1,R_2, hence it is a $\mathcal{MLS}$-invariant. We deduce that $b=\partial_{12}S_{(0,0)}$ is a $\mathcal{MLS}$-invariant.

[7] See the definitions of h_∞ and ξ_∞ in Section 3.

It follows that each of the three quantities $\mathrm{tr}(d^2S_{(0,0)})$, $\partial_{12}S_{(0,0)}$ and $\xi_\infty^2\big(\frac{1+3\lambda^2}{1-\lambda^2}\mathrm{tr}(d^2S_{(0,0)})-2\partial_{12}S_{(0,0)}\big)$ is a $\mathcal{MLS}$-invariant. Therefore, we conclude that the parameter ξ_∞ is a $\mathcal{MLS}$-invariant too. □

Remark 5.6. *Note that the parameter ξ_∞ can be interpreted in terms of some area in parameter space. Indeed, considering the area of the region of the (s,r)-plane bounded by the stable/unstable manifolds of the fixed point $(s(2,1),0)$ and the vertical line through $(s_n(n+1),0)$ in (s,r)-coordinates, and since the change of coordinates considered here is symplectic, we obtain*

$$\tan(\theta)s_n(n+1)^2 \sim 2\lambda^n\xi_\infty^2,$$

where $2\theta\in(0,\pi)$ is the angle between the stable/unstable subspaces at $(s(2,1),0)$.

The above result allows us to conclude:

Corollary 5.7. *Let $\mathcal{D}\in\mathbf{B}$, and let $\mathcal{F}=\mathcal{F}(\mathcal{D})$ be the associated billiard map. We consider a 2-periodic orbit (jk), with $j<k\in\{1,2,3\}$. Let $N=N(\mathcal{D},j,k)\colon(\xi,\eta)\mapsto(\Delta(\xi\eta)\xi,\Delta(\xi\eta)^{-1}\eta)$ be the Birkhoff Normal Form of $\mathcal{F}^2$ associated to the orbit (jk), with $\Delta=\Delta(\mathcal{D},j,k)\colon z\mapsto\lambda+\sum_{\ell=1}^{+\infty}a_\ell z^\ell$. If $a_1\neq 0$, then*

- *the Birkhoff Normal Form N is a $\mathcal{MLS}$-invariant;*
- *the differential of the gluing map $\mathcal{G}$ at any point $(\xi,\eta)\in\Gamma_\infty$ is also a $\mathcal{MLS}$-invariant, where $\mathcal{G}=\mathcal{G}(\mathcal{D},j,k)$ and $\Gamma_\infty=\Gamma_\infty(\mathcal{D},j,k)$ are taken as in Subsection 4.1.*

Proof. We assume that $(jk)=(12)$ and use the notation of Sections 3-4. Recall that by Corollary 4.21, the parameters $\{\bar a_\ell,\bar\gamma_\ell,\bar g_\ell\}_{\ell\geq 0}$ are $\mathcal{MLS}$-invariants, provided that $\bar a_1\neq 0$; by the above corollary (recall (4.7)) we thus conclude that $\{a_\ell,\gamma_\ell,g_\ell\}_{\ell\geq0}$ are $\mathcal{MLS}$-invariants, as well as the expressions of $\Delta\colon z\mapsto\lambda+\sum_{\ell=1}^{+\infty}a_\ell z^\ell$ and of the Birkhoff Normal Form $N=R_0\mathcal{F}^2R_0^{-1}$ in a neighborhood of $(s(2,1),0)$, but also of γ and $g=\partial_2G^-(\cdot,\gamma(\cdot))$. By (4.4), we deduce that the differential $D_{(\xi,\eta)}\mathcal{G}$ of the gluing map $\mathcal{G}=R\circ\mathcal{F}^2\circ R^{-1}|_{\Omega_\infty}$ at any point $(\xi,\eta)\in\Gamma_\infty$ is also a $\mathcal{MLS}$-invariant, where $\mathcal{G},\Omega_\infty$ and Γ_∞ are taken as in Subsection 4.1. □

6. Reconstructing the geometry from the Marked Length Spectrum

Let us consider the case of billiard tables $\mathcal{D}$ which present additional symmetries in the sense of Definition 1.3, i.e., $\mathcal{D}\in\mathbf{B}_{\mathrm{sym}}$. It follows from the previous part that if $T:=\mathcal{F}^2\colon(s,r)\mapsto(s'',r'')$ denotes the square of the billiard map $\mathcal{F}=\mathcal{F}(\mathcal{D})\colon(s,r)\mapsto(s',r')$, then under some twist condition, the Birkhoff Normal Form $N=N(\mathcal{D},1,2)=RTR^{-1}$ of T in a neighborhood of $(s(2,1),0)$ is completely determined by the Marked Length Spectrum $\mathcal{MLS}(\mathcal{D})$, assuming that $s(2,1)$ is the arc-length parameter of the point of $\mathcal{M}_2$ in the 2-periodic orbit (12).

In this part, our goal is to see which information on the geometry of the billiard table $\mathcal{D}$ can be reconstructed. In the following, we conclude the proof of our Main Theorem:

Theorem 6.1. *For an open and dense set of billiard tables $\mathcal{D}\in\mathbf{B}_{\mathrm{sym}}$, the Marked Length Spectrum $\mathcal{MLS}(\mathcal{D})$ determines completely the geometry of $\mathcal{D}$.*

Fix a billiard table $\mathcal{D}=\mathbb{R}^2\backslash\bigcup_{i=1}^3\mathcal{O}_i\in\mathbf{B}_{\mathrm{sym}}$, and let $\mathcal{F}:=\mathcal{F}(\mathcal{D})$ be the associated billiard map. After possibly applying some isometry, we assume that in the plane with $(\bar{x}, \bar{y})$-coordinates, the trace of the point in the 2-periodic orbit (12) which is on the first obstacle, resp. second obstacle, has coordinates $(-\frac{1}{2}\ell, 0)$, resp. $(\frac{1}{2}\ell, 0)$, where $\ell = \ell(\mathcal{D}) := \frac{1}{2}\mathcal{L}(12)$ is the half-length of the orbit (12). In particular, the axis of symmetry is the vertical axis $\{\bar{x} = 0\}$.

6.1. **A construction for symmetric billiard tables.** We consider the following construction. If $1, 2, 3$ are the labels of the three obstacles of $\mathcal{D}$, we define a new billiard table $\mathcal{D}^* = \mathcal{D}^*(\mathcal{D})$ formed by three obstacles $1^*, 2^*, 3^*$: (see Figure 8 below)

- we consider a half-plane $\{\bar{x} \leq 0\}$ or $\{\bar{x} \geq 0\}$ such that it contains the trace of at least one of the two points $x_\infty^{(j)}(0) \in \{r = 0\}$, $j \in \{1, 2\}$, in the respective homoclinic orbits $h_\infty^{(1)} = (\ldots 212131212\ldots)$ and $h_\infty^{(2)} = (\ldots 121232121\ldots)$; in the following we assume that this point is $x_\infty(0) = x_\infty^{(1)}(0)$ and that its trace is in the half-plane $\{\bar{x} \leq 0\}$;
- in this case, we let the obstacle with label 1^* in $\mathcal{D}^*$ be the same as the one of $\mathcal{D}$ with label 1;
- we define the obstacle 2^* as the vertical line segment $\{0\} \times [-\ell^*, \ell^*]$ for some $\ell^* > 0$ such that $\{0\} \times [-\ell^*, \ell^*]$ does not cross the third obstacle, and the intersection of $\{0\} \times (-\ell^*, \ell^*)$ and of the line segment between the points of parameters $x_\infty(1) = \mathcal{F}(x_\infty(0))$ and $x_\infty(2) = \mathcal{F}^2(x_\infty(0))$ is non-empty; we parametrize this line segment in arc-length in such a way that the image of the point in the 2-periodic orbit (1^*2^*) is associated to the parameter 0;
- we let the obstacle 3^* be some small arc in the obstacle 3 such that it is in the half-plane $\{\bar{x} \leq 0\}$ and contains a neighborhood of the point with coordinates $x_\infty(0)$.

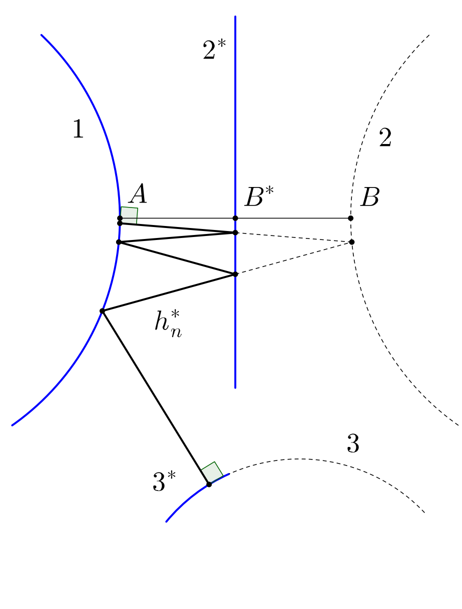


FIGURE 7. Defining a new table using the $\mathbb{Z}_2$-symmetry of the pair $\{\mathcal{O}_1, \mathcal{O}_2\}$.

By construction, the billiard table $\mathcal{D}^*$ satisfies the non-eclipse condition and is of the same type as $\mathcal{D}$, except that the obstacle 2^* is now flat. Moreover, by Proposition 2.1, for any sufficiently large integer n, each palindromic orbit $h_n = h_n^{(1)} = (31\underbrace{2121\dots 21}_{2n})$ as above in $\mathcal{D}$ shadows either $h_\infty = h_\infty^{(1)}$ or its image under $\mathcal{I}\colon (s,r)\mapsto(s,-r)$, and can thus be associated to a periodic orbit in $\mathcal{D}^*$, denoted by h_n^*, which is defined as follows:

- we start at the image of the point $x_n(0)\in\{r=0\}$ of h_n; it is close to the image of $x_\infty(0)$ so it is indeed on the obstacle 3^*;
- the trace of the first orbit's segment is the same as for h_n;
- the trace of the second orbit's segment is the first part of the trace of the second segment of h_n which is contained in the half-plane $\{\bar{x}\leq 0\}$;
- the trace of the third orbit's segment is the second part of the second segment of h_n, which is contained in $\{\bar{x}\geq 0\}$; it is also the continuation of the second segment of h_n^* under the billiard flow of $\mathcal{D}^*$, after it gets reflected on 2^* according to the usual law of reflection of angles;
- we repeat this folding procedure along the trajectory each time we hit the axis $\{\bar{x}=0\}$ until we reach the image of the point $x_n(2n+2)$, so that the trace of the orbit of h_n^* is contained in $\{\bar{x}\geq 0\}$.

The points of h_n^* which are on the boundary $\partial\mathcal{D}^*$ of the new table still define an orbit under the dynamics of the associated billiard map $\mathcal{F}^*=\mathcal{F}(\mathcal{D}^*)\colon (s,r)\mapsto(s',r')$, with the same length $\mathcal{L}(h_n^*)=\mathcal{L}(h_n)$ as the original orbit h_n. The symbolic coding of h_n^* is

$$h_n^* = (3^*1^*\underbrace{(2^*1^*)(2^*1^*)(2^*1^*)(2^*1^*)\dots(2^*1^*)(2^*1^*)}_{2\times 2n=4n}),$$

where each word (2^*1^*) with even index replaces a 1 and each word (2^*1^*) with odd index replaces a 2 in the previous coding. Formally, we obtain

Lemma 6.2. *The maps $\mathbf{B}_{\mathrm{sym}}\ni\mathcal{D}\mapsto\mathcal{D}^*(\mathcal{D})$ and $h_n\mapsto h_n^*$ satisfy the following properties:*

- *there exists an integer $m_0\geq 0$ such that the subset of palindromic orbits $(h_{2m-1})_{m\geq m_0}$ of $\mathcal{F}$ embeds into the set of palindromic orbits of $\mathcal{F}^*$ by the map $h_n\mapsto h_n^*$ defined above;*
- *for each $n=2m-1$, $m\geq m_0$, we have $\mathcal{L}(h_n)=\mathcal{L}(h_n^*)$;*
- *for each $n=2m-1$, $m\geq m_0$, we have $\mathrm{LE}(h_n)=\mathrm{LE}(h_n^*)$.*

Proof. The fact that each h_n^* is palindromic follows from the preservation of angles under reflections. It remains to show the third point about Lyapunov exponents. Indeed (see for example [CM]), the matrix of the differential of $\mathcal{F}^{2n+2}$ at the point $x_n(0)$ is an alternating product of parabolic matrices $\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$ between two collisions separated by a segment of length t, and of parabolic matrices $\begin{pmatrix}1 & 0\\ -\frac{2\mathcal{K}}{\cos(\varphi)} & 1\end{pmatrix}$, when the trajectory hits the boundary of the table, where $\varphi=\arcsin(r)$ and $\mathcal{K}$ are respectively the angular parameter and the curvature at the point of collision. Then, each new collision created by the introduction of the auxiliary obstacle 2^* at a shorter distance does not affect the product, nor the differential, since the curvature vanishes on 2^*, and then, the additional matrices are just the identity. □

As a consequence of the previous observations, we obtain:

Proposition 6.3. *We consider a billiard table $\mathcal{D} \in \mathbf{B}_{\mathrm{sym}}$, with Marked Length Spectrum $\mathcal{MLS}(\mathcal{D})$. We let $\mathcal{D}^* = \mathcal{D}^*(\mathcal{D})$ be the billiard table defined above and denote by $\mathcal{F}^* = \mathcal{F}^*(\mathcal{D}^*)$ the associated billard map. Let $N^* = N^*(\mathcal{D}^*, 1, 2)\colon (\xi, \eta) \mapsto (\Delta^*(\xi\eta)\xi, \Delta^*(\xi\eta)^{-1}\eta)$ be the Birkhoff Normal Form of $T^* := (\mathcal{F}^*)^2\colon (s, r) \mapsto (s'', r'')$ in a neighborhood of the point $(0_{1^*}, 0)$ in the period two orbit (1^*2^*) which is on the first obstacle 1^*, with $\Delta^*\colon z \mapsto \sum_{j=0}^{+\infty} a_j z^j$. If $a_1 \neq 0$, then $\mathcal{MLS}(\mathcal{D})$ determines the Birkhoff Normal Form N^*.*

Proof. There is a new 2-periodic orbit (1^*2^*) for the map $\mathcal{F}^*$ which bounces perpendicularly at the points with $(\bar{x}, \bar{y})$-coordinates $(-\frac{1}{2}\ell, 0)$ and $(0, 0)$. Moreover, the image h_∞^* in $\mathcal{D}^*$ of the homoclinic trajectory h_∞ in $\mathcal{D}$ can be defined following the same "folding" procedure as above. It is also homoclinic to (1^*2^*), and similarly, it is accumulated by the orbits $(h_n^*)_n$ defined above. The point $(0_{1^*}, 0)$ with trace $(-\frac{1}{2}\ell, 0)$ is a saddle fixed point for the dynamics of $T^* = (\mathcal{F}^*)^2$, hence we may consider the Birkhoff Normal Form $N^*\colon (\xi, \eta) \mapsto (\Delta^*(\xi\eta)\xi, \Delta^*(\xi\eta)^{-1}\eta)$ of T^* in a neighborhood of this point. The orbits $(h_n^*)_n$ are still palindromic, hence the analog of Lemma 4.6 remains true is this case. We also note that the homoclinic parameter ξ_∞ is preserved by the unfolding construction. By Lemma 6.2, the Lyapunov exponent of each orbit h_n^* is $\mathcal{MLS}(\mathcal{D})$-invariant. Therefore, if $a_1 \neq 0$, then by the same method as in Lemmata 4.8, 4.10, 4.11, 4.15, 4.20, Corollary 4.21 and Corollary 5.5, we can recover the Birkhoff invariants of N^* by considering the series expansion of $\mathrm{LE}(h_n^*)$ with respect to n. As a result, the Birkhoff Normal Form N^* is entirely determined by the Marked Length Spectrum $\mathcal{MLS}(\mathcal{D})$ of the initial table. □

Let $(0_{2^*}, 0)$ be the (s, r)-coordinates of the point in the orbit (1^*2^*) whose trace is on the second obstacle 2^*. The Birkhoff Normal Forms of $(\mathcal{F}^*)^2$ at the two points $(0_{1^*}, 0)$ and $(0_{2^*}, 0)$ in the orbit (1^*2^*) coincide:

Lemma 6.4. *Let $\mathcal{D} \in \mathbf{B}_{\mathrm{sym}}$ and let $\mathcal{F}^* = \mathcal{F}^*(\mathcal{D}^*)$. The Birkhoff Normal Form of $T^* = (\mathcal{F}^*)^2$ in a neighborhood of the point $\mathcal{F}^*(0_{1^*}, 0) = (0_{2^*}, 0)$ coincides with the map $N^* = N^*(\mathcal{D}^*, 1, 2)$ defined in Proposition 6.3.*

Proof. Let $\mathcal{U}^* \subset \mathbb{R}^2$ be an open neighborhood of $(0_{1^*}, 0)$, and let $R^*\colon \mathcal{U}^* \to \mathbb{R}^2$ be a conjugacy map such that $R^*T^*|_{\mathcal{U}^*} = N^*R^*|_{\mathcal{U}^*}$. Then $\mathcal{F}^*(\mathcal{U}^*)$ is an open neighborhood of $(0_{2^*}, 0)$, since $\mathcal{F}^*(0_{1^*}, 0) = (0_{2^*}, 0)$, and the map $\widetilde{R}^* := R^* \circ (\mathcal{F}^*)^{-1}$ is symplectic, and for any $y = \mathcal{F}^*(x)$ with $x \in \mathcal{U}^*$, it holds

$$\widetilde{R}^* \circ T^*(y) = R^* \circ (\mathcal{F}^*)^{-1} \circ (\mathcal{F}^*)^2(\mathcal{F}^*(x)) = R^* \circ T^*(x) = N^* \circ R^*(x) = N^* \circ \widetilde{R}^*(y),$$

which concludes, by uniqueness of the Birkhoff Normal Form. □

6.2. **Recovering the geometry of a symmetric billiard table.** In the following, we use the same notation as in the last part. Given $\mathcal{D} \in \mathbf{B}_{\mathrm{sym}}$, then by definition, after rotation by an angle of $-\frac{\pi}{2}$, near the point $(0, \frac{\ell}{2})$, the first obstacle 1 (which is the same as the obstacle 1^*) can be represented as a graph

$$\mathscr{C} = \left\{ \left(t, \frac{\ell}{2} + \beta_2 t^2 + \beta_4 t^4 + \dots \right) : t \in I \right\},$$

for some open interval $I \ni 0$. Indeed, this follows from our assumption that the obstacles $\mathcal{O}_1, \mathcal{O}_2$ have some axial symmetry with respect to the trace of the 2-periodic orbit (12), and then, there are only even coefficients in the above expansion.

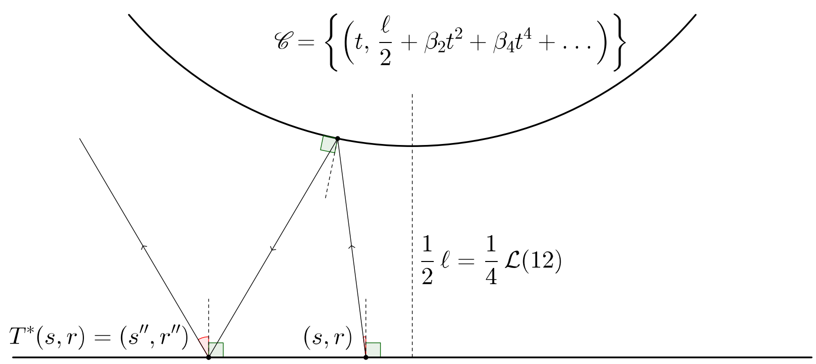


FIGURE 8. The map T^* near the point $(0_{2^*}, 0)$.

Let us recall the following result in the paper [CdV] of Colin de Verdière:

Lemma 6.5 ([CdV, Lemma 1])**.** *The jet of $T^* \colon (s,r) \mapsto (s'', r'')$ at $(0_{2^*}, 0)$ is in one-to-one correspondence with the coefficients $(\beta_{2k})_{k \geq 1}$ of the graph $\mathscr{C}$ defined above. Besides, the linear part of $D_{(0_{2^*},0)}T^*$ is associated to the hyperbolic matrix*

$$\begin{pmatrix} A-1 & -A \\ 2-A & A-1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}), \qquad A := 2(2\beta_2 + 1) > 2.$$

Indeed, by the strict convexity of $\mathcal{O}_1$, we have $\beta_2 > 0$. More precisely, for $k \geq 1$, and for some $C \neq 0$, it holds

$$T^*_{(2k+1)}(s,r) = T^{*,(0)}_{(2k+1)}(s,r) + C(s-r)^{2k+1}\beta_{2(k+1)} + O(|s| + |r|)^{2k+2},$$

where $T^{*,(0)}_{(2k+1)}$ denotes the jet of T^* of order $2k+1$ at $(0_{2^*}, 0)$ for $\beta_{2(k+1)} = 0$.

Lemma 6.6. *For any billiard table $\mathcal{D} \in \mathbf{B}_{\mathrm{sym}}$, the first Birkhoff invariant $a_1 = a_1(\mathcal{D})$ of the Birkhoff Normal Form $N^* = N^*(\mathcal{D}^*, 1, 2)$ satisfies*

$$a_1 = c^*\mathcal{K}'' + f^*(\ell, \mathcal{K}), \tag{6.1}$$

for some constant $c^* \neq 0$ and some continuous function $f^* \colon \mathbb{R}^2 \to \mathbb{R}$, where $\mathcal{K}$, $\mathcal{K}''$ respectively denote the curvature and its second derivative at the bouncing points of the 2-periodic orbit (12).

In particular, for an open and dense set of billiard tables $\mathcal{D} \in \mathbf{B}_{\mathrm{sym}}$, $\mathcal{D}$ satisfies the non-degeneracy condition

$$a_1(\mathcal{D}) \neq 0. \tag{$\star$}$$

Proof. For each integer $j \geq 0$, we define a map $a_j\colon \mathbf{B}_{\mathrm{sym}} \to \mathbb{R}$ in such a way that for all $\mathcal{D} \in \mathbf{B}_{\mathrm{sym}}$, the j^{th} Birkhoff invariant of $N^* = N^*(\mathcal{D}^*, 1, 2)$ is equal to $a_j(\mathcal{D})$, i.e., $N^*\colon (\xi, \eta) \mapsto (\Delta^*(\xi\eta)\xi, \Delta^*(\xi\eta)^{-1}\eta)$, with $\Delta^*\colon z \mapsto \sum_{j=0}^{+\infty} a_j(\mathcal{D}) z^j$.

As for the Birkhoff Normal Form $N(\mathcal{D}, 1, 2)$, the coefficient $\lambda := a_0(\mathcal{D})$ is related to the Lyapunov exponent $\mathrm{LE}(1^*2^*)$ and only depends on $\mathcal{L}(12)$ and on the curvature $\mathcal{K}$ at the bouncing points of the 2-periodic orbit (12).

Besides, following the construction of the Birkhoff Normal Form given by Moser [Mos], the first coefficient $a_1 := a_1(\mathcal{D})$ of N^* is determined by the jet of order three of T^*. Together with Lemma 6.5, we thus have

$$a_1 = c_0^* \beta_4 + f_0^*(\ell, \beta_2),$$

for some constant $c_0^* \neq 0$ and some continuous function $f_0^*\colon \mathbb{R}^2 \to \mathbb{R}$. Equivalently, β_2, β_4 can be interpreted in terms of the curvature $\mathcal{K}$ and its second derivative $\mathcal{K}''$ at the bouncing points of the 2-periodic orbit (12),[8] as

$$\mathcal{K} = 2\beta_2, \qquad \mathcal{K}'' = 24(\beta_4 - \beta_2^3),$$

which gives (6.1).

Therefore, we may ensure that the first Birkhoff invariant a_1 is non-zero by modifying the shape of the obstacles $\mathcal{O}_1, \mathcal{O}_2$ so as to change the value of $\mathcal{K}''$, but keeping ℓ, $\mathcal{K}$ fixed. More precisely, with the notations introduced in Definition 1.4, after possibly applying some isometry, we have $\mathcal{O}_1 = \mathcal{O}(f)$, with $f \in C^\omega(\mathbb{T}, \mathbb{R}^2)$, $\theta \mapsto \varrho(\theta)(\cos(\theta), \sin(\theta))$, for some even[9] function $\varrho \in C^\omega(\mathbb{T}, \mathbb{R})$, $\theta \mapsto \sum_{j=0}^{+\infty} \hat{\varrho}_j e^{\mathrm{i}j\theta}$. The curvature $\mathcal{K} = \mathcal{K}(0)$ and its second derivative $\mathcal{K}'' = \mathcal{K}''(0)$ satisfy

$$\begin{aligned}
\mathcal{K} &= \frac{1}{\varrho(0)} - \frac{\varrho''(0)}{\varrho^2(0)},\\
\mathcal{K}'' &= -\frac{\varrho''(0)}{\varrho^2(0)} + \frac{3(\varrho''(0))^2}{\varrho^3(0)} + \frac{3(\varrho''(0))^3}{\varrho^4(0)} - \frac{\varrho''''(0)}{\varrho^2(0)},
\end{aligned}$$

with $\varrho(0) = \sum_{j=0}^{+\infty} \hat{\varrho}_j$, $\varrho''(0) = -\sum_{j=0}^{+\infty} j^2 \hat{\varrho}_j$, and $\varrho''''(0) = \sum_{j=0}^{+\infty} j^4 \hat{\varrho}_j$. For any table $\mathcal{D}$ such that $a_1(\mathcal{D})$ vanishes, then it is sufficient to perturb the first Fourier coefficients of ϱ to get a new function $\tilde{\varrho}$ such that the associated table $\widetilde{\mathcal{D}}$ satisfies $a_1(\widetilde{\mathcal{D}}) \neq 0$. In this way, we see that for the topology introduced in Definition 1.4, the condition $a_1 \neq 0$ holds for a dense subset of $\mathbf{B}_{\mathrm{sym}}$, and clearly, this condition is also open (note that the map $a_1\colon \mathbf{B}_{\mathrm{sym}} \to \mathbb{R}$ is continuous). □

By Lemma 6.6, in order to prove Theorem 6.1, it is sufficient to show that the Marked Length Spectrum determines the geometry for the set of billiard tables in $\mathbf{B}_{\mathrm{sym}}$ such that the first invariant a_1 is non-zero. In the following, we fix a table $\mathcal{D} \in \mathbf{B}_{\mathrm{sym}}$ satisfying the non-degeneracy condition $(\star)$ and show that the geometry of $\mathcal{D}$ is determined by the Marked Length Spectrum $\mathcal{MLS} := \mathcal{MLS}(\mathcal{D})$.

Corollary 6.7. *The coefficients $(\beta_{2k})_{k\geq 1}$ of the graph $\mathscr{C}$ are $\mathcal{MLS}$-invariants. Therefore, by analyticity, the geometry of $\mathcal{O}_1, \mathcal{O}_2$ can be reconstructed from $\mathcal{MLS}$.*

Proof. By $(\star)$, Proposition 6.3 and Lemma 6.4, the Birkhoff Normal Form N^* of the map T^* in a neighborhood of the point $(0_{2^*}, 0)$ is determined by the Marked Length Spectrum $\mathcal{MLS}$. By the construction of the Normal Form given by Moser [Mos]

[8] The first derivative of the curvature vanishes due to the symmetries of the table.

[9] due to the $\mathbb{Z}_2$-symmetry of $\mathcal{O}_1$.

(see in particular the equations (3.2), (3.3) and (3.4) on pp. 680–681), the Birkhoff invariants are determined inductively by the jet of T^*. More precisely, for each $k_1 \geq 1$, the coefficients $(a_k(\mathcal{D}))_{1\leq k\leq k_1}$ of N^* are related to the $(2k_1+1)^{\text{th}}$ jet of T^* by some invertible triangular system, and thus, there is a one-to-one correspondence between the jets of T^* and N^* at the point $(0,0)$. By the previous discussion, we deduce that the jet of T^* at the point $(0_{2^*},0)$ is determined by $\mathcal{MLS}$. Now, by Lemma 6.5, the jet of T^* at the point $(0_{2^*},0)$ is also in one-to-one correspondence with the coefficients $(\beta_{2k})_{k\geq 1}$. Therefore, the coefficients $(\beta_{2k})_{k\geq 1}$ can be recovered from the Marked Length Spectrum, which concludes the proof. □

To conclude the proof of Theorem 6.1, it remains to show that the geometry of the third scatterer can also be recovered. While the auxiliary table $\mathcal{D}^*$ and the associated Birkhoff Normal Form N^* were useful to determine the geometry of $\mathcal{O}_1,\mathcal{O}_2$, now, we focus again on the initial billiard table $\mathcal{D}$. We denote by $\mathcal{F}=\mathcal{F}(\mathcal{D})$ and $T:=\mathcal{F}^2$ the billiard map of $\mathcal{D}$ and its square, let $N=N(\mathcal{D},1,2)$ be the Birkhoff Normal Form of $\mathcal{F}^2$ associated to the 2-periodic orbit (12), and assume that the first Birkhoff invariant of N is non-zero.

Corollary 6.8. *The geometry of $\mathcal{O}_3$ can be reconstructed from $\mathcal{MLS}$.*

Proof. In the following, we use the notation introduced in Subsections 4.1-4.2. By Corollary 5.7, the Marked Length Spectrum $\mathcal{MLS}$ determines the function γ and the differential $D_{(\xi,\eta)}\mathcal{G}$ of the gluing map $\mathcal{G}=R_-\circ T\circ R_+^{-1}|_{\Omega_\infty}$, at any point $(\xi,\eta)\in\Gamma_\infty$. Restricted to $\mathscr{O}_\infty:=R_+^{-1}(\Omega_\infty)$, resp. $T(\mathscr{O}_\infty)$, we have $R_+=R_{m_0}$, resp. $R_-=R_{-m_0}$, for some integer $m_0\geq 1$, where $R_{\pm m_0}:=N^{\pm m_0}R_0T^{\mp m_0}$, and R_0 is the canonical conjugacy map given by Lemma 2.5. By Corollary 5.7, the map N is a $\mathcal{MLS}$-invariant. Similarly, the map R_0 is also a $\mathcal{MLS}$-invariant, as it only depends on the obstacles $\mathcal{O}_1,\mathcal{O}_2$ whose geometry is also known, by Corollary 6.7. Besides, in the definition of $R_{\pm m_0}$, we take iterates of T between the first two obstacles, so their expression is also known. In other words, restricted to $\mathscr{O}_\infty$, resp. $T(\mathscr{O}_\infty)$, the map $R_+=R_{m_0}$, resp. $R_-=R_{-m_0}$ is determined by $\mathcal{MLS}$, as it only depends on the obstacles $\mathcal{O}_1,\mathcal{O}_2$ whose geometry is $\mathcal{MLS}$-invariant. Since γ and R_+ are $\mathcal{MLS}$-invariants, then the arc $\mathscr{A}_\infty:=R_+^{-1}(\Gamma_\infty)=\{R_+^{-1}(\eta,\xi_\infty+\gamma(\eta)):|\eta|\text{ small}\}$ can be recovered. Moreover, we know the differential $D_{(\xi,\eta)}\mathcal{G}$ at any point $(\xi,\eta)\in\Gamma_\infty$, with $\mathcal{G}=R_-\circ T\circ R_+^{-1}|_{\Omega_\infty}$, hence we can determine $D_x\mathcal{F}^2$, for any $x=(s,r)=R^{-1}(\eta,\xi_\infty+\gamma(\eta))\in\mathscr{A}_\infty$, with $|\eta|$ small. By definition, given any such point x, we have $\mathcal{F}(x)=x'=(s',0)$ for some parameter $s'\in\mathbb{R}$, and $\mathcal{F}^2(x)=(s,-r)$, by Lemma 4.2. Let us denote by $\mathcal{K}(s),\mathcal{K}(s')$ the respective curvatures at the points $x=(s,r)$, $x'=(s',0)$, by $\mathscr{L}:=h(s,s')$ the length of the line segment between their traces on the table, and set $\nu:=\sqrt{1-r^2}$. By (1.2), we have

$$D_x\mathcal{F}^2=\begin{pmatrix}\frac{2aa'}{\nu}-1 & \frac{2a'\mathscr{L}}{\nu^2}\\ \frac{2a}{\mathscr{L}}(aa'-\nu) & \frac{2aa'}{\nu}-1\end{pmatrix},$$

where

$$a:=\mathscr{L}\mathcal{K}(s)+\nu,\qquad a':=\mathscr{L}\mathcal{K}(s')+1.$$

The values of $\mathcal{K}(s)$ and ν are already known, thus by considering the first line in the expression of this differential, we deduce the value of aa' and $a'\mathscr{L}$, and then, of $\mathscr{L}$ and $\mathcal{K}(s')$. In particular, the geometry of $\mathcal{O}_3$ is completely determined: for any such $x\in\mathscr{A}_\infty$, we draw a line segment of length $\mathscr{L}$ starting from the associated point in the table, such that the angle of this segment with the normal to $\partial\mathcal{O}_2$ at this point is equal to $\varphi = \arcsin(r)$. Then, the endpoint belongs to $\partial\mathcal{O}_3$, and the curvature at this point is equal to $\mathcal{K}(s')$. In particular, we recover the geometry of an arc in $\mathcal{O}_3$, hence the third obstacle $\mathcal{O}_3$ is entirely determined by $\mathcal{MLS}$, by analyticity.

Alternatively, we may argue as follows. Since N and γ are $\mathcal{MLS}$-invariants, then for any sufficiently large integer $n = 2m-1 \geq 0$, we can compute the coordinates $(\xi_n, \eta_n) = R_-(x_n(1))$ of the associated point in the orbit h_n. The conjugacy map R_- is entirely determined by the obstacles $\mathcal{O}_1, \mathcal{O}_2$, whose geometry is known, thus we can recover the coordinates of $x_n(1) = (s_n(1), r_n(1))$, and also of the other points in this orbit which are on $\partial\mathcal{O}_1, \partial\mathcal{O}_2$, i.e., $x_n(k)$, for $k = 2, \cdots, 2n+1$. Let $\mathscr{L}_n$ be the length of the line segment connecting the points of parameters $x_n(0)$ and $x_n(1)$. Then $\mathscr{L}_n$ is determined by $\mathcal{MLS}$: indeed, we know the total length $\mathcal{L}(h_n)$, as well as the length of the other orbit segments in h_n, as they are associated to the points $x_n(k)$, for $k = 1, \cdots, 2n+1$. Therefore, starting from the trace of the point $x_n(1)$, then the endpoint of the outward line segment of length $\mathscr{L}_n$ based at this point and making an angle $-\varphi_n(1) = -\arcsin(r_n(1))$ with the normal to $\partial\mathcal{O}_2$ gives a point on $\partial\mathcal{O}_3$, associated to the parameter $x_n(0)$. For different values of n, the corresponding points are pairwise distinct (they have different periods and all start perpendicularly to $\partial\mathcal{O}_3$), and since they accumulate to the trace of the homoclinic point $x_\infty(0)$ as $n \to +\infty$, this determines completely the geometry of $\mathcal{O}_3$, again by analyticity. □

6.3. **Further remarks about general palindromic periodic orbits.** Actually, part of the argument that was detailed previously in the case of the 2-periodic orbit (12) can be adapted to general periodic palindromic trajectories. In particular, we will see that it is *a priori*[10] possible to recover a lot of information on the Birkhoff Normal Form in this case too. Recall that a periodic trajectory of period $p = 2q \geq 2$ is palindromic if it can be represented by an admissible word $\hat{\sigma} = (\sigma_1\sigma_2 \ldots \sigma_{q-1}\sigma_q\sigma_{q-1} \ldots \sigma_1\sigma_0)$ (see the definition given in Section 1). In particular, by expansiveness of the billiard dynamics, and since the past and future at the points x_0 and x_q with respective indices σ_0 and σ_q have the same symbolic coding, then as in (2.3), the angle with the normal has to vanish at those two points, and the orbit has two symmetries.

Remark 6.9. *Palindromic periodic orbits are dense among the set of non-escaping orbits in the following sense: given any orbit* $\mathcal{O} = (\ldots, y(-1), y(0), y(1), \ldots)$, *any* $n_1 \leq n_2$, *and any* $\varepsilon > 0$, *there exists a palindromic periodic orbit* $(y_\varepsilon(1), \ldots, y_\varepsilon(p_\varepsilon))$ *of period* $p_\varepsilon \geq 2$ *such that for some integer* $m_\varepsilon \geq 0$, *it holds*

$$d(y(k), y_\varepsilon(m_\varepsilon + k)) < \varepsilon, \qquad \forall\, k = n_1, n_1+1, \cdots, n_2.$$

Indeed, if $\sigma = (\ldots \sigma_{n_1}\sigma_{n_1+1} \ldots \sigma_{n_2} \ldots)$ *is a coding of* $\mathcal{O}$ *such that the points* $y(n_1)$ *and* $y(n_2)$ *are respectively indexed by* σ_{n_1} *and* σ_{n_2} *in the above word, then for any symbols* $\tau_1 \neq \sigma_{n_1}$, $\tau_2 \neq \sigma_{n_2}$, *and for any integer* $n \geq 0$, *the word* $(\sigma_{n_2}\sigma_{n_2-1} \ldots \sigma_{n_1})^n \tau_1 (\sigma_{n_1}\sigma_{n_1+1} \ldots \sigma_{n_2})^n \tau_2$ *is palindromic at the points indexed by* τ_1 *and* τ_2. *Moreover, by expansiveness of the dynamics, for* $n \gg 1$, *the points associated to symbols far away from* τ_1, τ_2 *can be made arbitrarily close to the points of* $\mathcal{O}$ *with the same symbol.*

[10] i.e., based only on the Marked Length Spectrum, without assuming analyticity of the scatterers nor symmetries.

Let us consider a billiard table $\mathcal{D} \in \mathbf{B}$. Let $\mathcal{F} := \mathcal{F}(\mathcal{D})$ be its billiard map, and assume that $\mathcal{MLS}(\mathcal{D})$ is known. In the case of a palindromic orbit $\hat{\sigma} = (\sigma_1\sigma_2 \dots \sigma_{q-1}\sigma_q\sigma_{q-1} \dots \sigma_1\sigma_0)$ of period $p = 2q$ as above, we consider the map $\hat{T} := \mathcal{F}^p$ in place of $T = \mathcal{F}^2$. Similarly, the point x_0 is a saddle fixed point for the dynamics of $\hat{T}$, and we can consider the associated Birkhoff Normal Form $\hat{N} = \hat{R} \circ \hat{T} \circ \hat{R}^{-1} \colon (\xi, \eta) \mapsto (\hat{\Delta}(\xi\eta)\xi, \hat{\Delta}(\xi\eta)^{-1}\eta)$ defined in a neighborhood of x_0, with $\hat{\Delta}(z) = \hat{\lambda} + \sum_{k\geq 1} \hat{a}_k z^k$, where $\hat{\lambda} < 1 < \hat{\lambda}^{-1}$ are the eigenvalues of $\hat{T}$ (by Theorem 1.12, the value of $\hat{\lambda}$ is a $\mathcal{MLS}$-invariant).

Corollary 6.10. *From the Marked Length Spectrum, we can recover the value of $\hat{a}_k\hat{\xi}_\infty^{2k}$ for each $k \geq 1$, where $\hat{\xi}_\infty \in \mathbb{R}$ is some (unknown) homoclinic parameter.*

Proof. The proof proceeds as in Corollary 4.21. Indeed, we pick a letter τ such that $\tau \neq \sigma_0$, and we note that the words $\hat{h}_n := (\tau\sigma_0\hat{\sigma}^n)$, $n \geq 0$ are palindromic. Similarly, by the palindromic symmetry, there is an analog of the relation obtained in Lemma 4.6. The Lyapunov exponent of each orbit $(\hat{h}_n)_n$ is a $\mathcal{MLS}$-invariant, hence following the same method as in Lemmata 4.8, 4.10, 4.11, 4.15, 4.20 and Corollary 4.21, we can recover the value of each coefficient $\hat{\lambda}^{-1}\hat{a}_\ell\hat{\xi}_\infty^{2\ell}$, $\ell \geq 1$. Here, $\hat{\xi}_\infty \in \mathbb{R}$ is some parameter associated to the homoclinic orbit $\hat{h}_\infty = (\dots\hat{\sigma}\hat{\sigma}\hat{\sigma}|\tau\sigma_0|\hat{\sigma}\hat{\sigma}\hat{\sigma}\dots)$. $\square$

Remark 6.11. *In order to recover the value of $\hat{\xi}_\infty$ and then of $\hat{a}_1, \hat{a}_2 \dots$, we may similarly consider functions $S^1, S^2, \cdots, S^p$ in place of S^1, S^2 in Section 5, and we get estimates of the analogous sums $\Sigma_n^1 + \Sigma_n^2$ in the new coordinates (ξ, η). As in Proposition 5.1, we can get estimates of $\mathcal{L}(\hat{h}_n) - n\mathcal{L}(\hat{\sigma}) - \mathcal{L}^\infty(\hat{\sigma})$, and thus deduce from the Marked Length Spectrum the value of $\hat{\xi}_\infty^2 \mathrm{tr}(d^2S)$, where $S = S^1 + \cdots + S^p$.*

Again, by invariance under symplectic conjugacy, we may compute $\mathrm{tr}(d^2S)$ in terms of the (s, r)-coordinates, which is a symmetric combination of the geometric information at the different points of the orbit $\hat{\sigma}$. Therefore, if we were able to compute this trace, then we could determine the Birkhoff Normal Form $\hat{N}$. But for general palindromic orbits, there is no analog of Corollary 1.11, i.e., we do not know a priori how to recover the curvature at each point in the orbit separately, so unless we already have additional information on the geometry, it seems more complicated to compute the value of $\mathrm{tr}(d^2S)$.

7. Conclusions and further questions

In this paper we showed that for a generic class of chaotic billiard obtained by removing from the plane $m \geq 3$ convex analytic scatterers with some symmetries, the Marked Length Spectrum determines the geometry of the billiard.

Our result leads to a number of natural questions:

Question 1. *Is it possible to remove the assumption about the mirror symmetry between $\mathcal{O}_1$ and $\mathcal{O}_2$?*

The analysis carried out in [BDKL] suggests that a more careful asymptotic analysis should yield the desired result, but more details need to be checked to ensure that this approach is feasible.

Question 2. *Is it possible to remove the assumption about each of the scatterers $\mathcal{O}_1$ and $\mathcal{O}_2$ being symmetric around the period-two collision?*

This would be a major step (similar to the one leading from [Z1] to [Z2, Z3]), and the techniques involving classical Birkhoff normal forms seem inadequate to this task.

Question 3. *Is it possible to obtain similar results for the unmarked Length Spectrum?*

As in [Z2, Z3], one could ask if simply marking the length of the 2-periodic orbit, of the associated Lyapunov exponent and possibly requiring some non-degeneracy of the Spectrum for those values, would suffice to recover the coefficients that describe the dynamics in the Birkhoff Coordinates. It is unclear to us if such an approach could be carried out successfully, since it is not easy to distinguish between the many homoclinic orbits that accumulate on the periodic orbit, and our strategy hinges on a very fine asymptotic analysis of a very special family of approximating homoclinic orbits. We ultimately believe that such an approach should be possible, but for sure our strategy would have to be modified to deal with all such homoclinic orbits at the same time.

Let us also note that several quantities are length-spectral invariants. For instance, any periodic orbit of period at least three bounces on the three scatterers; as 2-periodic orbits correspond to minimizers of the distance between two scatterers, the two smallest elements in the Length Spectrum are the lengths of two 2-periodic orbits.

Besides, given $0 < \tau_{\min} < \tau_{\max}$, we may estimate the number of periodic orbits of period $p \geq 3$ by counting the number $N(p)$ of elements in the Length Spectrum which are in the interval $[p\tau_{\min}, p\tau_{\max}]$. As the topological entropy of the subshift of finite type described in the first part is equal to $\log(m-1)$, where $m \geq 3$ is the number of obstacles, we expect $N(p)$ to grow as $(m-1)^p$, hence the number of obstacles to be a length-spectral invariant.

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