ON THE CENTRALIZER OF VECTOR FIELDS: CRITERIA OF TRIVIALITY AND GENERICITY RESULTS

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Abstract. In this paper, we investigate the question of whether a typical vector field on a compact connected Riemannian manifold M^d has a "small" centralizer. In the C^1 case, we give two criteria, one of which is C^1-generic, which guarantees that the centralizer of a C^1-generic vector field is indeed small, namely *collinear*. The other criterion states that a C^1 *separating* flow has a collinear C^1-centralizer. When all the singularities are hyperbolic, we prove that the collinearity property can actually be promoted to a stronger one, refered as *quasi-triviality*. In particular, the C^1-centralizer of a C^1-generic vector field is quasi-trivial. In certain cases, we obtain the triviality of the centralizer of a C^1-generic vector field, which includes C^1-generic Axiom A (or sectional Axiom A) vector fields and C^1-generic vector fields with countably many chain recurrent classes. For sufficiently regular vector fields, we also obtain various criteria which ensure that the centralizer is *trivial* (as small as it can be), and we show that in higher regularity, collinearity and triviality of the C^d-centralizer are equivalent properties for a generic vector field in the C^d topology. We also obtain that in the non-uniformly hyperbolic scenario, with regularity C^2, the C^1-centralizer is trivial.

Contents

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1. Introduction

Given a dynamical system, it is natural to try to understand the symmetries that it may have. Often times, they may give extra information which can be used to understand the dynamical behaviour. For example, towards the end of the 19th century, Lie was able to use the symmetries of some differential equations to derive their solutions, ans it was actually during this work that he introduced the notion of Lie groups. There are several different notions of symmetries that one may consider for a dynamical system. In this paper, the one we study is the so-called centralizer of a dynamical system.

The study of centralizers of diffeomorphisms has a long history. For a C^r diffeomorphism $f\colon M \to M$, where $r \geq 1$ and M is a compact Riemannian manifold, the centralizer of f is the set of all C^r diffeomorphisms $g\colon M \to M$ that commute with f. A diffeomorphism f has *trivial* centralizer if the only diffeomorphisms that commute with f are its powers, f^n, with $n \in \mathbb{Z}$. In [Sma91] and [Sma98], Smale asked the following question:

Question 1 ([Sma91], [Sma98]). Is the set of C^r diffeomorphisms with trivial centralizer a residual set? That is, does it contain a G_δ-dense subset of the space of C^r diffeomorphisms?

This question remains open in this generality. However, there are several partial answers. The first result related with Question 1 goes back to 1970, with the work of Kopell [Kop70], which gives an affirmative answer when $r \geq 2$ and M is the circle. Since then several partial answers have been given. In [BCW09], Bonatti-Crovisier-Wilkinson proved that the centralizer of a C^1-generic diffeomorphism is trivial, giving a positive answer to Question 1 for $r = 1$. For vector fields, the picture is much more incomplete.

Consider $\mathfrak{X}^r(M)$ to be the set of C^r vector fields of M. For $X \in \mathfrak{X}^r(M)$ and for any $1 \leq k \leq r$ we define the C^k-centralizer of X as

$$\mathfrak{C}^k(X) := \{Y \in \mathfrak{X}^k(M) : [X, Y] = 0\}, \tag{1.1}$$

where $[X, Y]$ is the Lie bracket of M. Notice that for vector fields there are different types of centralizers (or symmetries) that are natural to consider. For example, given a vector field X one may consider the set of diffeomorphisms f that fix X, that is, $f_*X = X$. However, in this paper we will only study the centralizer defined in (1.1).

There are different notions of "triviality" for the centralizers of vector fields, called *collinearity*, *quasi-triviality* and triviality (see Definitions 2.1, 2.6 and 2.7 below). We remark that the notion of collinearity is weaker than quasi-triviality, which itself is weaker than triviality.

There are some results about conditions for some "triviality" of the centralizer of a vector field, all of them related, to some extent with some hyperbolicity. In [KM73] the authors proved that an Anosov flow has quasi-trivial centralizer. This result was extended to (Bowen-Walters) expansive flows by Oka in [Oka76]. In [Sad79], Sad proved the "triviality" of a more general type of centralizer for an open and dense subset of C^∞-Axiom A vector fields that verify a strong transversality condition. More recently, in [BRV18], Bonomo-Rocha-Varandas proved the triviality of the centralizer of transitive Komuro expansive flows, which includes the Lorenz attractor.

This paper has two types of results: C^1-generic results, and some general results giving conditions for a vector field to have a centralizer exhibiting some of the types of "triviality" mentioned above. Let us summarize some of the results in this paper.

(1) A C^1-generic vector field has quasi-trivial centralizer (see Theorem B). Furthermore, if a C^1-generic vector field admits a countable spectral decomposition, then it has trivial centralizer. In particular, the centralizer of a C^1-generic vector field is trivial in the following cases: when M is a surface (see Corollary A), for Axiom A, or sectional Axiom A, vector fields (see Corollary B), and when M has dimension 3 and it is not approximated by vector fields with a homoclinic tangency (see Corollary C).

(2) For a compact manifold of dimension $d \geq 1$, a C^d-generic vector field has a collinear C^d-centralizer if and only if it has a trivial C^d-centralizer (see Theorem G).

(3) A *separating* vector field has collinear centralizer (see Proposition 2.4 and Definition 2.3 for the definition of the separating property).

(4) A C^1 vector field with collinear centralizer such that all its singularities are hyperbolic has quasi-trivial centralizer (see Theorem A).

(5) A C^2 vector field that preserves a non-uniformly hyperbolic measure with full support and with finitely many singularities has trivial centralizer (see Theorem D).

(6) In dimension 3, a C^3 vector field with some type of expansiveness (called *Kinematic expansiveness*) and such that all its singularities are hyperbolic has trivial centralizer (see Theorem E).

In the next section we give all the definitions and precise statements of our results. Let us make a few remarks. In result (1) we can actually obtain triviality in several scenarios, see Theorem 6.1. What is missing to obtain the triviality of the centralizer for a C^1-generic vector field is to prove that a C^1-generic vector field does not admit any non-trivial C^1 first integral (see section 5). This is a conjecture made by Thom [Thom]. With our result, a complete answer for Question 1 for C^1 vector fields is equivalent to answering Thom's conjecture. Result (2) states that at least C^d-generically the three notions of "triviality" coincide. For a manifold of dimension $d \geq 1$, the regularity C^d is needed because we use Sard's theorem in the proof.

An important point of our work is given in item (4). To conclude quasi-triviality from collinearity is a problem of extending an invariant function to the singularities. The previous works in this direction were dealing with vector fields with higher regularity, see for instance [BRV18]. In our work we are able to obtain a quite general criterion using only regularity C^1.

A natural direction is to understand what happens in the generic case in higher regularity. We conclude this section with the following question:

Question 2. Given any manifold M with dimension $\dim(M) \geq 3$, does there exist $r > 1$ sufficiently large and a C^r-open set $\mathcal{U} \subset \mathfrak{X}^r(M)$ such that for any $X \in \mathcal{U}$ the C^s-centralizer (for some $1 \leq s \leq r$) of X is not collinear?

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2. Definitions and statement of the main results

In this part, we introduce some definitions and notations, and we summarize some of the results that we will show in the following.

2.1. **General notions on vector fields.** Let M be a smooth manifold of dimension $d \geq 1$, which we assume to be compact and boundaryless. For any $r \geq 1$, we denote by $\mathfrak{X}^r(M)$ the space of vector fields over M, endowed with the C^r topology. A property $\mathcal{P}$ for vector fields in $\mathfrak{X}^r(M)$ is called C^r-generic if it is satisfied for any vector field in a *residual set* of $\mathfrak{X}^r(M)$. Recall that $\mathcal{R} \subset \mathfrak{X}^r(M)$ is *residual* if it contains a G_δ-dense subset of $\mathfrak{X}^r(M)$. In particular, it is dense in $\mathfrak{X}^r(M)$, by Baire's theorem.

In the following, given a vector field $X \in \mathfrak{X}^1(M)$, we denote by X_t the flow it generates. Recall that for any $Y \in \mathfrak{C}^1(X)$, and for any $s, t \in \mathbb{R}$, we have $Y_s \circ X_t = X_t \circ Y_s$. Differentiating this relation with respect to s at 0, we get

$$Y(X_t(x)) = DX_t(x) \cdot Y(x), \quad \forall x \in M. \tag{2.1}$$

We denote by $\mathrm{Zero}(X) := \{x \in M : X(x) = 0\}$ the set of *zeros*, or *singularities*, of the vector field X, and we set

$$M_X := M - \mathrm{Zero}(X). \tag{2.2}$$

For any $x \in M$ and any interval $I \subset \mathbb{R}$, we also let $X_I(x) := \{X_t(x) : t \in I\}$. In particular, we denote by $\mathrm{orb}^X(x) := X_{\mathbb{R}}(x)$ the orbit of the point x under X. Note that if $x \in M_X$, then $\mathrm{orb}^X(x) \subset M_X$ too.

Let $X \in \mathfrak{X}^1(M)$ be some C^1 vector field. The *non-wandering set* $\Omega(X)$ of X is defined as the set of all points $x \in M$ such that for any open neighbourhood $\mathcal{U}$ of x and for any $T > 0$, there exists a time $t > T$ such that $\mathcal{U} \cap X_t(\mathcal{U}) \neq \emptyset$.

Let us also recall another weaker notion of recurrence. Given two points $x, y \in M$, we write $x \sim_X y$ if for any $\varepsilon > 0$ and $T > 0$, there exists an (ε, T)-*pseudo orbit* connecting them, i.e., there exist $n \geq 2$, $t_1, t_2, \ldots, t_{n-1} \in [T, +\infty)$, and $x = x_1, x_2, \ldots, x_n = y \in M$, such that $d(X_{t_j}(x_j), x_{j+1}) < \varepsilon$, for $j \in \{1, \ldots, n-1\}$. The *chain recurrent set* $\mathcal{CR}(X) \subset M$ of X is defined as the set of all points $x \in M$ such that $x \sim_X x$. Restricted to $\mathcal{CR}(X)$, the relation $\sim_X$ is an equivalence relation. An equivalence class under the relation $\sim_X$ is called a *chain recurrent class*: $x, y \in \mathcal{CR}(X)$ belong to the same chain recurrent class if $x \sim_X y$. In particular, chain recurrent classes define a partition of the chain recurrent set $\mathcal{CR}(X)$.

A point $x \in M$ is *periodic* if there exists $T > 0$ such that $X_T(x) = x$. The set of all periodic points is denoted by $\mathrm{Per}(X)$, observe that we are also including the singularities in this set.

An X-invariant compact set Λ is hyperbolic if there is a continuous decomposition of the tangent bundle over Λ, $T_\Lambda M = E^s \oplus \langle X \rangle \oplus E^u$ into DX_t-invariant sub-bundles that verifies the following property: there exists $T > 0$ such that for any $x \in \Lambda$, we have

$$\|DX_T(x)|_{E^s_x}\| < \frac{1}{2} \text{ and } \|DX_{-T}(x)|_{E^u_x}\| < \frac{1}{2}.$$

A periodic point $x \in \mathrm{Per}(X)$ is hyperbolic if $\mathrm{orb}^X(x)$ is a hyperbolic set. Let γ be a hyperbolic periodic orbit. We denote by $W^s(\gamma)$ the stable manifold of the periodic orbit γ, which is defined as the set of points $y \in M$ such that $d(X_t(y), \gamma) \to 0$ as $t \to +\infty$. We define in an analogous way the unstable manifold of γ. It is well known that the stable and unstable manifolds are C^1-immersed submanifolds. A hyperbolic periodic orbit is a sink if the unstable direction is trivial. It is a source if the stable direction is trivial. A hyperbolic periodic orbit is a saddle if it is neither a sink nor a source. For a hyperbolic periodic point p we defined its *index* by $\mathrm{ind}(p) := \dim(E^s)$.

2.2. **Collinear centralizers.** In this part, we consider a compact Riemannian manifold M, and we let $r, k \geq 1$ be positive integers. Given $x \in M$ and $u, v \in T_xM$ we denote by $\langle u, v\rangle$ the subspace spanned by u and v in T_xM.

Definition 2.1 (Collinear centralizer). We say that $X \in \mathfrak{X}^r(M)$ has a *collinear* C^k-centralizer if

$$\dim\langle X(x), Y(x)\rangle \leq 1,$$

for every $x \in M$ and every $Y \in \mathfrak{C}^k(X)$.

We have the following elementary result:

Lemma 2.2. *Let $X \in \mathfrak{X}^r(M)$ and assume that the vector field $Y \in \mathfrak{C}^k(M)$ satisfies $\dim\langle X(x), Y(x)\rangle \leq 1$, for every $x \in M$. Then, there exists a function $f \in C^s(M_X, \mathbb{R})$, with $s := \min\{r, k\}$, such that*

$$Y(x) = f(x)X(x), \quad \forall\, x \in M_X.$$

Moreover, the function f is X-invariant, i.e.,

$$f(X_t(x)) = f(x), \quad \forall\, x \in M_X,\ \forall\, t \in \mathbb{R}.$$

Proof. Let us denote by $(\cdot,\cdot)$ the scalar product associated to the Riemannian structure on M. For any $x \in M_X$ and for any $v \in T_xM$, we set $\pi_X(x, v) := \frac{(X(x),v)}{(X(x),X(x))}$. In particular, $\pi_X(x, v)X(x)$ is the orthogonal projection of the vector v on the direction spanned by $X(x)$. Let $Y \in \mathfrak{C}^k(M)$ be a vector field that satisfies $\dim\langle X(x), Y(x)\rangle \leq 1$. The function $f\colon M_X \to \mathbb{R}$, $x \mapsto \pi_X(x, Y(x))$ is of class C^s, with $s = \min\{r, k\}$. Moreover, by the collinearity of the vector fields X and Y, we have $Y = fX$.

By (2.1), it holds $Y(X_t(\cdot)) = DX_t \cdot Y(\cdot)$. Therefore, for any $x \in M_X$ and for any $t \in \mathbb{R}$, we have

$$f(X_t(x))X(X_t(x)) = DX_t(x)\cdot(f(x)X(x)) = f(x)DX_t(x)\cdot X(x) = f(x)X(X_t(x)),$$

where the last equality follows from (2.1), with Y in place of X. Since $X(X_t(x)) \neq 0$, we obtain $f(X_t(x)) = f(x)$, which concludes the proof. $\square$

In this paper, we obtain a few different criteria which ensure that the C^1-centralizer of a C^1 vector field is collinear. The following definition is a very weak form of expansiveness for flows.

Definition 2.3. A vector field $X \in \mathfrak{X}^1(M)$ is *separating* if there exists $\varepsilon > 0$ such that the following holds: if $d(X_t(x), X_t(y)) < \varepsilon$ for every $t \in \mathbb{R}$, then $y \in \mathrm{orb}^X(x)$.

In Section 3 we will elaborate on this property. In Section 3, we prove the following criterion for collinearity:

Proposition 2.4. *If $X \in \mathfrak{X}^1(M)$ is separating, then X has collinear C^1-centralizer.*

We remark that the separating property is not generic (see Appendix A). So to obtain that the C^1-centralizer of a C^1-generic vector field is collinear we will need another criterion.

In Section 3, we define the notion of unbounded normal distortion (see Definition 3.3). This is an adaptation for flows of the definition of unbounded distortion used in [BCW09] to prove the triviality of the C^1-centralizer of a C^1-generic diffeomorphism. Using this property we obtain the following proposition.

Proposition 2.5. *Let $X \in \mathfrak{X}^1(M)$. Suppose that X verifies the following properties:*

- *X has unbounded normal distortion;*
- *every singularity and periodic orbit of X is hyperbolic;*
- $\mathcal{CR}(X) = \overline{\mathrm{Per}(X)}$.

Then X has collinear C^1-centralizer.

2.3. **Quasi-trivial centralizers.** Let M be a compact manifold.

Definition 2.6 (Quasi-trivial centralizer). Given two positive integers $1 \leq k \leq r$, we say that $X \in \mathfrak{X}^r(M)$ has a *quasi-trivial* C^k-centralizer if for every $Y \in \mathfrak{C}^k(X)$, there exists a C^1 function $f : M \to \mathbb{R}$ such that $X \cdot f \equiv 0$ and $Y(x) = f(x)X(x)$, for every $x \in M$.

Actually, by Lemma 2.2, if $X \in \mathfrak{X}^r(M)$ has a quasi-trivial C^k-centralizer, then for any $Y \in \mathfrak{C}^k(X)$, the function f in Definition 2.6 is in fact of class C^k in restriction to M_X.

The difference between collinear and quasi-trivial centralizers is to know whether or not a C^k invariant function defined on M_X admits a C^1 extension to M. This is not always the case; indeed, in Section 4 we construct an example of a vector field with collinear centralizer which is not quasi-trivial.

Nevertheless, when all the singularities of a C^1 vector field are hyperbolic, collinearity can actually be promoted to quasi-triviality:

Theorem A. *Let M be a compact manifold. If $X \in \mathfrak{X}^1(M)$ has collinear C^1-centralizer and all the singularities of X are hyperbolic, then X has quasi-trivial C^1-centralizer.*

A significant part of the present paper is dedicated to the proof of the C^1-genericity of the unbounded normal distortion property (see Section 6). Since the other assumptions of Proposition 2.5 and Theorem A are already known to be C^1-generic, this allows us to conclude:

Theorem B. *Let M be a compact manifold. There exists a residual subset $\mathcal{R} \subset \mathfrak{X}^1(M)$ such that any $X \in \mathcal{R}$ has quasi-trivial C^1-centralizer.*

2.4. **Trivial centralizers.** Let M be a compact manifold. Notice that for any $r \geq 1$ and $X \in \mathfrak{X}^r(M)$, we have that $cX \in \mathfrak{C}^k(X)$, for any $c \in \mathbb{R}$ and $1 \leq k \leq r$.

Definition 2.7 (Trivial centralizer)*.* For any $1 \leq k \leq r$, we say that $X \in \mathfrak{X}^r(M)$ has a *trivial* C^k-centralizer if $\mathfrak{C}^k(X)$ is as small as it can be, i.e.,

$$\mathfrak{C}^k(X) = \{cX : c \in \mathbb{R}\}.$$

It is easy to construct examples of vector fields whose centralizer is quasi-trivial but not trivial. In Section 5 we explain how example 3.1 has quasi-trivial centralizer, but not trivial.

The problem of knowing if a quasi-trivial centralizer is trivial is reduced to the problem of knowing when a X-invariant function is constant. This problem will be studied in Section 5.

Our first criterion to obtain triviality is based on the notion of *spectral decomposition.* We say that X admits a *countable spectral decomposition* if the non-wandering set, $\Omega(X)$, satisfies $\Omega(X) = \sqcup_{i\in\mathbb{N}}\Lambda_i$, where the sets Λ_i are pairwise disjoint, X-invariant and transitive, i.e., contains a dense orbit.

Theorem C. *Let M be a compact connected manifold and let $X \in \mathfrak{X}^1(M)$. Assume that all the singularities of X are hyperbolic, that X admits a countable spectral decomposition and that the C^1-centralizer of X is collinear. Then $\mathfrak{C}^1(X)$ is trivial.*

Theorem C has several interesting applications. We now present several scenarios where we immediately obtain the triviality of the centralizer of a C^1-generic vector field.

In [Pei60], Peixoto proved that a C^1-generic vector field on a compact surface is Morse-Smale. Recall that a vector field is Morse-Smale if the non-wandering set is the union of finitely many hyperbolic periodic orbits and hyperbolic singularities, and it verifies some transversality condition. In particular, the non-wandering set is finite. As a consequence of this result of Peixoto and Theorems B and C, we have the following corollary.

Corollary A. *Let M be a compact connected surface. Then, there exists a residual set $\mathcal{R}_\dagger \subset \mathfrak{X}^1(M)$ such that for any $X \in \mathcal{R}_\dagger$, the C^1-centralizer of X is trivial.*

A C^1-vector field X is *Axiom A* if the non-wandering set is hyperbolic and $\Omega(X) = \overline{\mathrm{Per}(X)}$. It is well known that Axiom A vector fields admits a spectral decomposition, with finitely many basic pieces.

Corollary B. *A C^1-generic Axiom A vector field has trivial C^1-centralizer.*

Remark 2.8*.* Corollary B actually holds for more a general type of hyperbolic system called *sectional Axiom A*. We refer the reader to [MM08] definition 2.14, for a precise definition.

Another corollary is for C^1-vector fields far from homoclinic tangencies in dimension three. Let us make it more precise. Recall that a vector field $X \in \mathfrak{X}^1(M)$ has a *homoclinic tangency* if there exists a hyperbolic non-singular closed orbit γ and a non-transverse intersection between $W^s(\gamma)$ and $W^u(\gamma)$. By the proof of Palis conjecture in dimension three given in [CY17], a C^1-generic $X \in \mathfrak{X}^1(M)$ which cannot be approximated by such vector fields admits a finite spectral decomposition, hence:

Corollary C. *Let M be a compact connected 3-manifold. Then there exists a residual subset $\mathcal{R}_\ddagger \subset \mathfrak{X}^1(M)$ such that any vector field $X \in \mathcal{R}_\ddagger$ which cannot be approximated by vector fields exhibiting a homoclinic tangency has trivial C^1-centralizer.*

As a simple application of Proposition 2.4 and Theorem C, we obtain the triviality of the centralizer of the flow introduced in [Art15]. This example is a transitive Komuro expansive flow on the three-sphere such that all its singularities are hyperbolic. In particular, by the discussion in [Art16], this flow is separating.

In higher regularity, Pesin's theory in the non-uniformly hyperbolic case and Sard's theorem give us two useful tools to verify triviality of the centralizer. Consider a probability measure μ on M and $X \in \mathfrak{X}^1(M)$. We say that μ is X-invariant if for any measurable set $A \subset M$ and any $t \in \mathbb{R}$ we have $\mu(A) = \mu(X_t(A))$. By Oseledets theorem, for μ-almost every point x, there exist a number $1 \leq l(x) \leq d$ and $l(x)$-numbers $\lambda_1(x) < \ldots < \lambda_{l(x)}(x)$ with the following properties: there exist $l(x)$-subspaces $E_1(x), \ldots, E_{l(x)}(x)$ such that $T_xM = E_1(x) \oplus \cdots \oplus E_{l(x)}(x)$ and for each $i = 1, \ldots, l(x)$ and for any non zero vector $v \in E_i(x)$ we have

$$\lim_{t \to \pm\infty} \frac{\log \|DX_t(x) \cdot v\|}{t} = \lambda_i(x).$$

The numbers λ_i are called Lyapunov exponents. We say that μ is *non-uniformly hyperbolic* if for μ-almost every point all the Lyapunov exponents are non-zero except the direction generated by the vector field X. Using Pesin's theory, in the non-uniformly hyperbolic scenario, we obtain:

Theorem D. *Let M be a compact manifold of dimension $d \geq 2$. Let $X \in \mathfrak{X}^2(M)$ be a vector field with finitely many singularities and let μ be a X-invariant probability measure such that* $\operatorname{supp}\mu = M$. *If μ is non-uniformly hyperbolic for X, then X has trivial C^1-centralizer.*

Theorem D can be applied for non-uniformly hyperbolic geodesic flows, like the ones constructed by Donnay [Don88] and Burns-Gerber [BG89]. In particular, we obtain that non-uniformly hyperbolic geodesic flows have trivial centralizer.

In dimension three, under higher regularity assumptions, we are also able to obtain triviality, for a slightly stronger notion of expansiveness.

Definition 2.9. We say that $X \in \mathfrak{X}^1(M)$ is *Kinematic expansive* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in M$ satisfy $d(X_t(x), X_t(y)) < \delta$, for every $t \in \mathbb{R}$ then there exists $0 < |s| < \varepsilon$ such that $y = X_s(x)$.

The difference between the separating property and Kinematic expansiveness is that for the later even points on the same orbit must eventually separate. In [Art16] it is described a vector field on the Möbius band which is separating but is not Kinematic expansive.

Theorem E. *Let M be a compact 3-manifold and consider $X \in \mathfrak{X}^3(M)$. If X is Kinematic expansive and all its singularities are hyperbolic, then its C^3-centralizer is trivial.*

Remark 2.10. The Kinematic expansive condition does not imply that the system admits a countable spectral decomposition. Hence, we cannot use Theorem C to conclude Theorem E.

The technique we use in the above theorem, which relies on Sard's Theorem, also leads to a criteria to obtain triviality from a collinear centralizer of high regularity.

Theorem F. *Let M be a compact, connected Riemannian manifold of dimension $d \geq 1$, and let $X \in \mathfrak{X}^d(M)$. Assume that every singularity and periodic orbit of X is hyperbolic, that $\Omega(X) = \overline{\mathrm{Per}(X)}$ and that the C^d-centralizer of X is collinear. Then X has trivial C^d-centralizer.*

This criterion is not sufficient if we want to obtain a generic result, due to the lack of a C^d-closing lemma. However, following the arguments of [Hur86, Man73], we can show that C^d-generically the triviality of the C^d-centralizer is equivalent to the collinearity of the C^d-centralizer.

Theorem G. *Let M be a compact, connected Riemannian manifold of dimension $d \geq 1$. There exists a residual set $\mathcal{R}_T \subset \mathfrak{X}^d(M)$ such that for any $X \in \mathcal{R}_T$, the C^d-centralizer $\mathfrak{C}^d(X)$ of X is collinear if and only if it is trivial.*

Organization of the paper: The structure of this paper has two parts. The first part deals with general criteria for collinearity, quasi-triviality and triviality of the centralizer (Sections 3,4 and 5). The second part deals with our generic results (Section 6). Propositions 2.4 and 2.5 are proved in Section 3. In Section 4 we prove Theorem A. Theorems C, D, E, F and G are proved in Section 5. Finally in Section 6 we prove Theorem B.

3. Collinearity

In this section we obtain three criteria for collinear centralizer. The first criterion is given by Proposition 2.4, which is based on the notion of being *separating* (see Definition 2.3). There are several different notions of "expansiveness for flows. The property of being separating is a very weak form of expansiveness. Indeed, all the usual definitions for flows (Bowen-Walters expansive or Komuro expansive) imply that the flow is separating, see [Art16] for a discussion. Let us give an example of a separating flow.

Example 3.1. Fix two positive real numbers $0 < a < b$ and consider the annulus on $\mathbb{R}^2$ given by $A := \{(x, y) \in \mathbb{R}^2 : a \leq \|(x, y)\| \leq b\}$. Using polar coordinates (r, θ) on A, we consider the vector field $X(r, \theta) := \frac{\partial}{\partial \theta}$. Observe that every orbit of X is periodic with different period. It is easy to see that this flow is separating.

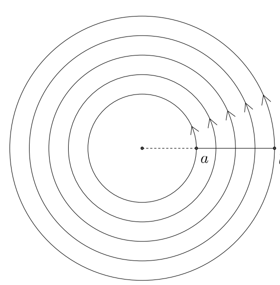

Figure 1. Example 3.1.

Proof of Proposition 2.4. Let $X \in \mathfrak{X}^1(M)$ be a separating vector field with separating constant $\varepsilon > 0$ and suppose that there exists $Y \in \mathfrak{C}^1(X)$ that is not collinear to X. Thus there is a point $x \in M$ such that $\dim\langle X(x), Y(x)\rangle = 2$.

Let (φ, U) be a small flow box for the flow X_t around x, that is, $\varphi\colon M \supset U \to W \subset \mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$ is a local chart such that $\varphi_* X = (1, 0)$. In particular we have that for every point $p \in U$ there is a positive number $\rho(p) > 0$ such that

$$\varphi(X_{(-\rho(p),\rho(p))}(p)) \subset \varphi(p) + (-\rho(p), \rho(p)) \times \{0\}.$$

Fix $\delta > 0$ small enough such that for each $s \in (-\delta, \delta)$, we have $Y_s(x) \in U$, $d_{C^0}(Y_s, \mathrm{id}) < \varepsilon$, and $\dim\langle Y(Y_s(x)), X(Y_s(x))\rangle = 2$. Thus in the flow box, we have that $D\varphi(Y_s(x)) \cdot Y(Y_s(x)) = (Y_1(s), Y_2(s))$, with $Y_1(s) \in \mathbb{R}$ and $Y_2(s) \in \mathbb{R}^d - \{0\}$. In particular, for each $t \in \mathbb{R}$ such that $X_t(x) \in Y_{(-\delta,\delta)}(x)$, there exists an open interval $I_t := (t - \rho(X_t(x)), t + \rho(X_t(x))) \subset \mathbb{R}$ such that $\# X_{I_t}(x) \cap Y_{(-\delta,\delta)}(x) = 1$.

We conclude that the set $\mathrm{orb}^X(x) \cap Y_{(-\delta,\delta)}(x)$ is at most countable. Then, since $(-\delta, \delta)$ is uncountable, there is $s \in (-\delta, \delta)$ such that $Y_s(x) \notin \mathrm{orb}^X(x)$. Now, by commutativity, and by our choice of δ, we obtain

$$d(X_t(Y_s(x)), X_t(x)) = d(Y_s(X_t(x)), X_t(x)) < \varepsilon, \text{ for every } t \in \mathbb{R},$$

which is a contradiction.

□

The following theorem will be used to prove Theorem D. It also gives another criterion to obtain collinearity of the C^1-centralizer.

Proposition 3.2. *Let $X \in \mathfrak{X}^1(M)$. Suppose that X verifies the following condition: there exists a dense set $\mathcal{D} \subset M$ such that for any $x \in \mathcal{D}$ and any non zero vector $v \in T_xM - \langle X(x)\rangle$, it holds*

$$\|DX_t(x) \cdot v\| \to +\infty, \text{ for } t \to +\infty \text{ or } t \to -\infty.$$

Then X has collinear centralizer.

Proof. Let $Y \in \mathfrak{C}^1(X)$. Then, by (2.1), for any $x \in M$, and $t \in \mathbb{R}$, it holds

$$Y(X_t(x)) = DX_t(x) \cdot Y(x).$$

Assume that $Y(x)$ is not collinear to $X(x)$. Since this is an open condition, we can take x belonging to the set $\mathcal{D}$. By compactness of M, we also have $\sup_{p \in M} \|Y(p)\| < +\infty$. However, by hypothesis,

$$\|DX_t(x) \cdot Y(x)\| \to +\infty, \text{ for } t \to +\infty \text{ or } t \to -\infty,$$

which is a contradiction. □

Some examples of vector fields that verify the conditions of Proposition 3.2 are non-uniformly hyperbolic divergence-free vector fields and quasi-Anosov flows.

We remark that the conditions of collinearity in Propositions 2.4 and 3.2 are not generic (see Appendix A). Therefore, to obtain that the C^1-centralizer of a C^1-generic vector field is collinear we will need another criterion, which is given by Proposition 2.5.

Let $X \in \mathfrak{X}^1(M)$ and let $M_X := M - \mathrm{Zero}(X)$ be as in (2.2). Over M_X we may consider the normal vector bundle N_X defined as $N_{X,p} := \langle X(p)\rangle^\perp$, for $p \in M_X$, where $\langle X(p)\rangle^\perp$ is the orthogonal complement of the direction $\langle X(p)\rangle$ inside T_pM. Let $\Pi^X : TM_X \to N_X$ be the orthogonal projection on N_X. On N_X we have a well defined flow, called the *linear Poincaré flow*, which is defined as follows: for any $p \in M_X$, any $v \in N_{X,p}$, and $t \in \mathbb{R}$, the image of v by the linear Poincaré flow is

$$P^X_{p,t}(v) := (\Pi^X_{X_t(p)} \circ DX_t(p)) \cdot v. \tag{3.1}$$

The key criterion to study the centralizer of C^1-generic vector fields is based on the following property.

Definition 3.3 (Unbounded normal distortion). Let $X \in \mathfrak{X}^1(M)$ be a C^1 vector field. We say that X verifies the *unbounded normal distortion* property if the following holds: there exists a dense subset $\mathcal{D} \subset M - \mathcal{CR}(X)$, such that for any $x \in \mathcal{D}$, $y \in M - \mathcal{CR}(X)$ such that $y \notin \mathrm{orb}^X(x)$ and $K \geq 1$, there is $n \in (0, +\infty)$, such that

$$|\log \det P^X_{x,n} - \log \det P^X_{y,n}| > K.$$

Proof of Proposition 2.5. Let $X \in \mathfrak{X}^1(M)$ be a vector field with the unbounded normal distortion property and let $\mathcal{D} \subset M - \mathcal{CR}(X)$ be the set given in Definition 3.3. Take $Y \in \mathfrak{C}^1(X)$. Assume by contradiction that Y is not collinear with X on $M - \mathcal{CR}(X)$. The set of points $x \in M$ such that $X(x)$ and $Y(x)$ are non-collinear is open, hence by density of the set $\mathcal{D}$, there exists a point $x \in \mathcal{D}$ such that $Y(x)$ and $X(x)$ are not collinear.

By the same argument as in the proof of Proposition 2.4, we can always find $s > 0$ arbitrarily close to 0 such that $Y_s(x) \notin \mathrm{orb}^X(x)$. Observe that for any $t \in \mathbb{R}$, it holds

$$|\det P^X_{Y_s(x),t}| = |\det \Pi^X_{X_t(Y_s(x))}. \det DX_t(Y_s(x))|_{N_{X,Y_s(x)}}|.$$

Since X commutes with Y, we have that

$$DX_t(Y_s(x)) = DY_s(X_t(x)) \circ DX_t(x) \circ (DY_s(x))^{-1}. \tag{3.2}$$

Using the coordinates $N_X \oplus \langle X \rangle$ on TM_X, for each $s \in \mathbb{R}$, we obtain a linear map $L_{s,x} \colon N_{X,x} \to \langle X(x) \rangle$ such that

$$(DY_s(x))^{-1}(N_{X,Y_s(x)}) = \mathrm{graph}(L_{s,x}).$$

Furthermore, $\|L_{s,x}\|$ can be made arbitrarily small as $s \to 0$, since Y_s is C^1-close to the identity. Using the coordinates $N_{X,x} \oplus \langle X(x) \rangle$, any vector $v \in \mathrm{graph}(L_{s,x})$ can be written as $v = (v_N, L_{s,x}(v_N))$, where $v_N := \Pi^X_x(v)$. For any such vector v, for each $t \in \mathbb{R}$ and using the coordinates $N_{X,X_t(x)} \oplus \langle X(X_t(x)) \rangle$, we have

$$DX_t(x)v = \left(P^X_{x,t}(v_N), L_{s,x}(v_N) \frac{\|X(X_t(x))\|}{\|X(x)\|} + \left(DX_t(x)v_N, \frac{X(X_t(x))}{\|X(X_t(x))\|} \right) \right), \tag{3.3}$$

where $(\cdot, \cdot)$ inside the second coordinate of the right side of (3.3) denotes the scalar product given by the Riemannian structure.

On the other hand, for any vector $v_N \in N_{X,x}$ and any $t \in \mathbb{R}$, we have

$$DX_t(x)v_N = \left(P^X_{x,t}(v_N), \left(DX_t(x) \cdot v_N, \frac{X(X_t(x))}{\|X(X_t(x))\|} \right) \right). \tag{3.4}$$

Set $c := \|X(x)\| > 0$, and let $\tilde{c} \geq 1$ be a constant such that $\sup_{p \in M} \|X(p)\| < \tilde{c}$. For any vector $v_N \in N_{X,x}$, we obtain

$$|L_{s,x}(v_N)| \frac{\|X(X_t(x))\|}{\|X(x)\|} < \|L_{s,x}\| \cdot \|v_N\| \frac{\tilde{c}}{c},$$

which can be made arbitrarily close to 0 by taking s small enough. This holds for any $t \in \mathbb{R}$. Hence, comparing (3.3) and (3.4) we conclude that $DX_t(x)|_{\mathrm{graph}(\mathrm{L}_{s,x})}$ is arbitrarily close to $DX_t(x)|_{N_{X,x}}$, for any $t \in \mathbb{R}$.

By (3.2), we obtain

$$
\begin{aligned}
&\left|\det P^X_{Y_s(x),t}\right| = \\
&\left|\det \Pi^X_{Y_s(X_t(x))}|_{DY_s(X_t(x))DX_t(x)\cdot \mathrm{graph}(L_{s,x})}\right| \cdot \left|\det DY_s(X_t(x))|_{DX_t(x)\cdot \mathrm{graph}(L_{s,x})}\right| \cdot \\
&\cdot \left|\det DX_t(x)|_{\mathrm{graph}(L_{s,x})}\right| \cdot \left|(\det(DY_s(x))^{-1}|_{N_{X,Y_s(x)}}\right| =: A\cdot B\cdot C\cdot D.
\end{aligned}
$$

Observe that

$$\left|\det P^X_{x,t}\right| = \left|\det \Pi^X_{X_t(p)}|_{DX_t(x)N_{X,x}}\right| \cdot \left|\det DX_t(x)|_{N_{X,x}}\right| =: \mathrm{I}\cdot \mathrm{II}.$$

Notice that B and D are arbitrarily close to 1 if $s \in \mathbb{R}$ is small enough. By our previous discussion, for any $t \in \mathbb{R}$ the value of C is arbitrarily close to the value of II, for s sufficiently small.

Our previous discussion also implies that $DY_s(X_t(x))DX_t(x)\cdot \mathrm{graph}(L_{s,x})$ is close to $DX_t(x)\cdot N_{X,x}$, since $Y_s(X_t(x))$ is close to $X_t(x)$. Thus, the value of A can be made arbitrarily close to the value of I, for $s \in \mathbb{R}$ small enough. Hence, we can take s small such that $Y_s(x) \notin \mathrm{orb}^X(x)$ and

$$\frac{1}{2} < \frac{\left|\det P^X_{Y_s(x),t}\right|}{\left|\det P^X_{x,t}\right|} < 2, \text{ for any } t \in \mathbb{R}.$$

This is a contradiction with the unbounded normal distortion property. We conclude that any vector field $Y \in \mathfrak{C}^1(X)$ verifies that $Y|_{M-\mathcal{CR}(X)}$ is collinear to $X|_{M-\mathcal{CR}(X)}$.

Suppose that for some $x \in \mathcal{CR}(X)$ we have that $Y(x)$ is not collinear to $X(x)$. Since this is an open condition and the hyperbolic periodic points are dense in $\mathcal{CR}(X)$, we can suppose that x is a periodic point. By a calculation similar to the one made in the proof of Proposition 3.2 we would then have that $\|Y(X_t(x))\| \to +\infty$ for $t \to +\infty$ or $t \to -\infty$, which contradicts the fact that $\sup_{p\in M}\|Y(p)\| < +\infty$. Thus we have that Y is also collinear to X on $\mathcal{CR}(X)$. $\square$

4. Quasi-triviality

This section has two parts. In the first part we construct an example of a vector field whose centralizer is collinear but not quasi-trivial. In the second part we prove that under the condition that every singularity is hyperbolic we can promote the collinearity to quasi-triviality.

4.1. **Collinear does not imply quasi-trivial.** To obtain a quasi-trivial centralizer from a collinear centralizer is an issue of knowing whether an invariant function $f\colon M_X \to \mathbb{R}$ admits a C^1 extension to the set $\mathrm{Zero}(X)$. The simple example below shows that this is not always possible. Indeed we construct an example of a vector field whose C^1-centralizer is collinear but not quasi-trivial.

Example 4.1. Let $V \in \mathfrak{X}^\infty(\mathbb{T}^2)$ generate an irrational flow. Fix a point $p \in \mathbb{T}^2$ and consider a function $\psi\colon \mathbb{T}^2 \to [0,1]$ such that $\psi(x) = 0 \iff x = p$. Let

$Z \in \mathfrak{X}^\infty(\mathbb{T}^2)$ be defined by $Z = \psi V$. As it is described in example 2.8 in [Art16], Z is separating. Now, consider $f, g\colon [0,1) \to [1,+\infty)$ be given, respectively, by

$$f(t) = \frac{1}{1-t} \quad \text{and} \quad g(t) = \frac{1}{1-t^2}.$$

Observe that both functions diverge to $+\infty$ when $t \to 1$, but the function $\frac{f}{g} = 1+t$ extends smoothly to $[0,1]$. Consider $M = [0,1] \times \mathbb{T}^2$ and extend Z to M by $Z(t,x) = Z(x)$. Define the vector field $X(t,x) = \frac{1}{g(t)} Z(t,x)$. Notice that X is tangent to the fiber $\{t\} \times \mathbb{T}^2$, and the trajectories on each fiber are the same, but travelled with different speeds. Then, the proof of Proposition 2.4 shows that $X \in \mathfrak{X}^\infty(M)$ has collinear centralizer.

Nevertheless the vector field $Y = \frac{f}{g} Z$ is smooth and commutes with X. Indeed, both vector fields vanish at the fiber $\{1\} \times \mathbb{T}^2$. Moreover, both f and g are constant on each fiber and for $t < 1$ one has

$$Y(t,x) = f(t) X(t,x).$$

As X is tangent to each fiber $\{t\} \times \mathbb{T}^2$, we conclude that $[X,Y] = 0$. Since $f(t) \to \infty$ when $t \to 1$, this proves that X has not a quasi-trivial centralizer.

The above example has uncountably many singularities, and thus it is not separating. This raises the following question.

Question 3. Is there a separating vector field whose centralizer is not quasi-trivial?

We do not know what to expect as an answer to this question.

4.2. **The case of hyperbolic zeros.** The main result of this section is Theorem 4.3 below, in which we obtain the quasi-triviality from collinearity of $\mathfrak{C}^1(X)$ assuming only that all the singularities of X are hyperbolic.

Definition 4.2. A function $f\colon M \to \mathbb{R}$ is called a *first integral* of X if it is of class C^1 and satisfies $X \cdot f \equiv 0$. We denote by $\mathfrak{I}^1(X)$ the set of all such maps.

In particular, for any $c \in \mathbb{R}$, the constant map $\underline{c}(x) := c$ is in $\mathfrak{I}^1(X)$, and then, we always have $\mathbb{R} \simeq \{\underline{c} : c \in \mathbb{R}\} \subset \mathfrak{I}^1(X)$. The following theorem is a reformulation in terms of $\mathfrak{I}^1(X)$ of Theorem A.

Theorem 4.3. *Let $X \in \mathfrak{X}^1(M)$. If X has collinear centralizer and all the singularities of X are hyperbolic, then X has quasi-trivial C^1-centralizer, in the sense of Definition 2.6. More precisely, we have*

$$\mathfrak{C}^1(X) = \{fX : f \in \mathfrak{I}^1(X)\}.$$

This theorem is an immediate consequence of Propositions 4.4, 4.5, and 4.6 below. We divide the proof into two subsections to emphasize that the technique to deal with singularities that are saddles is different from the technique to deal with sinks and sources. We also remark that Theorem 4.3 gives a significant improvement compared with previous works on centralizers of vector fields, since we only need C^1 regularity. The results that were known previously used Sternberg's linearisation results, which require higher regularity of the vector field and non-resonant conditions on the eigenvalues of the singularity, see for instance [BRV18].

4.3. **When the singularity is of saddle type.** Given any vector field $X \in \mathfrak{X}^1(M)$, and $Y \in \mathfrak{C}^1(X)$, by Lemma 2.2, we know that $Y|_{M_X} = fX|_{M_X}$, for some C^1, X-invariant function $f\colon M_X \to \mathbb{R}$. Assume that $\sigma \in \mathrm{Zero}(X)$ is a saddle type singularity. In Propositions 4.4 and 4.5, we show that f can be extended to a C^1 function in a neighbourhood of σ.

Proposition 4.4. *Let $X \in \mathfrak{X}^1(M)$ and let $f\colon M_X \to \mathbb{R}$ be an X-invariant continuous function. If $\sigma \in \mathrm{Zero}(X)$ is a saddle type singularity, then f admits a continuous extension to σ.*

Proof. Recall that M has dimension $d \geq 0$. Fix a point $p_s \in W^s_{\mathrm{loc}}(\sigma)$. We claim that for any point $q_u \in W^u(\sigma)$ we have that $f(p_s) = f(q_u)$. By the X-invariance of f, it is enough to consider $q_u \in W^u_{\mathrm{loc}}(\sigma)$. Let $(D^s_n)_{n\in\mathbb{N}}$ be a sequence of discs of dimension $\mathrm{ind}(\sigma)$, centred on q_u, with radius $\frac{1}{n}$ and transverse to $W^u_{\mathrm{loc}}(\sigma)$. Similarly, consider a sequence $(D^u_n)_{n\in\mathbb{N}}$ of discs of dimension $d - \mathrm{ind}(\sigma)$, centred on p_s, with radius $\frac{1}{n}$, and transverse to $W^s_{\mathrm{loc}}(\sigma)$.

For each $n \in \mathbb{N}$, by the lambda-lemma (see [PM82] chapter 2.7) there exists $t_n > 0$ such that $X_{t_n}(D^u_n) \pitchfork D^s_n \neq \emptyset$. In particular, there exists a point $x_n \in D^u_n$ that verifies $X_{t_n}(x_n) \in D^s_n$. It is immediate that $x_n \to p_s$, as $n \to +\infty$. Since the function f is continuous on M_X, we have that $f(x_n) \to f(p_s)$. We also have that $X_{t_n}(x_n) \to q_u$ as $n \to +\infty$. Hence, $f(X_{t_n}(x_n)) \to f(q_u)$. By the X-invariance of f, we have

$$f(p_s) = \lim_{n\to+\infty} f(x_n) = \lim_{n\to+\infty} f(X_{t_n}(x_n)) = f(q_u).$$

Analogously, we can prove that for a fixed $q'_u \in W^u_{\mathrm{loc}}(\sigma)$ and for any $p'_s \in W^s(\sigma)$, it is verified $f(p'_s) = f(q'_u)$. We conclude that $f|_{W^s(\sigma)-\{\sigma\}} = f|_{W^u(\sigma)-\{\sigma\}} = c$, for some constant $c \in \mathbb{R}$. In particular, we can define a continuous extension of f to the singularity σ by setting $f(\sigma) := c$. $\square$

Proposition 4.5. *Let $X \in \mathfrak{X}^1(M)$ and let $f\colon M_X \to \mathbb{R}$ be an X-invariant function of class C^1. If $\sigma \in \mathrm{Zero}(X)$ is a saddle type singularity, then f can be extended to a C^1 function in a neighbourhood of σ, by setting $\nabla f(\sigma) := 0$.*

Proof. By Proposition 4.4, we already know that the function f admits a continuous extension to σ. We claim that $\lim_{x\to\sigma} \nabla f(x) = 0$. Let us fix $r > 0$ sufficiently small such that $B(\sigma, 2r) \cap \mathrm{Zero}(X) = \{\sigma\}$ and set $K^* := W^*_{\mathrm{loc}}(\sigma) \cap \partial B(\sigma, r)$, for $* \in \{s, u\}$. In the following, given any two points $p_s \in K^s$ and $q_u \in K^u$, we let $(D^u_n)_{n\in\mathbb{N}}$ be a sequence of discs of dimension $d - \mathrm{ind}(\sigma)$, centred on p_s, with radius $\frac{1}{n}$, transverse to $W^s_{\mathrm{loc}}(\sigma)$, and we let $(D^s_n)_{n\in\mathbb{N}}$ be a sequence of discs of dimension $\mathrm{ind}(\sigma)$, centred on q_u, with radius $\frac{1}{n}$, transverse to $W^u_{\mathrm{loc}}(\sigma)$.

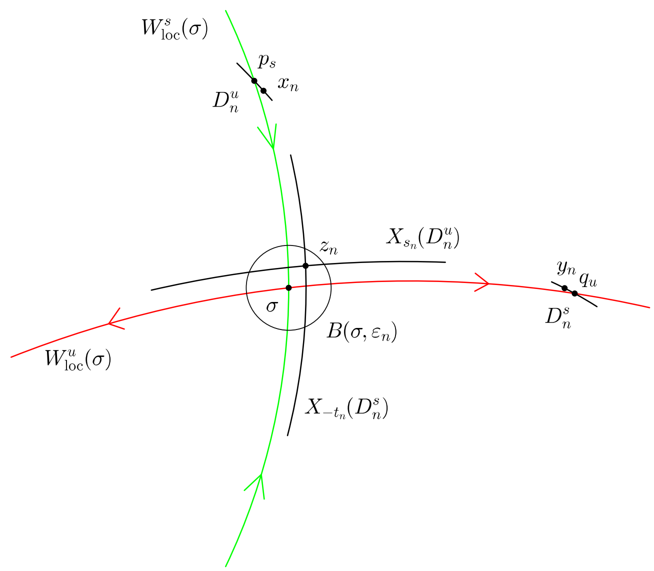

FIGURE 2. Proposition 4.5.

For any $n \geq 0$, by the lambda-lemma, there exists a sequence $(\varepsilon_n)_{n\geq 0} \in (\mathbb{R}_+^*)^{\mathbb{N}}$, with $\varepsilon_n \to 0$ as $n \to +\infty$, such that for any $z \in B(\sigma, \varepsilon_n)$, if $p_s \in K^s, q_u \in K^u$ are suitably chosen, and for D^u_n, D^s_n as defined previously, then there exist $x_n \in D^u_n$, $y_n \in D^s_n$, and $s_n, t_n > 0$, such that $z = X_{s_n}(x_n) = X_{-t_n}(y_n)$. Note that necessarily, $s_n, t_n \to +\infty$ as $n \to +\infty$. Fix $p_s \in K^s, q_u \in K^u$ and let $(z_n)_{n\geq 0}$ be a sequence of points such that $z_n = X_{s_n}(x_n) = X_{-t_n}(y_n) \in B(\sigma, \varepsilon_n)$, with $x_n \in D^u_n$, $y_n \in D^s_n$, $s_n, t_n > 0$, for all $n \geq 0$. It is immediate that $x_n \to p_s$ and $y_n \to q_u$, as $n \to +\infty$. Since the function f is C^1 on M_X, we deduce that

$$\lim_{n\to+\infty} \nabla f|_{D^u_n} = \nabla f(p_s), \qquad \lim_{n\to+\infty} \nabla f|_{D^s_n} = \nabla f(q_u). \tag{4.1}$$

We set $\mathbb{S}^u_n := \{v \in T_{x_n} D^u_n : \|v\| = 1\}$ and $\mathbb{S}^s_n := \{v \in T_{y_n} D^s_n : \|v\| = 1\}$. Let $v \in \mathbb{S}^u_n$. By the X-invariance, we have $f(z_n) = f(X_{s_n}(x_n)) = f(x_n)$. Differentiating the equation $f(.) = f \circ X_t(.)$ we obtain

$$(\nabla f(z_n), DX_{s_n}(x_n) \cdot v) = (\nabla f(X_{s_n}(x_n)), DX_{s_n}(x_n) \cdot v) = (\nabla f(x_n), v). \tag{4.2}$$

By the lambda-lemma, we know that $d_{C^1}(X_{s_n}(D^u_n), D^u) \to 0$, for some disc $D^u \subset W^u_{\mathrm{loc}}(\sigma)$. In particular, $\angle(DX_{s_n}(x_n) \cdot T_{x_n} D^u_n, E^u(\sigma)) \to 0$, and $\|DX_{s_n}(x_n) \cdot v\| \to +\infty$. By (4.1), by compactness of K^s, and since $\|v\| = 1$, the right hand side of (4.2) is uniformly bounded, independently of the choices of $p_s, q_u, (z_n)_n$ and n, thus,

$$\lim_{n\to+\infty} \left(\nabla f(z_n), \frac{DX_{s_n}(x_n) \cdot v}{\|DX_{s_n}(x_n) \cdot v\|} \right) = \lim_{n\to+\infty} \frac{(\nabla f(x_n), v)}{\|DX_{s_n}(x_n) \cdot v\|} = 0.$$

We deduce that $\lim_{n\to+\infty} \|\pi^u_n(\nabla f(z_n))\| = 0$, where $\pi^u_n \colon T_{z_n}M \to T_{z_n}(X_{s_n}(D^u_n))$ denotes the orthogonal projection onto $T_{z_n}(X_{s_n}(D^u_n))$. Arguing in the same way for

$X_{-t_n}(D_n^s)$, we also have $\lim_{n\to+\infty} \|\pi_n^s(\nabla f(z_n))\| = 0$, where π_n^s is the orthogonal projection onto $T_{z_n}(X_{-t_n}(D_n^s))$. Since $T_{z_n}M = T_{z_n}(X_{s_n}(D_n^u)) \oplus T_{z_n}(X_{-t_n}(D_n^s))$, then for some sequence $(\delta_n)_{n\geq 0}$ going to 0, and for any $z \in B(\sigma, \varepsilon_n)$, we have

$$\|\nabla f(z)\| \leq \delta_n.$$

We conclude that ∇f can be extended by continuity to σ, by setting $\nabla f(\sigma) := 0$. In particular, the extension of f is C^1 in a neighbourhood of σ. $\square$

4.4. **When the singularity is type sink or source.** We now deal with hyperbolic singularities of type sink or source.

Proposition 4.6. *Let $X, Y \in \mathfrak{X}^1(M)$ such that $[X,Y] = 0$ and $\dim\langle X(x), Y(x)\rangle \leq 1$, for every $x \in M$. Assume that $\sigma \in \mathrm{Zero}(X)$ is a hyperbolic sink. Then, there exists $c \in \mathbb{R}$ such that $Y(x) = cX(x)$, for every $x \in W^s(\sigma)$.*

In the proof of Proposition 4.6 we shall use the following elementary lemma.

Lemma 4.7. *Let $(E, \|\cdot\|)$ be a finite-dimensional vector space endowed with a norm. Let Λ be an infinite set and assume that for each $\lambda \in \Lambda$, there exists a non-empty compact subset $K_\lambda \subset \mathbb{S} := \{v \in E : \|v\| = 1\}$ of the sphere of unit vectors in $(E, \|\cdot\|)$, such that*

$$\lambda' \neq \lambda \text{ in } \Lambda \quad \Longrightarrow \quad K_\lambda \cap K_{\lambda'} = \emptyset.$$

Suppose that $\dim E \geq 2$. Then, there exist a finite subset $\{\lambda, \lambda_1, \ldots, \lambda_k\} \subset \Lambda$ and vectors $\{u, u_1, \ldots, u_k\}$ such that

(1) $u \in K_\lambda$ and $u_\ell \in K_{\lambda_\ell}$, for each $\ell = 1, \ldots, k$;
(2) u belongs to the subspace spanned by $\{u_1, \ldots, u_k\}$;
(3) $\{u_1, \ldots, u_k\}$ is a linearly independent set.

Proof. We begin with a simple observation that we will use repeatedly in this proof: for each $u \in \mathbb{S}$, $-u$ is the only other vector in $\mathbb{S}$ which is collinear with u.

Now, since Λ is infinite, we can pick a sequence $(\lambda_n)_{n\geq 0} \subset \Lambda$, whose terms are distinct. For each n, choose a vector $u_n \in K_{\lambda_n}$. Since $\dim E \geq 2$, and the sets K_λ are pairwise disjoint, by the simple observation above, we can assume without lost of generality that the set $\{u_1, u_2\}$ is linearly independent. Assume by contradiction that the conclusion does not hold. Then, we conclude by induction that for every n the set $\{u_1, u_2, u_3, \ldots, u_n\}$ must be linearly independent. But this is absurd as E is finite dimensional. $\square$

Proof of Proposition 4.6. By Lemma 2.2, for any $x \in M_X = M - \mathrm{Zero}(X)$, we have $Y(x) = f(x)X(x)$, for some C^1 function $f\colon M_X \to \mathbb{R}$. Moreover, $f(X_t(x)) = f(x)$ for every $x \in M_X$ and $t \in \mathbb{R}$. Notice that, as σ is an isolated zero of X, we have $\sigma \in \mathrm{Zero}(Y)$. Take $\varepsilon > 0$ small so that $\overline{B}(\sigma, \varepsilon) \subset W^s(\sigma)$ and let $S := \partial B(\sigma, \varepsilon)$. In particular, notice that $x \in S$ implies $\lim_{t\to+\infty} X_t(x) = \sigma$. Also, for every $x \in W^s(\sigma)$, there exists $T \in \mathbb{R}$ such that $X_T(x) \in S$.

By the above remarks, the proof of the proposition is reduced to the proof of the following claim.

Claim 1. *$Df(p) = 0$ for every $p \in S$.*

We shall postpone the proof of Claim 1. Take a point $p \in S$ and consider the set

$$V(p) := \left\{ u \in T_\sigma M : \exists\, t_n \to \infty,\, u = \lim_{n\to\infty} \frac{X(X_{t_n}(p))}{\|X(X_{t_n}(p))\|} \right\}.$$

By compactness, $V(p)$ is non-empty, and every $u \in V(p)$ is a unit vector; in particular, $0 \notin V(p)$. The following claims are the key arguments for this proof.

Claim 2. *If $u \in V(p)$ then $DY(\sigma) \cdot u = f(p)DX(\sigma) \cdot u$.*

Proof. Fix some $t \in \mathbb{R}$. Since $Y(X_{t+s}(p)) = f(p)X(X_{t+s}(p))$ for every $s \in \mathbb{R}$, taking the derivative with respect to s on both sides we obtain

$$DY(X_t(p)) \cdot \left(\frac{X(X_t(p))}{\|X(X_t(p))\|} \right) = f(p)DX(X_t(p)) \cdot \left(\frac{X(X_t(p))}{\|X(X_t(p))\|} \right).$$

By using this formula with $t = t_n$ and letting $n \to \infty$ we conclude that $DY(\sigma) \cdot u = f(p)DX(\sigma) \cdot u$, proving the claim. □

Claim 3. *If $p, q \in S$ and $V(p) \cap V(q) \neq \emptyset$ then $f(p) = f(q)$.*

Proof. Assume that there exists $u \in V(p) \cap V(q)$. Then, by Claim 2, one has

$$DY(\sigma) \cdot u = f(p)DX(\sigma) \cdot u = f(q)DX(\sigma) \cdot u.$$

As $DX(\sigma)$ is an invertible linear map (because all eigenvalues are negative) this implies that $(f(p) - f(q))u = 0$, and since $u \neq 0$, the claim is proved. □

We are now in position to give the proof of Claim 1. Assume by contradiction that the claim is not true. Then, there exists $U \subset S$ and real numbers $a < b$ such that $f\colon U \to [a, b]$ is surjective.

Now, for every $t \in [a, b]$, we choose some point $p_t \in U \cap f^{-1}(t)$, and we consider the family of compact subsets $\{V(p_t)\}_{t\in[a,b]} \subset T_\sigma M$ of unit vectors. As $t \neq s$ implies $f(p_t) \neq f(p_s)$, one obtains from Claim 3 that the family $\{V(p_t)\}_{t\in[a,b]}$ satisfies all the assumptions of Lemma 4.7.

Thus, there exists a finite set $\{p, p_1, \ldots, p_k\} \subset U$ and vectors $u \in V(p)$, $u_\ell \in V(p_\ell)$, $\ell = 1, \ldots, k$, with $u \in \langle u_1, \ldots, u_k\rangle$ and $\{u_1, \ldots, u_k\}$ linearly independent, and such that $f(p_i) \neq f(p_j) \neq f(p)$, for every $i, j \in \{1, \ldots, k\}$.

Take $\alpha^1, \ldots, \alpha^k \in \mathbb{R}$ such that $u = \sum_{\ell=1}^k \alpha^\ell u_\ell$. Using Claim 2 we can write

$$DY(\sigma) \cdot u = f(p)DX(\sigma) \cdot u = DX(\sigma) \cdot \left(\sum_{\ell=1}^k f(p)\alpha^\ell u_\ell \right).$$

Also

$$DY(\sigma) \cdot u_\ell = f(p_\ell)DX(\sigma) \cdot u_\ell,\ \forall\, \ell = 1, \ldots, k,$$

which implies that

$$DY(\sigma) \cdot u = DX(\sigma) \cdot \left(\sum_{\ell=1}^k f(p_\ell)\alpha^\ell u_\ell \right).$$

Since $DX(\sigma)$ is invertible we must have $\sum_{\ell=1}^k f(p)\alpha^\ell u_\ell = \sum_{\ell=1}^k f(p_\ell)\alpha^\ell u_\ell$, and as $\{u_1, \ldots, u_k\}$ is linearly independent, this gives

$$f(p)\alpha^\ell = f(p_\ell)\alpha^\ell, \text{ for every } \ell = 1, \ldots, k.$$

Since $u \neq 0$ there exists some $\alpha^\ell \neq 0$. However, this implies that $f(p) = f(p_\ell)$, a contradiction. □

We now give the proof of Theorem 4.3.

Proof of Theorem 4.3. Assume that $\mathfrak{C}^1(X)$ is collinear and that each singularity $\sigma \in \mathrm{Zero}(X)$ is hyperbolic. Let us consider $Y \in \mathfrak{C}^1(X)$. By Lemma 2.2, there exists a C^1 function $f\colon M_X \to \mathbb{R}$ which satisfies $X \cdot f \equiv 0$ on M_X and such that $Y(x) = f(x)X(x)$, for every $x \in M_X$. By assumption, the singularities of X are hyperbolic, hence they are isolated, and $Y(\sigma) = 0$, for all $\sigma \in \mathrm{Zero}(X)$. By Propositions 4.4, 4.5 and 4.6, we can extend f to a C^1 invariant function on M. We conclude that f is a first integral of X, and $Y = fX$.

Conversely, assume that $f\colon M \to \mathbb{R}$ is a first integral of X. We define a vector field $Y \in \mathfrak{X}^1(M)$ as $Y(x) := f(x)X(x)$, for every $x \in M$. Indeed, both f and X are of class C^1, thus Y is C^1 too. Moreover, we have $Y \in \mathfrak{C}^1(X)$, since

$$[X, Y] = (X \cdot f)X + f[X, X] = 0. \quad \square$$

5. The study of invariant functions and trivial centralizers

The main focus of this section is the study of invariant functions. An invariant function is also called a first integral of the system. There are several works that study the existence of non trivial (non constant) first integrals, see for instance [ABC16, FP15, FS04, Hur86, Man73, Pag11]. In this work we study dynamical conditions that imply the non-existence of first integrals.

First, it is easy to obtain examples of vector fields with quasi-trivial C^1-centralizer which is not trivial. Indeed consider the vector field in example 3.1. Since X is separating, it has collinear C^1-centralizer. This flow is non-singular, hence it has quasi-trivial C^1-centralizer. Now take any non-constant C^1-function f which is constant on each orbit, that is, a function which depends only on the coordinate r. The vector field $Y = fX$ belongs to the C^1-centralizer of X, therefore the centralizer of X is only quasi-trivial.

Let $X \in \mathfrak{X}^1(M)$. Recall that a compact set Λ is a basic piece for X if Λ is X-invariant and transitive, that is, it has a dense orbit. We say that X admits a countable spectral decomposition if $\Omega(X) = \sqcup_{i\in\mathbb{N}}\Lambda_i$, where the sets Λ_i are pairwise disjoint basic pieces.

Theorem 5.1. *Let $X \in \mathfrak{X}^1(M)$. If X admits a countable spectral decomposition then any continuous X-invariant function is constant.*

Proof. Let $f\colon M \to \mathbb{R}$ be a continuous X-invariant function. Suppose that f is not constant. Since M is connected, there exist two real numbers $a < b$ such that $f(M) = [a, b]$. It is easy to see that in each basic piece the function f is constant: this follows from the transitivity of each basic piece. For each $i \in \mathbb{N}$ define $c_i := f(\Lambda_i)$. Since X admits a countable spectral decomposition, the set $C := \{c_1, c_2, \dots\}$ is at most countable and in particular $[a, b] - C$ is non-empty. Take any value $c \in [a, b] - C$ and consider $\Lambda := f^{-1}(\{c\})$.

The set Λ is compact and X-invariant. Hence, for any point $p \in \Lambda$ we must have $\omega(p) \subset \Lambda$, where $\omega(p)$ is the set of all accumulations points of the future orbit of p. By the countable spectral decomposition, $\omega(p)$ must be contained in some basic piece Λ_i, which implies that $\Lambda \cap \Lambda_i \neq \emptyset$. Since Λ is a level set of f, this implies that $c_i = f(\Lambda_i) = f(\Lambda) = c$ and this is a contradiction with our choice of c. $\square$

Theorem C follows easily from Theorems A and 5.1.

5.1. **First integrals and trivial C^1-centralizers.** Let M be a compact connected manifold. Recall that for any $X \in \mathfrak{X}^1(M)$, we let $\mathfrak{I}^1(X) := \{f \in C^1(M, \mathbb{R}) : X \cdot f \equiv 0\}$ be the set of all C^1 functions which are invariant under X. As an easy consequence of Theorem A, we obtain the following lemma.

Lemma 5.2. *Let $X \in \mathfrak{X}^1(M)$. Assume that the singularities of X are hyperbolic and that the C^1-centralizer of X is collinear. Then X has trivial C^1-centralizer if and only if the set of first integrals of X is trivial, i.e., $\mathfrak{I}^1(X) \simeq \mathbb{R}$.*

As an immediate consequence of Theorem 5.1 and Lemma 5.2, we obtain:

Corollary 5.3. *Let $X \in \mathfrak{X}^1(M)$ be such that X admits a countable spectral decomposition and all its singularities are hyperbolic. If the C^1-centralizer of X is collinear, then it is trivial.*

The following lemma will be used several times in this section.

Lemma 5.4. *Let M be a compact manifold of dimension $d \geq 1$ and let $X \in \mathfrak{X}^1(M)$. Then, for any $f \in \mathfrak{I}^1(X)$ and for any hyperbolic singularity or hyperbolic periodic point $p \in \mathrm{Zero}(X) \cup \mathrm{Per}(X)$, it holds that $\nabla f(p) = 0$.*

Proof. Let $X \in \mathfrak{X}^1(X)$ be as above and let $f \in \mathfrak{I}^1(X)$. If $\sigma \in \mathrm{Zero}(X)$ is a hyperbolic singularity, then it follows from Propositions 4.5 and 4.6 that $\nabla f(\sigma) = 0$. Assume now that for some regular hyperbolic periodic point $p \in \mathrm{Per}(X)$, we have $\nabla f(p) \neq 0$. Then, we have the hyperbolic decomposition along its orbit given by

$$T_{\mathrm{orb}^X(p)} M = E^s \oplus \langle X \rangle \oplus E^u.$$

Note that $f|_{W^s(p)} = f|_{W^u(p)} = f(p)$: this follows easily from the X-invariance of f. Since $\nabla f(p) \neq 0$, by the local form of submersion, we have that $\Sigma := f^{-1}(\{f(p)\})$ is locally contained in a submanifold D of dimension $d-1$. In particular, $T_p D$ is a subspace of dimension $d-1$ contained in $T_p M$. However, our previous observation implies that $W^s_{\mathrm{loc}}(p) \subset \Sigma$ and $W^u_{\mathrm{loc}}(p) \subset \Sigma$. This implies that $E^s(p) \oplus \langle X(p) \rangle \oplus E^u(p) \subset T_p D$. By the hyperbolicity of p, we have that $T_p M = E^s(p) \oplus \langle X(p) \rangle \oplus E^u(p)$, but this is a contradiction with the fact that $T_p D$ has dimension $d-1$. $\square$

For surfaces where the Poincaré-Bendixson Theorem holds true, any level set of an invariant function f has to contain a singularity or a periodic orbit, which forces f to be constant in the generic case where the latter are hyperbolic.

Proposition 5.5. *Let $M := \mathbb{S}^2$ be the two dimensional sphere, and let $X \in \mathfrak{X}^1(M)$ be such that every singularity and periodic orbit of X is hyperbolic. Then any continuous function that is invariant under the flow X is constant.*

Proof. Let $X \in \mathfrak{X}^1(M)$ be as above, and let $f \colon X \to \mathbb{R}$ be a continuous function which satisfies $f(X_t(x)) = f(x)$ for all $x \in M$ and $t \in \mathbb{R}$. Assume that f is non-constant. Then $f(M) = [a, b]$, with $a < b \in \mathbb{R}$. By assumption, each singularity of X is hyperbolic, hence there are finitely many of them. Let $c \in [a, b] - f(\mathrm{Zero}(X))$. For any $x \in f^{-1}(\{c\})$, it follows from Poincaré-Bendixson Theorem that $\omega(x)$ is a closed orbit formed by regular points, and by our assumption, $\omega(x)$ is hyperbolic. Moreover, $\omega(x) \subset f^{-1}(\{c\})$, since f is invariant under X. In particular, for each $c \in [a, b] - f(\mathrm{Zero}(X))$, the level set $f^{-1}(\{c\})$ contains a hyperbolic periodic orbit. This is a contradiction, since $[a, b] - f(\mathrm{Zero}(X))$ is uncountable, while there can be at most countably many hyperbolic periodic orbits. $\square$

5.2. **Some results in higher regularity.** As we mentioned in Section 2, using Sard's theorem and Pesin's theory we can obtain more information about the invariant functions.

Theorem 5.6. *Let M be a compact, connected Riemannian manifold of dimension $d \geq 1$ and let $X \in \mathfrak{X}^1(M)$. Suppose that X verifies the following conditions:*

- *every singularity and periodic orbit of X is hyperbolic;*
- $\Omega(X) = \overline{\mathrm{Per}(X)}$.

Then any function $f\colon M_X \to \mathbb{R}$ which is X-invariant and such that $f|_{M_X}$ is of class C^d is constant.

Proof. Let $f\colon M_X \to \mathbb{R}$ be an X-invariant function such that $f|_{M_X}$ is of class C^d. By assumption, each singularity $\sigma \in \mathrm{Zero}(X)$ is hyperbolic, thus by Propositions 4.4 and 4.6, f admits a continuous extension to the whole manifold M. Suppose that f is not constant. Then, there exist two real numbers $a < b$ such that $f(M) = [a, b]$. All the singularities are hyperbolic, hence there are at most finitely many of them. In particular, $I \subset f(M) - f(\mathrm{Zero}(X))$ for some non-trivial open interval $I \subset \mathbb{R}$. Since $f|_{M_X}$ is of class C^d, then by Sard's theorem, there exists a set $R \subset I$ of full Lebesgue measure, such that each $c \in R$ is a regular value of f, that is, any $x \in f^{-1}(\{c\})$ verifies $\nabla f(x) \neq 0$.

Fix a value $c \in R - f(\mathrm{Zero}(X))$. By the same reason as in the proof of Theorem 5.1, we have that $f^{-1}(\{c\}) \cap \Omega(X) \neq 0$. The fact that c is a regular value implies that there exists $y \in \Omega(X) \cap M_X$ such that $\nabla f(y) \neq 0$, thus by the continuity of X and ∇f, there exists a neighbourhood $\mathcal{V} \subset M_X$ of y such that the gradient of f is non-zero at any $q \in \mathcal{V}$. Using the density of periodic points in the non-wandering set, we conclude that there exists a regular periodic point $p \in \mathrm{Per}(X) \cap \mathcal{V}$ such that $\nabla f(p) \neq 0$. By Lemma 5.4, we get a contradiction, since by assumption, the point p is hyperbolic. $\square$

As a consequence of Theorem 5.6, we can prove Theorem F.

Proof of Theorem F. Let $X \in \mathfrak{X}^d(M)$ be as above and let $Y \in \mathfrak{C}^d(X)$. By the collinearity of $\mathfrak{C}^d(X)$, and since all the singularities of X are hyperbolic, Lemma 2.2 and Theorem 4.3 imply that $Y = fX$, where f is a X-invariant C^1 function such that $f|_{M_X}$ is of class C^d. We deduce from Theorem 5.6 that f is constant. Therefore, $\mathfrak{C}^d(X)$ is trivial. $\square$

Using the ideas from [Man73], we are able to prove Theorem G.

Proof of Theorem G. By Kupka-Smale Theorem, there exists an open and dense subset $\mathcal{U}_{KS} \subset \mathfrak{X}^d(M)$ such that for any $X \in \mathcal{U}_{KS}$, any singularity of X is hyperbolic. Let $S(M)$ be the pseudometric space of subsets of M with the Hausdorff pseudometric. By [Tak71], there exists a residual subset $\mathcal{R}_d \subset \mathfrak{X}^d(M)$ such that the function $\Omega\colon \mathcal{R}_d \to S(M)$ which assigns to $X \in \mathcal{R}_d$ its non-wandering set is continuous. Let us define the residual set $\mathcal{R}_T := \mathcal{U}_{KS} \cap \mathcal{R}_d \subset \mathfrak{X}^d(M)$, and let $X \in \mathcal{R}_T$. Notice that X has finitely many singularities, since they are hyperbolic.

Suppose that X has collinear C^d-centralizer and let $Y \in \mathfrak{C}^d(X)$. By the collinearity, as a consequence of Lemma 2.2 and Theorem 4.3, we have $Y = fX$, for some X-invariant C^1 function f such that $f|_{M_X}$ is of class C^d. Assume that f is non-constant. Then, as in the proof of Theorem 5.6, $f(M) - f(\mathrm{Zero}(X))$ contains a non-trivial open interval $I \subset \mathbb{R}$. Consider a regular value $c \in I$ (this set is non-empty by Sard's theorem) and let $M_c = f^{-1}(\{c\})$. We now describe Mañé's argument from Theorem 1.2 in [Man73]. Let U be a small open neighbourhood of M_c. Since $\Omega(X) \cap U \neq \emptyset$, by the continuity of $\Omega(\cdot)$ at X, for any X' in a neighbourhood of X verifies $\Omega(X') \cap U \neq \emptyset$. Consider the gradient $\nabla f|_{M_c}$, since it is nonzero on M_c we can extend it to a vector field $V : U \to TU$ without singularities. We can take a C^1-vector field Z C^1-arbitrarily close to the zero vector field, with the following property: for any $x \in U$, $(Z(x), V(x)) > 0$. For the vector field $X' = X + Z$, it is easy to verify that $\Omega(X') \cap U = \emptyset$, a contradiction. We conclude that f is constant, and thus, $\mathfrak{C}^d(X)$ is trivial. $\square$

Using Pesin's theory and ideas similar to the proof of Lemma 5.4, we can prove Theorem D.

Proof of Theorem D. Since the support of μ is the entire manifold, and by non-uniform hyperbolicity, we have that X verifies the conditions of Proposition 3.2, in particular, $\mathfrak{C}^1(X)$ is collinear. Let Y be a vector field in the C^1-centralizer of X. there exists a C^1-function $f : M_X \to \mathbb{R}$ such that $Y = fX$ on M_X.

Notice that M_X is a connected open and dense subset of M. If f were not constant, then it would exist a point $p \in M_X$ such that $\nabla f(p) \neq 0$. Since this condition is open we may take the point p to be a regular point of the measure μ. By Pesin's stable manifold theorem, there exists a C^1-stable manifold, $W^s_{\text{loc}}(p)$, which is tangent to $E^-(p) \oplus \langle X(p) \rangle$ on p. Similarly, there exists a C^1-unstable manifold which on p is tangent to $\langle X(p) \rangle \oplus E^+(p)$. The non-uniform hyperbolicity implies that $E^-(p) \oplus \langle X(p) \rangle \oplus E^+(p) = T_pM$.

Since p is a non-singular point, we have that $f|_{W^s_{\text{loc}}(p)} = f|_{W^u_{\text{loc}}(p)} = f(p)$. An argument similar to the one in the proof of Theorem 5.6 gives a contradiction and we conclude that $f|_{M_X}$ is constant. This implies that the centralizer of X is trivial. $\square$

5.2.1. *The C^3 centralizer of a C^3 Kinematic expansive vector field.* In this part we prove Theorem E. The proof is a combination of two results: Sard's Theorem and the proposition below.

Proposition 5.7. *Let $\mathbb{T}^2$ denote the two dimensional torus. If $X \in \mathfrak{X}^2(\mathbb{T}^2)$ and if $\mathrm{Zero}(X) = \emptyset$ then X is not Kinematic expansive.*

Proof. The argument follows closely some ideas in [Art16]. We present it here for the sake of completeness.

Assume by contradiction that there exists $X \in \mathfrak{X}^2(\mathbb{T}^2)$ a Kinematic expansive vector field. In particular it is separating. We fix $\varepsilon > 0$ to be the separation constant. Since X is C^2 we can apply Denjoy-Schwartz's Theorem [Sch63] and we have three possibilities for the dynamics:

(1) each orbit is periodic and X is a suspension of the identity map $\mathrm{id} : \mathbb{S}^1 \to \mathbb{S}^1$;
(2) there exist two distinct periodic orbits γ^s, γ^u and a non-periodic point x such that $\omega(x) = \gamma^s$ and $\alpha(x) = \gamma^u$;
(3) X is a suspension of a C^3 diffeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$, which is topologically conjugate to an irrational rotation.

We shall prove that each case leads us to a contradiction. In the first case, let $\tau : \mathbb{S}^1 \to (0, +\infty)$ be the first return time function. Then, $\tau(x)$ is the period of the orbit of x. As τ is a continuous function on the circle, there exists a maximum point x_0 and arbitrarily close to x_0 there are points x_1, x_2 such that $\tau(x_1) = \tau(x_2)$. This implies that one can choose those points so that

$$d(X_t(x_1), X_t(x_2)) \leq \varepsilon, \ \forall\, t \in \mathbb{R},$$

a contradiction.

Let us deal now with case (2). Fix an arbitrarily small number $\delta > 0$.

Take a small segment I transverse to X at a point $p \in \gamma^s$ and let $f\colon I \to I$ be the first return map, with $\tau\colon I \to (0, +\infty)$ the first return time function. There exists a time $T^s > 0$ such that $X_{T^s}(x) \in I$. Consider the fundamental domain $I_0^s = [f(x), x]$ for the dynamics of f and the sequence of image intervals $I_n^s = [f^{n+1}(x), f^n(x)]$, $n \geq 0$. Then, there exists $N^s > 0$ such that for $n \geq N^s$, it holds that $I_n^s \subset B(p, \delta)$. Pick $a, b \in I_0^s$ arbitrarily close.

Let $C > 0$ be the Lipschitz constant of τ. Then,

$$\left| \sum_{\ell=0}^{n} \tau(f^\ell(a)) - \sum_{\ell=0}^{n} \tau(f^\ell(b)) \right| \leq C \sum_{\ell=0}^{n} |f^\ell(a) - f^\ell(b)|.$$

The hight-hand side of above inequality is bounded by $\sum_n |I_n^s| = |I| < \infty$. Therefore, the left-hand side converges. Moreover, by continuity of f, if $d(a, b)$ is small enough then $\sum_{\ell=0}^{N^s} |f^\ell(a) - f^\ell(b)| < \delta$. Since $I_n^s \subset B(p, \delta)$ for every $n \geq N^s$, we have $\sum_{\ell=N^s}^{\infty} |f^\ell(a) - f^\ell(b)| < \delta$. We conclude that

$$\left| \sum_{\ell=0}^{\infty} \tau(f^\ell(a)) - \sum_{\ell=0}^{\infty} \tau(f^\ell(b)) \right| \leq 2C\delta.$$

Taking δ small enough, as the flow of X is the suspension of f with return time τ, we conclude that $d(X_t(a), X_t(b)) < \varepsilon$, for every $t \geq 0$.

Considering a small transverse segment to a point $q \in \gamma^u$ and arguing similarly with backwards iteration we obtain two arbitrarily close points a, b whose orbits are distinct and such that $d(X_t(a), X_t(b)) < \varepsilon$ for every $t \in \mathbb{R}$, a contradiction.

Finally, let us see that case (3) leads to a contradiction. This is essentially contained in the proof of Theorem 4.11 from [Art16] with a minor adaptation. We will sketch the main points of the proof. Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a C^3 diffeomorphism with irrational rotation number θ, and let $\tau\colon \mathbb{S}^1 \to (0, +\infty)$ be a C^1 function. It is well known that the Lebesgue measure is the only ergodic measure for an irrational rotation. Since f is C^3 by the usual Denjoy's theorem on the circle, f is conjugated with an irrational rotation, in particular, f has only one ergodic f-invariant probability measure μ.

Write $T := \int_{S^1} \tau(x) d\mu(x)$ and let $\left(\frac{p_n}{q_n}\right)_{n \in \mathbb{N}}$ be the approximation of θ by rational numbers given by the continued fractions algorithm. From the corollary in [NT13], which is a version of Denjoy-Koksma inequality (Corollary C in [AK11]), we obtain the following

$$\lim_{n \to +\infty} \sup_{x \in S^1} \left| \sum_{l=0}^{q_n - 1} \tau(f^l(x)) - T q_n \right| = 0.$$

Following the same calculations in the proof of Theorem 4.11 from [Art16], for any $\epsilon > 0$ and for $n \in \mathbb{N}$ large enough, the points x and $f^{q_n}(x)$ are always ϵ-close for the future. One can argue similarly for f^{-1} and find points that are not separated for the past. Therefore, the flow cannot be Kinematic expansive. $\square$

Remark 5.8. We do not know if there exists a separating suspension of an irrational rotation. The above proof shows that this is the only possibility for a separating non-singular vector field on $\mathbb{T}^2$.

Proof of Theorem E. Since all the singularities are hyperbolic, by Proposition 3.2 and Theorem 4.3, we have that $\mathfrak{C}^3(X)$ is quasi-trivial. Let $f\colon M \to \mathbb{R}$ be a C^1, X-invariant function such that $f|_{M_X}$ is C^3. We will prove that f is constant. Suppose not.

Since there are only finitely many singularities, then as in the proof of Theorem 5.6, if f were not constant, we would have $I \subset f(M) - f(\mathrm{Zero}(X))$, for some non-trivial open interval $I \subset \mathbb{R}$. By Sard's theorem, almost every value in I is a regular value.

Take a regular value $c \in I$. Hence, $S_c := f^{-1}(\{c\})$ is a compact surface that does not contain any singularity of X. Furthermore, since f is X-invariant, we have that $X|_{S_c}$ is a C^3 non-singular vector field on S_c. Up to considering a double orientation covering, this implies that S_c is a torus, since it is the only orientable closed surface that admits a non-singular vector field.

Notice that $X|_{S_c}$ induces a Kinematic expansive flow. However this contradicts Proposition 5.7. We conclude that f is constant, and this implies that the C^3-centralizer of X is trivial. □

In the higher dimensional case, and at a point of continuity of $\Omega(\cdot)$, we also have:

Proposition 5.9. *Let M be a compact manifold of dimension $d \geq 1$. Assume that $X \in \mathfrak{X}^d(M)$ is separating, that all its singularities are hyperbolic, and that X is a point of continuity of the map $\Omega(\cdot)$. Then the C^d-centralizer of X is trivial.*

Remark 5.10. As noted in the proof of Theorem G, the last two assumptions are satisfied by a residual subset of vector fields in $\mathfrak{X}^d(M)$.

Proof of Proposition 5.9. Since X is separating and its singularitis are hyperbolic, it follows from Proposition 2.4 and Theorem A that its C^1-centralizer is quasi-trivial. Take any vector field Y in the C^d-centralizer of X. By the quasi-triviality, and by Lemma 2.2, there exists a C^1 function $f\colon M \to \mathbb{R}$ such that $f|_{M_X}$ is of class C^d and $Y = fX$. If f is not constant, then as in the proof of Theorem G, by continuity of $\Omega(\cdot)$ at X, and by considering a regular value $c \in f(M) - f(\mathrm{Zero}(X))$ of $f|_{M_X}$, we reach a contradiction. We conclude that the C^d-centralizer is trivial. □

6. The generic case

Our goal in this section is to prove the following theorem:

Theorem 6.1. *There exists a residual subset $\mathcal{R} \subset \mathfrak{X}^1(M)$ such that if $X \in \mathcal{R}$ then X has quasi-trivial C^1-centralizer. Furthermore, if X has at most countably many chain recurrent classes then its C^1-centralizer is trivial.*

In particular, this theorem implies Theorem B. To prove this theorem, we will use a few generic results. In the following statement we summarize all the results we will need.

Theorem 6.2 ([BC04], [Cro06] and [PR83])**.** *There exists a residual subset $\mathcal{R}_*$ such that if $X \in \mathcal{R}_*$, then the following properties are verified:*

(1) $\overline{\mathrm{Per}(X)} = \Omega(X) = \mathcal{CR}(X)$*;*

(2) every periodic orbit, or singularity, is hyperbolic;

(3) if $\mathcal{C}$ is a chain recurrent class, then there exists a sequence of periodic orbits $(\gamma_n)_{n\in\mathbb{N}}$ such that $\gamma_n \to C$ in the Hausdorff topology.

We first prove that C^1-generically the centralizer is collinear. This proof is an adaptation for flows of Theorem A in [BCW09]. Once we have collinearity, using the criterion for quasi-triviality given by Theorem 4.3, we conclude that quasi-triviality of the C^1-centralizer is a C^1-generic property. At the end of this section we will show that for a C^1-generic vector field X that has at most countably many chain recurrent classes has trivial C^1-centralizer.

Idea of the proof of collinearity–. In [BCW09] the authors prove that a version of the unbounded normal distortion holds C^1-generically for diffeomorphisms. To prove this, the key perturbative result is a perturbation made on a linear cocycle over $\mathbb{Z}$. To reduce the proof to this linear cocycle scenario, after several reductions, they introduce some change of coordinates that linearizes the dynamics around the orbit of a point for a finite time. Using the compactness of the manifold, they get uniform estimates on the C^1-norm of these changes of coordinates.

Our strategy is to reduce our problem to a perturbation of a linear cocycle over $\mathbb{Z}$. In order to do that, we study the Poincaré maps between a sequence of transverse sections. Since we are dealing with wandering points, this can be defined for a sequence of times arbitrarily large. Using these Poincaré maps we also introduce some change of coordinates to linearize the dynamics given by these maps for a finite time. However, the space where this can be defined is no longer compact, since the Poincaré map is only defined over non-singular points. Nevertheless, we can obtain uniform estimates for the C^1-norm of these change of coordinates.

We also need to prove that any perturbation of a Poincaré map, that verifies some conditions, can be realized as the Poincaré map of a perturbed vector field. All of these perturbations have to be done with precise control on the estimates that appear. These two ingredients are given in Lemma 6.9. Once we have that, we can adapt the proof of Bonatti-Crovisier-Wilkinson in [BCW09] and obtain that the unbounded normal distortion property is C^1-generic.

6.1. Unbounded normal distortion is C^1-generic.

The goal of this section is to prove the following theorem:

Theorem 6.3. *There exists a residual subset of $\mathcal{R} \subset \mathfrak{X}^1(M)$ such that if $X \in \mathcal{R}$ then X has unbounded normal distortion.*

6.1.1. *Linearizing coordinates.* Let $X \in \mathfrak{X}^1(M)$, and as before, set $M_X := M - \mathrm{Zero}(X)$. For $p \in M_X$ and $t \in \mathbb{R}$, for any two submanifolds Σ_1 and Σ_2 which are transverse to the orbit segment $O := X_{[0,t]}(p)$, each of which intersects O only at one point, we define the *Poincaré map* between these two transverse sections as follows: let $p_1 := O \cap \Sigma_1$ and $p_2 := O \cap \Sigma_2$. If a point $q \in \Sigma_1$ is sufficiently close to p_1, then $X_{[-t,2t]}(q)$ intersects Σ_2 at a unique point $\mathcal{P}^X_{\Sigma_1,\Sigma_2}(q)$. The map $q \mapsto \mathcal{P}^X_{\Sigma_1,\Sigma_2}(q)$ is called the Poincaré map between Σ_1 and Σ_2.

This map is a C^1-diffeomorphism between a neighbourhood of p_1 and its image. It also holds that for any vector field $Y \in \mathfrak{X}^1(M)$ sufficiently C^1-close to X, the Poincaré map $\mathcal{P}^Y_{\Sigma_1,\Sigma_2}$ for Y is well defined in some neighbourhood of p_1 in Σ_1.

Let $R > 0$ be smaller than the radius of injectivity of M. Using the exponential map, for each $p \in M_X$ and $r \in (0, R)$, we define the submanifold $\mathcal{N}_{X,p}(r) =$

$\exp_p(N_{X,p}(r))$, where $N_{X,p}(r)$ is the ball of center 0 and radius r contained in $N_{X,p}$.

Remark 6.4. Considering R to be small enough, for each $p \in M_X$ and for each $q \in \mathcal{N}_{X,p}(R)$ we have that the C^1-norm of $\Pi_q^X|_{T_q\mathcal{N}_{X,p}(R)}$ is close to 1.

It is known that for each $t \in \mathbb{R}$, there exists a constant $\beta_t = \beta(X,t) > 0$ such that for any point $p \in M_X$, the Poincaré map is a C^1 diffeomorphism from $\mathcal{N}_{X,p}(\beta_t\|X(p)\|)$ to its image inside $\mathcal{N}_{X,X_t(p)}(R)$. We denote this map by $\mathcal{P}_{p,t}^X$ and we write $\beta := \beta_1$. For a fixed $\delta > 0$, we can choose β sufficiently small such that for any $p \in M_X$ and any $q \in \mathcal{N}_{X,p}(\beta\|X(p)\|)$, it holds

$$\|D\mathcal{P}_{p,1}^X(q) - D\mathcal{P}_{p,1}^X(p)\| < \delta, \tag{6.1}$$

we refer the reader to Section 2.2 in [GY18] for more details. By our choices of transversals, we remark that $D\mathcal{P}_{p,1}^X(p) = P_{p,1}^X$.

Definition 6.5. For any $C > 1$ we say that a vector field $X \in \mathfrak{X}^1(M)$ is *bounded by* C if it holds

- $\sup\limits_{x\in M} |X(p)| < C$;
- $\sup_{p\in M} \|DX(p)\| < C$;
- $C^{-1} < \inf\limits_{x\in M} \inf\limits_{t\in[-1,1]} \|(DX_t(x))^{-1}\|^{-1} \leq \sup\limits_{x\in M} \sup\limits_{t\in[-1,1]} \|DX_t(x)\| < C$;
- $C^{-1} < \inf\limits_{p\in M_X} \inf\limits_{t\in[-1,1]} \|(P_{p,t}^X)^{-1}\|^{-1} \leq \sup\limits_{p\in M_X} \sup\limits_{t\in[-1,1]} \|P_{p,t}^X\| < C$;
- there exists $\beta > 0$ small, such that

$$C^{-1} < \|(D\mathcal{P}_{p,1}^X(q))^{-1}\|^{-1} \leq \|D\mathcal{P}_{p,1}^X(q)\| < C, \text{ for any } q \in \mathcal{N}_{X,p}(\beta\|X(p)\|).$$

By (6.1), for any vector field $X \in \mathfrak{X}^1(M)$, there is a constant $C > 1$ such that X is bounded by C.

Let $X \in \mathfrak{X}^1(M)$ be a vector field bounded by $C > 1$. Using the exponential map, for $p \in M_X$, we consider the *lifted Poincaré map*

$$\widetilde{\mathcal{P}}_{p,1}^X = \exp_{X_1(p)}^{-1} \circ \mathcal{P}_{p,1}^X \circ \exp_p,$$

which goes from $N_{X,p}(\beta\|X(p)\|)$ to $N_{X,X_1(p)}(R)$. Observe that

$$\|X(X_1(p))\| > C^{-1}\|X(p)\|. \tag{6.2}$$

By (6.2) and the last item in Definition 6.5, for any $n \in \mathbb{N}$, the map $\mathcal{P}_{p,n}^X$ is well defined on $\mathcal{N}_{X,p}\big(\frac{\beta}{C^n}\|X(p)\|\big)$, while the lifted map $\widetilde{\mathcal{P}}_{p,n}^X$ is well defined on $V_{p,n}^X := N_p\big(\frac{\beta}{C^n}\|X(p)\|\big)$.

For each $n \in \mathbb{N}$ and $p \in M_X$, we define the change of coordinates $\psi_{p,n} = P_{X_{-n}(p),n}^X \circ (\widetilde{\mathcal{P}}_{X_{-n}(p),n}^X)^{-1}$, which is a C^1 diffeomorphism from $\widetilde{\mathcal{P}}_{p,n}^X(V_{X_{-n}(p),n}^X)$ to $P_{X_{-n}(p),n}^X(V_{X_{-n}(p),n}^X) \subset N_{X,p}$. Observe that $\psi_{p,0} = \mathrm{id}$. The sequence $(\psi_{X_j(p),j})_{j\in\mathbb{N}}$ verifies the following equality:

$$\psi_{X_n(p),n} \circ \widetilde{\mathcal{P}}_{p,n}^X = P_{p,n}^X \circ \psi_{p,0},$$

which holds on $V_{p,n}^X$. In other words, this change of coordinates linearizes the dynamics of $\widetilde{\mathcal{P}}_{p,n}^X$.

For all $y \in \mathcal{N}_{X,p}(\frac{\beta}{C^n}\|X(p)\|)$, we define the *hitting time* $\tau^X_{p,n}(y)$ as the first positive time where the trajectory starting at y hits the transverse section $\mathcal{N}_{X,X_n(p)}(R)$, that is,

$$\tau^X_{p,n}(y) := \min\{t \geq 0 : X_t(y) \in \mathcal{N}_{X,X_n(p)}(R)\}.$$

Notation. Let $p \in M_X$ and $n \in \mathbb{N}$. Suppose that for $Y \in \mathfrak{X}^1(M)$ the submanifolds $\mathcal{N}_{X,p}\big(\frac{\beta}{C^n}\|X(p)\|\big)$ and $\mathcal{N}_{X,X_n(p)}(R)$ are transverse to Y, and that the Poincaré map for Y between these transverse sections is well defined on $\mathcal{N}_{X,p}\big(\frac{\beta}{C^n}\|X(p)\|\big)$. Then we denote this Poincaré map for Y by $\mathcal{P}^Y_{X,p,n}$. Accordingly, we denote its lift by $\widetilde{\mathcal{P}}^Y_{X,p,n}$ and its hitting time by $\tau^Y_{X,p,n}$. We also extend those notations for non-integer times: given an integer $n \geq 1$ and $t \in [n-1, n]$, we let $\mathcal{P}^Y_{X,p,t}$ be the Poincaré map between the transversals $\mathcal{N}_{X,p}\big(\frac{\beta}{C^n}\|X(p)\|\big)$ and $\mathcal{N}_{X,X_t(p)}(R)$.

In the next definition we introduce the type of perturbations of the Poincaré map that we will consider in the sequel.

Definition 6.6. For each $\delta > 0$ and given an open set $U \subset \mathcal{N}_{X,p}(\beta\|X(p)\|)$, a C^1 map $g \colon \mathcal{N}_{X,p}(\beta\|X(p)\|) \to \mathcal{N}_{X,X_1(p)}(R)$ is called a *δ-perturbation of $\mathcal{P}^X_{p,1}$ with support in U* if the following holds:

- $d_{C^1}(\mathcal{P}^X_{p,1}, g) < \delta$;
- the image of g coincides with the image of $\mathcal{P}^X_{p,1}$;
- the map g is a C^1 diffeomorphism into its image;
- the support of $(\mathcal{P}^X_{p,1})^{-1} \circ g$ is contained in U.

For any $n \in \mathbb{N}$ and any $U \subset \mathcal{N}_{X,p}\big(\frac{\beta}{C^n}\|X(p)\|\big)$, we define

$$\mathcal{I}^X(p, U, n) := \{(y,t) : y \in U,\ t \in [0, \tau^X_{p,n}(y)]\}, \tag{6.3}$$

and we let $\mathcal{U}^X(p,U,n)$ be the image of $\mathcal{I}^X(p,U,n)$ under the map $(y,t) \mapsto X_t(y)$:

$$\mathcal{U}^X(p,U,n) := \bigcup_{y \in U} \bigcup_{t \in [0, \tau^X_{p,n}(y)]} X_t(y). \tag{6.4}$$

Remark 6.7. For a vector field $X \in \mathfrak{X}^1(M)$ we can fix a constant $\alpha = \alpha(X)$ small enough such that for any $t \in [-\alpha, \alpha]$ and $p \in M_X$, it holds that $|\det P^X_{p,t} - 1| < \frac{\log 2}{2}$.

Remark 6.8. Let $\alpha > 0$ be as in Remark 6.7. Then for $\beta > 0$ sufficiently small, for any $p \in M_X$ and $q \in \mathcal{N}_{X,p}\big(\frac{\beta}{C^n}\|X(p)\|\big)$, it holds that $\tau^X_{p,n}(q) \in [n-\alpha, n+\alpha]$. From now on we will always assume that β verifies this condition for this choice of α.

6.1.2. *A realization lemma.* We state and prove below a lemma that allows us to realise a non-linear perturbation of the linear Poincaré flow as the lifted Poincaré map of a vector field nearby.

Lemma 6.9. *For any $C, \varepsilon > 0$, there exists $\delta = \delta(C, \varepsilon) > 0$ that verifies the following. For any vector field $X \in \mathfrak{X}^1(M)$ that is bounded by C, any $0 < \delta_1 < \delta$ and any integer $n \in \mathbb{N}$, there is $\rho = \rho(X, \varepsilon, \delta_1) > 0$ with the following property.*

For any $p \in M_X$ and $U \subset \mathcal{N}_{X,p}(\rho\|X(p)\|)$ such that the map $(y,t) \mapsto X_t(y)$ is injective restricted to the set $\mathcal{I}^X(p,U,n)$, then the following holds:

(1) Set $\widetilde{U} := \exp_p^{-1}(U)$. Then for every $i \in \{0, \dots, n\}$, the map $\Psi_{X_i(p),i} := \psi_{X_i(p),i} \circ \exp^{-1}_{X_i(p)}$ induces a C^1 diffeomorphism from $\mathcal{P}^X_{p,i}(U)$ onto $P^X_{p,i}(\widetilde{U})$ such that

$$\max\{\|D\Psi_{X_i(p),i}\|, \|D\Psi^{-1}_{X_i(p),i}\|, |\det D\Psi_{X_i(p),i}|, |\det D\Psi^{-1}_{X_i(p),i}|\} < 2. \tag{6.5}$$

(2) *For $i \in \{1, \dots, n\}$, let $\tilde{g}_i \colon N_{X,X_{i-1}(p)} \to N_{X,X_i(p)}$ be any C^1 diffeomorphism such that the support of $(P^X_{X_i(p),1})^{-1} \circ \tilde{g}_i$ is contained in $P^X_{p,i-1}(\widetilde{U})$, and which satisfies $d_{C^1}(\tilde{g}_i, P^X_{X_{i-1}(p),1}) < \delta_1$. Let g_i be the map defined as follows:*
   - $g_i(y) := \mathcal{P}^X_{X_{i-1}(p),1}(y)$*, if* $y \notin \mathcal{P}^X_{p,i-1}(U)$*;*
   - $g_i(y) := \Psi^{-1}_{X_i(p),i} \circ \tilde{g}_i \circ \Psi_{X_{i-1}(p),i-1}(y)$*, if* $y \in \mathcal{P}^X_{p,i-1}(U)$*.*

   Then the map g_i is a δ-perturbation of $\mathcal{P}^X_{X_{i-1}(p),1}$ with support in $\mathcal{P}^X_{p,i-1}(U)$.

(3) *There exists $Y \in \mathfrak{X}^1(M)$ such that $d_{C^1}(X,Y) < \varepsilon$, and the Poincaré map $\mathcal{P}^Y_{X,X_i(p),1}$ for the vector field Y between $\mathcal{N}_{X_{i-1}(p)}(\rho\|X(p)\|)$ and $\mathcal{N}_{X_i(p)}(R)$ is well defined and is given by g_i, for each $i \in \{1, \dots, n\}$. Moreover, the support of $X - Y$ is contained in $\mathcal{U}^X(p, U, n)$ and the image of $\tau^Y_{X,p,n}$ coincides with the image of $\tau^X_{p,n}$. In particular, it is contained in $[n-\alpha, n+\alpha]$.*

Before proving this lemma, let us say a few words on items 2 and 3 in the statement. Item 2 states that we can obtain perturbations of the Poincaré map by perturbing its lift, with precise estimates on the size of each of these perturbations we consider. Observe that this only gives C^1 diffeomorphisms between certain transverse sections. Item 3 states that any such perturbation can be realized as the Poincaré map of a vector field C^1-close to X, with precise estimates on its distance to X. Furthermore, the hitting time is the "same" as the hitting time of X, in the sense that they have the same image as a function of the transverse section to $\mathbb{R}$. These two properties will be crucial in our proof, because it will allow us to reduce the proof of the theorem to the discrete case, after several adaptations.

Proof. We will obtain δ later, as consequence of a finite number of inequalities. In the following, we always assume that $0 < \rho \leq \frac{\beta}{C^n}$. By the previous discussion, this ensures that $\mathcal{P}^X_{p,n}$ is well defined on $\mathcal{N}_{X,p}\big(\rho\|X(p)\|\big)$, for all $p \in M_X$.

Point (1) follows from the following facts:

- It holds
$$C^{-n} < \inf_{p \in M_X,\ t \in [-n,n]} \|(P^X_{p,t})^{-1}\|^{-1} \leq \sup_{p \in M_X,\ t \in [-n,n]} \|P^X_{p,t}\| < C^n, \tag{6.6}$$
and by (6.1), we have similar estimates for the Poincaré maps $\mathcal{P}^X_{p,t}$, uniformly in $p \in M_X$ and $t \in [-n,n]$.
- By choosing $\rho > 0$ sufficiently small, the set
$$\bigcup_{y \in N_p(\rho\|X(p)\|)} \bigcup_{t \in [0,n]} P^X_{p,t}(y)$$
can be made arbitrarily close to the 0 section, uniformly in $p \in M_X$. Similarly, the set $\mathcal{U}^X(p, \mathcal{N}_{X,p}(\rho\|X(p)\|), n)$ defined in (6.4) can be made arbitrarily close to the orbit segment $\{X_t(p) : 0 \leq t \leq n\}$, uniformly in $p \in M_X$.
- The map $D\exp_p^{-1}$ is uniformly close to the identity in a neighbourhood of p.
- Since the vector field X is of class C^1 and by choosing $\rho > 0$ sufficiently small, the maps $\psi_{X_i(p),i}$ used to linearize the dynamics can be made uniformly C^1-close to the identity for $i \in \{0, \dots, n\}$ and $p \in M_X$. Therefore, the map $\Psi_{X_i(p),i}$ can be made arbitrarily C^1 close to $\exp^{-1}_{X_i(p)}$.

In particular, we obtain a uniform control of $\Psi_{X_i(p),i}$ for $p \in M_X$ and $i \in \{0, \ldots, n\}$ even though the space M_X is not compact.

By Definition 6.6, the proof of (2) follows easily from the first point. Indeed, given $i \in \{1, \ldots, n\}$ and $p \in M_X$, we use the maps $\Psi_{X_{i-1}(p),i-1}$ and $\Psi_{X_i(p),i}$ to conjugate $\mathcal{P}^X_{X_{i-1}(p),1}$ to the linear Poincaré map $P^X_{X_{i-1}(p),1}$. By the previous discussion, for $\rho > 0$ small enough, the maps $\Psi_{X_{i-1}(p),i-1}$ and $\Psi_{X_i(p),i}$ are arbitrarily C^1-close to $\exp^{-1}_{X_{i-1}(p)}$ and $\exp^{-1}_{X_i(p)}$ respectively. The estimate on the C^1 distance between g_i and $\mathcal{P}^X_{X_{i-1}(p),1}$ follows, since we assume $d_{C^1}(\tilde{g}_i, P^X_{X_{i-1}(p),1}) < \delta_1$, and $\delta_1 < \delta$.

The proof of point (3) follows from arguments similar to those presented in Pugh-Robinson [PR83] (see in particular Lemma 6.5 in that paper).

More precisely, let $i \in \{1, \ldots, n\}$, and let $\tilde{g}_i \colon N_{X_{i-1}(p)} \to N_{X_i(p)}$ be a C^1 diffeomorphism satisfying the assumptions of point (2). We pull back $\tilde{g}_i$ to a C^1 diffeomorphism $\hat{g}_i \colon N_{X_{i-1}(p)} \to N_{X_{i-1}(p)}$ by letting $\hat{g}_i := P^X_{X_i(p),-1} \circ \tilde{g}_i$. By assumption, the support of $\hat{g}_i$ is contained in $P^X_{p,i-1}(\widetilde{U})$, with $\widetilde{U} := \exp_p^{-1}(U)$ and $U \subset \mathcal{N}_{X,p}(\rho \|X(p)\|)$, hence by (6.6), we get

$$d_{C^0}(\hat{g}_i, \mathrm{id}) \leq 2C\rho \max_{p \in M} \|X(p)\|. \tag{6.7}$$

Then for all $t \in [i-1, i]$, we define a map $\tilde{g}_t \colon N_{X_{i-1}(p)} \to N_{X_t(p)}$ as $\tilde{g}_t := P^X_{X_{i-1}(p),t-i+1} \circ \hat{g}_i$. By the above estimate, and by (6.6), we deduce that

$$d_{C^0}(\tilde{g}_t, P^X_{X_{i-1}(p),t-i+1}) \leq 2C^2\rho \max_{p \in M} \|X(p)\|, \qquad \forall t \in [i-1, i]. \tag{6.8}$$

Moreover, for any $t \in [i-1, i]$, we have $D\tilde{g}_t = P^X_{X_{i-1}(p),t-i+1} \cdot D\hat{g}_i = P^X_{X_{i-1}(p),t-i+1} \circ P^X_{X_i(p),-1} \cdot D\tilde{g}_i$. Since $d_{C^1}(\tilde{g}_i, P^X_{X_{i-1}(p),1}) < \delta_1$, we obtain

$$d_{C^1}(\tilde{g}_t, P^X_{X_{i-1}(p),t-i+1}) \leq C^2\delta_1, \qquad \forall t \in [i-1, i]. \tag{6.9}$$

Let us fix a C^∞ bump function $\chi \colon \mathbb{R} \to [0,1]$ which is 0 near 0 and 1 near 1. Fix $i \in \{1, \ldots, n\}$ and set $\chi_{i-1}(\cdot) := \chi(\cdot - i + 1)$. For $k \in \{0, \ldots, n\}$, we also let $\mathcal{N}_{p,k} := \mathcal{N}_{X,X_k(p)}\big(\frac{\beta}{C^{n-k}}\|X(p)\|\big)$. Then for any $t \in [i-1, i]$, we let $h^{(i)}_t \colon \mathcal{N}_{p,i-1} \to \mathcal{N}_{p,i}$ be the map defined as

- $h^{(i)}_t(y) := \mathcal{P}^X_{X_{i-1}(p),t-i+1}(y)$, if $y \notin \mathcal{P}^{i-1}_p(U)$;
- $h^{(i)}_t(y) := \Psi^{-1}_{X_t(p),t} \circ \Big(\chi_{i-1}(t)\tilde{g}_t + (1-\chi_{i-1}(t))P^X_{X_{i-1}(p),t-i+1}\Big) \circ \Psi_{X_{i-1}(p),i-1}(y)$, if $y \in \mathcal{P}^X_{p,i-1}(U)$,

where we have extended the previous notation by setting

$$\Psi_{X_t(p),t} := P^X_{p,t} \circ \widetilde{\mathcal{P}}^X_{X_t(p),-t} \circ \exp^{-1}_{X_t(p)}.$$

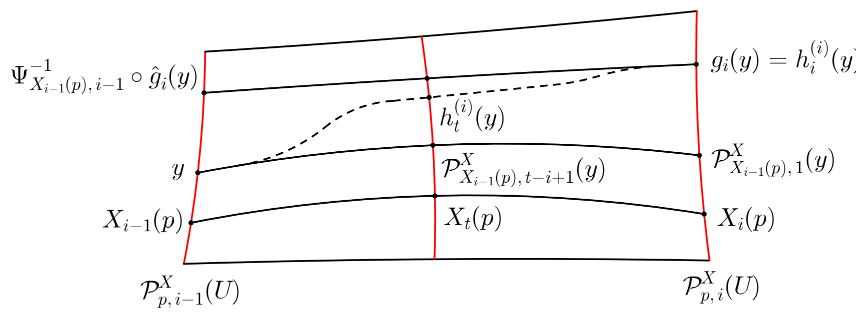

FIGURE 3. Interpolation between the initial Poincaré map and g_i.

In particular, we note that for $t = i-1$, we have $h^{(i)}_t = h^{(i)}_{i-1} = \mathrm{id}$, while for $t = i$, $h^{(i)}_t = h^{(i)}_i$ coincides with the map g_i defined in item (2).

By (6.8), for all $t \in [i-1, i]$, we have

$$d_{C^0}\left(\chi_{i-1}(t)\tilde{g}_t + (1-\chi_{i-1}(t))P^X_{X_{i-1}(p),t-i+1}, P^X_{X_{i-1}(p),t-i+1}\right) \le 2C^2\rho \max_{p\in M}\|X(p)\|.$$

Since $\mathcal{P}^X_{X_{i-1}(p),t-i+1} = \Psi^{-1}_{X_t(p),t} \circ P^X_{X_{i-1}(p),t-i+1} \circ \Psi_{X_{i-1}(p),i-1}$, by the definition of $h^{(i)}_t$ and by (6.5), we can thus make the C^0 distance between $h^{(i)}_t$ and $\mathcal{P}^X_{X_{i-1}(p),t-i+1}$ arbitrarily small, provided that $\rho > 0$ is taken small enough.

For any $t \in [i-1, i]$, we have

$$\begin{aligned} &D\left(\chi_{i-1}(t)\tilde{g}_t + (1-\chi_{i-1}(t))P^X_{X_{i-1}(p),t-i+1}\right) \\ &= DP^X_{X_{i-1}(p),t-i+1} + \chi_{i-1}(t)\big(D\tilde{g}_t - P^X_{X_{i-1}(p),t-i+1}\big). \end{aligned}$$

By (6.6) and (6.9), we thus get

$$d_{C^1}(h^{(i)}_t, \mathcal{P}^X_{X_{i-1}(p),t-i+1}) \le 4C^2\delta_1. \tag{6.10}$$

For any $t \in [i-1, i]$, we also have:

$$\begin{aligned} &\partial_t\left(\chi_{i-1}(t)\tilde{g}_t + (1-\chi_{i-1}(t))P^X_{X_{i-1}(p),t-i+1}\right) - \partial_t P^X_{X_{i-1}(p),t-i+1} \\ &= \chi'_{i-1}(t)\big(\tilde{g}_t - P^X_{X_{i-1}(p),t-i+1}\big) + \chi_{i-1}(t)\partial_t\big(\tilde{g}_t - P^X_{X_{i-1}(p),t-i+1}\big) \\ &= \chi'_{i-1}(t)P^X_{X_{i-1}(p),t-i+1} \circ \big(\hat{g}_i - \mathrm{id}\big) + \chi_{i-1}(t)\partial_t P^X_{X_{i-1}(p),t-i+1} \circ \big(\hat{g}_i - \mathrm{id}\big). \end{aligned}$$

By (6.5), (6.6) and (6.7), we deduce that

$$\max_{t\in[i-1,i]}\max_{y\in U}|\partial_t\mathcal{P}^X_{X_{i-1}(p),t-i+1}(y)-\partial_t h^{(i)}_t(y)|$$
$$\leq 8C\max\left(C,\sup_{t\in[0,1]}\|\partial_t P^X_{X_{i-1}(p),t}\|\right)\|\chi\|_{C^1}\rho\max_{p\in M}\|X(p)\|. \tag{6.11}$$

Recall that for $k\in\{0,\dots,n\}$, we denote $\mathcal{N}_{p,k}:=\mathcal{N}_{X,X_k(p)}\big(\frac{\beta}{C^{n-k}}\|X(p)\|\big)$. As in (6.3), given a set $V\subset\mathcal{N}_{p,0}$, we set

$$\mathcal{I}^X(p,V,n):=\left\{(y,t):y\in V,\ t\in[0,\tau^X_{p,n}(y)]\right\}.$$

Let us assume that $U\subset\mathcal{N}_{X,p}(\rho\|X(p)\|)$ is such that the map $(y,t)\mapsto X_t(y)$ is injective on the set $\mathcal{I}^X(p,U,n)$. For $\rho>0$ small, the hitting time function $\tau^X_{p,n}$ is uniformly close to n on $\mathcal{N}_{X,p}(\rho\|X(p)\|)$, and the C^1 distance between the maps $(y,t)\mapsto\mathcal{P}^X_{p,t}(y)$ and $(y,t)\mapsto X_t(y)$ restricted to $\mathcal{I}^X(p,\mathcal{N}_{X,p}(\rho\|X(p)\|),n)$ is small. Given $i\in\{1,\dots,n\}$, let us consider the map $h^{(i)}\colon(y,t)\mapsto h^{(i)}_t(y)$ defined on $\mathcal{N}_{p,i-1}\times[i-1,i]$ as above. By (6.10) and (6.11), and since $0<\delta_1<\delta$, the maps $\mathcal{N}_{p,i-1}\times[i-1,i]\ni(y,t)\mapsto\mathcal{P}^X_{X_{i-1}(p),t-i+1}(y)$ and $h^{(i)}$ can be made arbitrarily C^1-close by taking $\delta>0$ small enough. For $\delta>0$ sufficiently small, we deduce that the map $h^{(i)}$ is locally injective on the interior of $\mathcal{P}^X_{p,i-1}(U)\times[i-1,i]$. Besides, as we have seen, $h^{(i)}_{i-1}|_{\mathcal{N}_{p,i-1}}=\mathrm{id}|_{\mathcal{N}_{p,i-1}}$, while $h^{(i)}_i|_{\mathcal{N}_{p,i-1}}=g_i|_{\mathcal{N}_{p,i-1}}$ is a C^1 diffeomorphism.

Now, we define a map H on $\mathcal{N}_{p,0}\times[0,n]$ by setting

$$H(y,t):=h^{(i)}_t\circ g_{i-1}\circ g_{i-2}\circ\cdots\circ g_1(y), \tag{6.12}$$
$$\forall y\in\mathcal{N}_{p,0},\ t\in[i-1,i],\ i\in\{1,\dots,n\}.$$

By what precedes, the map H is locally injective on the interior of the set $U\times[0,n]$. Moreover, $\partial(U\times[0,n])=(U\times\{0\})\cup(U\times\{n\})\cup(\partial U\times[0,n])$. On the one hand, we have $H(\cdot,0)|_U=\mathrm{id}|_U$, and by construction, the map $H(\cdot,n)|_U$ coincides with $g_n\circ g_{n-1}\circ\cdots\circ g_1|_U$, hence it is a C^1 diffeomorphism from U to $\mathcal{P}^X_{p,n}(U)\subset\mathcal{N}_{p,n}$. On the other hand, by point (2), each diffeomorphism g_i is a δ_2-perturbation of $\mathcal{P}^X_{X_{i-1}(p),1}$ with support in $\mathcal{P}^X_{p,i-1}(U)$. Therefore the restriction of H to the set $\partial U\times[0,n]$ coincides with the restriction of the map $(y,t)\mapsto\mathcal{P}^X_{p,t}(y)$. In particular, we deduce that the restriction $H|_{\partial(U\times[0,n])}$ of H to the boundary of $U\times[0,n]$ is injective. From Lemma 6.5 in Pugh-Robinson [PR83], we conclude that H embeds $U\times[0,n]$ into the set $\mathcal{U}^X(p,U,n)$ introduced in (6.4).

In the same way as before, for any $y\in\mathcal{N}_{X,p}(\rho\|X(p)\|)$ and $t\in[0,n]$, we set

$$\tau^X_{p,t}(y):=\min\{s\geq 0:X_s(y)\in\mathcal{N}_{X,X_t(p)}(R)\}.$$

By definition, $\mathcal{P}^X_{p,t}(y)=X_{\tau^X_{p,t}(y)}(y)$, for any $(y,t)\in\mathcal{N}_{X,p}(\rho\|X(p)\|)\times[0,n]$, thus

$$X(\mathcal{P}^X_{p,t}(y))=(\partial_t\tau^X_{p,t}(y))^{-1}\partial_t\mathcal{P}^X_{p,t}(y). \tag{6.13}$$

Moreover, $\tau^X_{p,\cdot}(p)=\mathrm{id}$, and the map $(y,t)\mapsto\tau^X_{p,t}(y)$ is C^1 on $\mathcal{N}_{X,p}(\rho\|X(p)\|)\times[0,n]$, hence for $\rho>0$ sufficiently small, we have

$$\frac{1}{2}<|\partial_t\tau^X_{p,t}(y)|<2,\qquad\forall p\in M_X,\ y\in\mathcal{N}_{X,p}(\rho\|X(p)\|),\ t\in[0,n]. \tag{6.14}$$

As we have noted above, on the complement of $U\times[0,n]$, the maps H and $(y,t)\mapsto\mathcal{P}^X_{p,t}(y)$ coincide. We thus define a vector field $Y\in\mathfrak{X}^1(M)$ on M by setting

- $Y(q) := X(q)$, if $q \in M - \mathcal{U}^X(p, U, n)$;
- $Y(q) := (\partial_t|_{t=t_0} \tau^X_{p,t}(y))^{-1} \partial_t|_{t=t_0} H(y_0, t)$, if $q \in \mathcal{U}^X(p, U, n)$, where $(y_0, t_0) := H^{-1}(q) \in U \times [0, n]$.

For each $i \in \{1, \dots, n\}$, by the definition of H in (6.12) and since $h_i^{(i)} = g_i$, it follows that the Poincaré map $\mathcal{P}^Y_{X, X_{i-1}(p),1}$ for the vector field Y between $\mathcal{N}_{p,i-1}$ and $\mathcal{N}_{p,i}$ is given by g_i. By definition, the support of $X - Y$ is contained in $\mathcal{U}^X(p, U, n)$. Moreover, given any point $q = \mathcal{P}^X_{p,t}(y) = H(y', t) \in \mathcal{U}^X(p, U, n)$, say $(y, t) \in U \times [i-1, i]$, letting $z := \mathcal{P}^X_{p,i-1}(y)$ and $z' := g_{i-1} \circ g_{i-2} \circ \dots \circ g_1(y')$, we obtain

$$
\begin{aligned}
\mathcal{P}^X_{p,t}(y) &= \mathcal{P}^X_{X_{i-1}(p), t-i+1}(z) \\
&= \Psi^{-1}_{X_t(p),t} \circ P^X_{X_{i-1}(p), t-i+1} \circ \Psi_{X_{i-1}(p), i-1}(z); \\
H(y', t) &= h_t^{(i)}(z') \\
&= \Psi^{-1}_{X_t(p),t} \circ \left(\chi_{i-1}(t) \tilde{g}_t + (1 - \chi_{i-1}(t)) P^X_{X_{i-1}(p), t-i+1} \right) \circ \Psi_{X_{i-1}(p), i-1}(z') \\
&= \Psi^{-1}_{X_t(p),t} \circ P^X_{X_{i-1}(p), t-i+1} \circ (\chi_{i-1}(t)(\hat{g}_i - \mathrm{id}) + \mathrm{id}) \circ \Psi_{X_{i-1}(p), i-1}(z').
\end{aligned}
$$

Set

$$
w := \Psi_{X_{i-1}(p), i-1}(z) = (\chi_{i-1}(t)(\hat{g}_i - \mathrm{id}) + \mathrm{id}) \circ \Psi_{X_{i-1}(p), i-1}(z').
$$

We deduce that

$$
\begin{aligned}
\partial_t \mathcal{P}^X_{p,t}(y) &= \partial_t \left(\Psi^{-1}_{X_t(p),t} \circ P^X_{X_{i-1}(p), t-i+1} \right)(w), \\
\partial_t H(y', t) &= \partial_t \left(\Psi^{-1}_{X_t(p),t} \circ P^X_{X_{i-1}(p), t-i+1} \right)(w) + D_w \left(\Psi^{-1}_{X_t(p),t} \circ P^X_{X_{i-1}(p), t-i+1} \right) \\
&\qquad \cdot \partial_t \left((\chi_{i-1}(t)(\hat{g}_i - \mathrm{id}) + \mathrm{id}) \circ \Psi_{X_{i-1}(p), i-1}(z') \right) \\
&\quad = \partial_t \mathcal{P}^X_{p,t}(y) + \chi'_{i-1}(t) D_{\Psi_{X_t(p),t}(q)} \Psi^{-1}_{X_t(p),t} \circ P^X_{X_{i-1}(p), t-i+1} \circ \\
&\qquad \circ (\hat{g}_i - \mathrm{id}) \left(\Psi_{X_{i-1}(p), i-1}(z') \right),
\end{aligned}
$$

and

$$
\begin{aligned}
&Y(q) - X(q) = \\
&\frac{\chi'_{i-1}(t)}{\partial_t \tau^X_{p,t}(y)} D_{\Psi_{X_t(p),t}(q)} \Psi^{-1}_{X_t(p),t} \circ P^X_{X_{i-1}(p), t-i+1} \circ (\hat{g}_i - \mathrm{id}) \left(\Psi_{X_{i-1}(p), i-1}(z') \right),
\end{aligned}
$$

where the last equality follows from (6.13) and the definition of Y. In particular, the difference between the vector fields X and Y is essentially controlled by the C^0 distance between $\hat{g}_i$ and id. More precisely, by (6.5), (6.6), (6.7), and (6.14), we deduce that

$$
|X(q) - Y(q)| \leq 8 \|\chi\|_{C^1} C^2 \rho \max_{p \in M} \|X(p)\|,
$$

and we argue similarly for the derivatives. Therefore, by taking ρ sufficiently small, we can ensure that $d_{C^1}(X, Y) < \varepsilon$, which concludes the proof of point (3), and then, of Lemma 6.9. □

6.1.3. *Producing unbounded normal distortion by perturbation.* We are now in position to prove the main perturbation result (Proposition 6.12 below) that will allow us to obtain unbounded normal distortion generically. The key tool behind this is a perturbation result for linear cocycles taken from [BCW09].

Proposition 6.10. *For any $d \geq 2$, $C > 1$, $K, \varepsilon > 0$, let $\delta = \delta(C, \varepsilon)$ be the constant given by Lemma 6.9. There exists $n_0 = n_0(d, C, K, \varepsilon) \in \mathbb{N}$ with the following property.*

For any d-dimensional manifold M, any vector field $X \in \mathfrak{X}^1(M)$ which is bounded by C, there exists $\rho_0 = \rho_0(d, C, K, \varepsilon) > 0$ such that for any $\eta > 0$, any compact set $\Delta \subset M_X$ and $x, p \in M_X$ satisfying:

(a) there exists an open set U inside $\mathcal{N}_{X,p}(\rho_0 \|X(p)\|)$, such that $\Delta \subset U$;

(b) the map $(y, t) \mapsto X_t(y)$ is injective on $\mathcal{I}^X(p, U, n_0)$ (see (6.3));

(c) $\operatorname{orb}^X(x) \cap U = \emptyset$,

there exists a vector field $Y \in \mathfrak{X}^1(M)$ such that

(1) the support of $X - Y$ is contained in $\mathcal{U}^X(p, U, n_0)$ (see (6.4));

(2) $d_{C^1}(X, Y) < \varepsilon$;

(3) for any $i \in \{0, \dots, n_0 - 1\}$, it is verified $d_{C^1}(\mathcal{P}^X_{X_i(p),1}, \mathcal{P}^Y_{X,X_i(p),1}) < \delta$, where $\mathcal{P}^Y_{X,X_i(p),1}$ is the Poincaré map between $\mathcal{N}_{X,X_i(p)}(\beta\|X(X_i(p))\|)$ and $\mathcal{N}_{X,X_{i+1}(p)}(R)$;

(4) $d_{C^0}(\mathcal{P}^X_{X_i(p),t}, \mathcal{P}^Y_{X,X_i(p),t}) < \eta$, for all $t \in [0, 1]$;

(5) for all $y \in \Delta$, there exists an integer $n \in \{1, \dots, n_0\}$ such that

$$|\log \det P^Y_{x,n} - \log \det P^Y_{y,n}| > K.$$

Proposition 6.10 is the analogous for flows of Proposition 8 in [BCW09]. Using Lemma 6.9, we will reduce the proof of this proposition to a discrete scenario where we can apply the following proposition from [BCW09].

Proposition 6.11 (Proposition 9 in [BCW09])**.** *For any $d \geq 1$, and any $C, K, \varepsilon > 0$, there exists $n_1 = n_1(d, C, K, \varepsilon) \geq 1$ with the following property.*

Consider any sequence $(A_i) \in \mathrm{GL}(d, \mathbb{R})$ with $\|A_i\|, \|A_i^{-1}\| < C$ and the associated cocycle $\tilde{f}$ on $\mathbb{Z} \times \mathbb{R}^d$ defined by $\tilde{f}(i, v) := (i+1, A_i v)$. Then, for any open set $U \subset \mathbb{R}^d$, for any compact set $\Delta \subset U$ and any $\eta > 0$, there exists a diffeomorphism $\tilde{g}$ of $\mathbb{Z} \times \mathbb{R}^d$ such that:

- *$d_{C^1}(\tilde{f}, \tilde{g}) < \varepsilon$;*
- *$d_{C^0}(\tilde{f}, \tilde{g}) < \eta$;*
- *$\tilde{f} = \tilde{g}$ on the complement of $\bigcup_{i=0}^{2n_1 - 1} \tilde{f}^i(\{0\} \times U)$;*
- *for all $y \in \{0\} \times \Delta$, there exists $n \in \{1, \dots, n_1\}$ such that*

$$|\log \det D\tilde{f}^n(y) - \log \det D\tilde{g}^n(y)| > K.$$

Proof of Proposition 6.10 from Proposition 6.11. Fix any $\delta_1 \in (0, \delta)$ and $K_0 > 2K + 10 \log 2$. Let $n_1 = n_1(d - 1, C, K_0, \delta_1)$ be the constant given by Proposition 6.11 for $d - 1$, C, K_0, ε and let $n_0 = 2n_1$. Let $X \in \mathfrak{X}^1(M)$ be a vector field bounded by C and let $\rho > 0$ be the constant given by Lemma 6.9 for C, ε, δ_1, n_0 and X. Fix $\rho_0 \in (0, \frac{\rho}{C^{n_0}})$.

Let $\Delta \subset M_X$, $x, p \in M_X$ and $\eta > 0$ be such that conditions (a), (b) and (c) in Proposition (6.10) are verified. Let $U \subset \mathcal{N}_{X,p}(\rho_0 \|X(x)\|)$ be the open set given by condition (a). Consider $O_{X_1}(p) = \{\dots, X_{-1}(p), p, X_1(p), \dots\}$ and observe that this set is naturally identified with $\mathbb{Z}$. We consider the normal bundle, with respect to X, over $O_{X_1}(p)$ and the linear cocycle defined as follows: for $i \in \mathbb{Z}$ and $v \in N_{X,X_i(p)}$ set $\tilde{f}(i, v) := (i + 1, P^X_{X_i(p),1} v)$.

Recall that $\widetilde{U} = \exp_p^{-1}(U)$. By item (1) in Lemma 6.9, for any $i \in \{0, \dots, n_0\}$, we obtain C^1 diffeomorphisms $\Psi_i := \Psi_{X_i(p),i} \colon \mathcal{P}^X_{p,i}(U) \to P^X_{p,i}(\widetilde{U})$, such that for any $q \in \mathcal{P}^X_{p,i}(U)$ it holds that

$$P^X_{X_i(p),1}(\Psi_i(q)) = \Psi_{i+1}(\mathcal{P}^X_{X_i(p),1}(q)). \tag{6.15}$$

Write $\Psi \colon \bigcup_{i=0}^{n_0} \mathcal{P}^X_{p,i}(U) \to \bigcup_{i=0}^{n_0} P^X_{p,i}(\widetilde{U})$ as the C^1 diffeomorphism which is equal to Ψ_i on $\mathcal{P}^X_{p,i}$.

For the cocycle $\tilde{f}$, we apply Proposition 6.11 and obtain a δ_1-perturbation $\tilde{g}$ of $\tilde{f}$ supported on $\bigcup_{i=0}^{n_0-1} \tilde{f}^i(\{0\} \times \widetilde{U})$, such that for every $q \in \Psi_0(\Delta)$, it holds:

- $d_{C^0}(\tilde{f}, \tilde{g}) < \frac{\eta}{2}$;
- $\tilde{f} = \tilde{g}$ on the complement of $\bigcup_{i=0}^{n_0-1} \tilde{f}^i(\{0\} \times \widetilde{U})$;
- for every $q \in \Psi_0(\Delta)$, there exists $n \in \{1, \dots, n_0\}$ such that

$$|\log\det D\tilde{f}^n(q) - \log\det D\tilde{g}^n(q)| > K_0.$$

For each $i \in \{0, \dots, n_0 - 1\}$, let $\tilde{g}_i := g|_{\{i\} \times N_{X, X_i(p)}}$ and observe that $d_{C^1}(\tilde{g}_i, P^X_{X_i(p),1}) < \delta_1$. By item (2) of Lemma 6.9, we obtain a δ-pertubation g_i of $\mathcal{P}^X_{X_i(p),1}$. By (6.5) and (6.15), we have

$$d_{C^0}(g_i, \mathcal{P}^X_{X_i(p),1}) < 2 d_{C^0}(\tilde{g}_i, P^X_{X_i(p),1}) < \eta.$$

Moreover, by the estimates in (6.5), we conclude that for any $q \in \Delta$, there exists $n \in \{1, \dots, n_0 - 1\}$ such that

$$|\log\det D\mathcal{P}^X_{p,n}(q) - \log\det D(g^n)(q)| > K_0 - 4\log 2, \tag{6.16}$$

where $g^n(q) := g_n \circ \cdots \circ g_1(q)$.

Recall that for $n \in \{0, \dots, n_0 - 1\}$, the maps $\mathcal{P}^X_{p,n}$ and $P^X_{p,n}$ are conjugated on Δ by Ψ. By (6.5), we obtain that for any $q \in \Delta$, it holds

$$|\log\det D\mathcal{P}^X_{p,n}(q) - \log\det P^X_{p,n}| \le 2\log 2. \tag{6.17}$$

Suppose there exists $n \in \{0, \dots, n_0 - 1\}$ such that $|\log\det P^n_p - \log\det P^n_x| > K + 3\log 2$. By (6.17) and Remark 6.4, for any $q \in \Delta$ it holds that

$$\left|\log\det P^X_{q,\tau^X_{p,n}(q)} - \log\det P^X_{x,n}\right| > K + \log 2.$$

By Remark 6.7 and item 3 of Lemma 6.9, we conclude that

$$|\log\det P^X_{q,n} - \log\det P^X_{x,n}| > K.$$

In this case we do not make any perturbation. Suppose that for every $n \in \{0, \dots, n_0 - 1\}$ and every $q \in \Delta$ we have

$$|\log\det P^X_{q,n} - \log\det P^X_{x,n}| \le K + 3\log 2.$$

Consider the maps $g_1, \dots, g_{n_0}$ as it was explained above (obtained using Proposition 6.11). Applying Lemma 6.9, we obtain a C^1 vector field Y that verifies the following properties:

- $d_{C^1}(X, Y) < \varepsilon$;
- the support of $X - Y$ is contained in $\mathcal{U}^X(p, U, n_0)$;
- for each $i \in \{1, \dots, n_0\}$, we have that $\mathcal{P}^Y_{X, X_i(p),1} = g_i$.

By (6.16) and (6.17), we conclude that for each $q \in \Delta$, there exists $n \in \{1, \dots, n_0\}$ such that

$$\begin{aligned}\left|\log\left(\frac{\det P^Y_{x,n}}{\det P^Y_{q,n}}\right)\right| &\geq \left|\log\left(\frac{\det D\mathcal{P}^X_{p,n}(q)}{\det P^Y_{q,n}}\right)\right| - \left|\log\left(\frac{\det P^X_{x,n}}{\det D\mathcal{P}^X_{p,n}(q)}\right)\right| \\ &\geq \left|\log\left(\frac{\det D\mathcal{P}^X_{p,n}(q)}{\det Dg^n(q)}\right)\right| - \left|\log\left(\frac{\det Dg^n(q)}{\det P^Y_{q,n}}\right)\right| - \\ &- \left|\log\left(\frac{\det P^X_{x,n}}{\det P^X_{q,n}}\right)\right| - \left|\log\left(\frac{\det P^X_{q,n}}{\det D\mathcal{P}^X_{p,n}(q)}\right)\right| \\ &> K_0 - \log 2 - K - 4\log 2 - \log 2 > K.\end{aligned}$$

This concludes the proof of Proposition 6.10. □

The following proposition is the version for flows of Proposition 7 in [BCW09].

Proposition 6.12. *Consider a vector field $X \in \mathfrak{X}^1(M)$, a compact set $D \subset M_X$, an open set $O \subset M_X$ and a point $x \in M_X$ satisfying:*

- *for any $y \in O$, any $t \geq 0$, we have $X_t(y) \in O$ and $X_1(\overline{O}) \subset O$;*
- *$D \subset O - X_1(\overline{O})$;*
- *$\mathrm{orb}^{\mathrm{X}}(x) \cap D = \emptyset$.*

Then for any $K, \varepsilon > 0$, there exists a vector field $Y \in \mathfrak{X}^1(M)$ with $d_{C^1}(X, Y) < \varepsilon$ which satisfies the following property: for all $y \in D$, there exists $n \geq 1$ such that

$$|\log\det P^Y_{x,n} - \log\det P^Y_{y,n}| > K.$$

Moreover, the support of $X - Y$ is contained in the complement of the chain recurrent set of X.

Proof. Let X, D, O, x be as in the statement of Proposition 6.12. Let $C > 1$ be chosen such that the vector field X is bounded by C, and let $n_0 = n_0(d, C, 3K, \varepsilon)$, $\rho_0 = \rho_0(d, C, K, \varepsilon)$ be chosen according to Proposition 6.10. We set $N := 2^d n_0$. Without loss of generality, we also assume that K satisfies $K > 2d\log(2C) > 0$.

We fix a finite cover $\mathcal{F} = \{D_1, \dots, D_\ell\}$ of D by compact sets satisfying:

(1) $D \subset \bigcup_{j=1}^{\ell} \mathrm{int}(D_j) \subset O - X_1(\overline{O})$;
(2) for each $j \in \{1, \dots, \ell\}$, there exists a real number $\tau_j \in (0, 1)$, a point $p_j \in O - X_1(\overline{O})$, an open set $U_j \subset \mathcal{N}_{X,p_j}(\rho_0|X(p_j)|)$, and a compact set $\Delta_j \subset U_j$, such that the following properties hold:
   (a) we have
$$D_j = \{X_t(y) : y \in \Delta_j,\ t \in [0, \tau_j]\}, \tag{6.18}$$
   and
$$\mathrm{int}(D_j) \subset \{X_t(y) : y \in U_j,\ t \in (0, \tau_j)\} \subset O - X_1(\overline{O});$$
   (b) for each $t \in [0, N]$, we have $\mathcal{P}^X_{p_j,t}(U_j) \subset \mathcal{N}_{X, X_t(p_j)}(\rho_0|X_t(p_j)|)$;
   (c) for each $t \in [0, N-1]$, for each $t' \in [0, 1]$, and for each $y_1, y_2 \in \mathcal{P}^X_{p_j,t}(\Delta_j)$, it holds
$$d(\mathcal{P}^X_{X_t(p_j),t'}(y_1), \mathcal{P}^X_{X_t(p_j),t'}(y_2)) \leq 2Cd(y_1, y_2); \tag{6.19}$$

(3) $\mathrm{orb}^{\mathrm{X}}(x) \cap \bigcup_{j=1}^{\ell} U_j = \emptyset$;
(4) for each $j \in \{1, \ldots, \ell\}$, the map $(y,t) \mapsto X_t(y)$ is injective restricted to the set $U_j \times [0,1]$, and thus, it is also injective on the whole set $\mathcal{I}^X(p_j, U_j, N)$;[1]
(5) there exists a partition $\{1, \ldots, \ell\} = J_0 \sqcup \cdots \sqcup J_{2^d-1}$ such that for each $k \in \{0, \ldots, 2^d-1\}$, and for each $j_1 \neq j_2 \in J_k$, we have

$$\mathcal{U}^X(p_{j_1}, U_{j_1}, 1) \ \cap \ \mathcal{U}^X(p_{j_2}, U_{j_2}, 1) = \emptyset.$$

One can obtain $\mathcal{F}$ by tiling the compact set D by arbitrarily small cubes as in (6.18), i.e., obtained by flowing small transversals Δ_j under X, for $j = 1, \ldots, \ell$. Besides, since we assume that $D \subset M_X$, properties (1)-(4) are satisfied provided that D_j, U_j and Δ_j are chosen sufficiently small, for all $j \in \{1, \ldots, \ell\}$. In particular, (6.19) is true provided that D_j and Δ_j are chosen small enough, for all $j \in \{1, \ldots, \ell\}$, since X is bounded by C. Moreover, item (5) holds true provided that the diameter of the sets $U_1, \ldots, U_\ell$ is small enough, since M has dimension d.

For each $j \in \{1, \ldots, \ell\}$, and for each $i, m \geq 0$, we set

$$\mathcal{V}_j^X(i,m) := \mathrm{int}(\mathcal{U}^X(X_i(p_j), \mathcal{P}^X_{p_j,i}(U_j), m)).$$

Each set $\mathcal{V}_j^X(i,m)$ is open: it is the interior of the "tube" obtained by flowing points in the transversal $\mathcal{P}^X_{p_j,i}(U_j)$ under X until they hit the section $\mathcal{P}^X_{p_j,i+m}(U_j)$. We have the following properties:

- for each $j \in \{1, \ldots, \ell\}$, the sets $\mathcal{V}_j^X(0,1), \mathcal{V}_j^X(1,1), \ldots, \mathcal{V}_j^X(N-1,1)$ are pairwise disjoint;
- for each $j \in \{1, \ldots, \ell\}$, the orbit $\mathrm{orb}^{\mathrm{X}}(x)$ is disjoint from $\mathcal{U}^X(p_j, U_j, N)$;
- for each $(k_1, j_1) \neq (k_2, j_2)$ with $k_1, k_2 \in \{0, \ldots, 2^d-1\}$ and $j_1 \in J_{k_1}$, $j_2 \in J_{k_2}$, we have

$$\mathcal{V}^X_{j_1}(n_0 k_1, n_0) \cap \mathcal{V}^X_{j_2}(n_0 k_2, n_0) = \emptyset. \tag{6.20}$$

Indeed, the first item is a consequence of point (4) above, the second one follows from point (3) above, and the third one is a consequence of points (4) and (5) above.

[1] Indeed, for $t > 1$, we have $X_t(U_j) = X_1(X_{t-1}(U_j))$, and $X_{t-1}(U_j) \subset O - X_1(\overline{O})$.

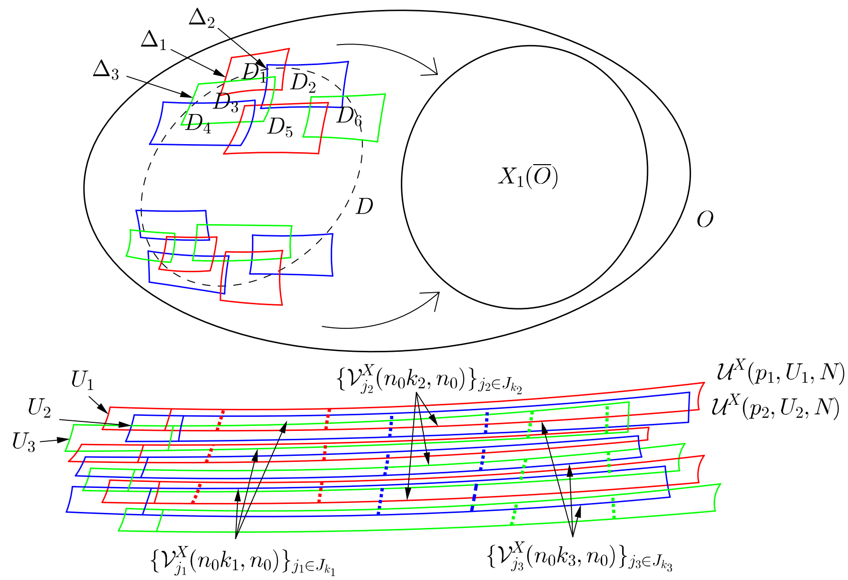

FIGURE 4. Selection of the perturbation times for the different tiles.

Claim 4. *There exists $\lambda > 0$ such that for each $y \in \bigcup_{j=1}^{\ell} D_j$, there exist $j \in \{1, \dots, \ell\}$, $z \in \Delta_j$ and $u \in [0, 1]$ such that $y = X_u(z)$, and $\mathcal{N}_{X,z}(2\lambda) \subset \Delta_j$.*

Proof. Let $\lambda_1 > 0$ be a Lebesgue number of the cover $\mathcal{F}$. We choose $\lambda_2 > 0$ such that $\mathcal{N}_{X,y}(\lambda_2) \subset B(y, \lambda_1)$, for any $y \in \bigcup_{j=1}^{\ell} D_j$, and we take $\lambda > 0$ such that $\mathcal{P}^X_{z,u}(\mathcal{N}_{X,z}(2\lambda)) \subset \mathcal{N}_{X,X_u(z)}(\lambda_2)$ for any $z \in \bigcup_{j=1}^{\ell} \Delta_j$ and $u \in [0, 1]$. The existence of $\lambda > 0$ follows from the compactness of $\bigcup_{j=1}^{\ell} \Delta_j$ and from the fact that X is bounded $C > 0$. By the definition of λ_1 and $D_1, \dots, D_\ell$, for each $y \in \bigcup_{j=1}^{\ell} D_j$, there exist $j \in \{1, \dots, \ell\}$, $z \in \Delta_j$, and $u \in [0, 1]$ such that $y = X_u(z)$, and $B(y, \lambda_1) \subset D_j$. By the definition of λ_2, we also have $\mathcal{N}_{X,y}(\lambda_2) \subset B(y, \lambda_1)$. Then, by the definition of λ and D_j, and since $y = X_u(z) \in \mathcal{N}_{X,y}(\lambda_2) \subset D_j$, we deduce that $\mathcal{N}_{X,z}(2\lambda) \subset (\mathcal{P}^X_{z,u})^{-1}(\mathcal{N}_{X,y}(\lambda_2)) \subset \Delta_j$. □

For any $\eta > 0$, we define a sequence $(a_\eta(m))_{m\geq 0}$ inductively as follows:

$$a_\eta(0) := 0; \qquad a_\eta(m + 1) := 2Ca_\eta(m) + \eta.$$

Note that $\lim_{\eta\to 0} a_\eta(N) = 0$. In the following, we fix $\eta_0 > 0$ small enough that

$$a_{\eta_0}(N) < (2C)^{-N}\lambda, \qquad \eta_0 < \frac{\lambda}{2}.$$

For each $k \in \{0, \dots, 2^d - 1\}$ and $j \in J_k$, the set $\mathcal{P}^X_{p_j,n_0k}(\Delta_j)$ and the point $X_{n_0k}(x)$ satisfy the hypotheses of Proposition 6.10. We obtain a vector field $\widetilde{Y} \in \mathfrak{X}^1(M)$ such that the support of $X - \widetilde{Y}$ is contained in $\overline{\mathcal{V}^X_j(n_0k, n_0)}$. Moreover, for distinct choices of (k, j), (6.20) guarantees that the associated perturbations will be disjointly supported. Hence, applying Proposition 6.10 over all pairs (k,j) with $k \in \{0,\dots,2^d-1\}$ and $j \in J_k$, we obtain a vector field $Y \in \mathfrak{X}^1(M)$ with the following properties:

- the support of $X - Y$ is contained in
$$\bigcup_{k=0}^{2^d-1} \bigcup_{j \in J_k} \overline{\mathcal{V}_j^X(n_0 k, n_0)} \subset \bigcup_{j=1}^{\ell} \mathcal{U}^X(p_j, U_j, N);$$
- $d_{C^1}(X,Y) < \varepsilon$;
- $d_{C^1}(\mathcal{P}^X_{X_i(p_j),1}, \mathcal{P}^Y_{X,X_i(p_j),1}) < \delta(\varepsilon)$, for all $i \in \{0,\dots,N\}$ and $j \in \{1,\dots,\ell\}$;
- $d_{C^0}(\mathcal{P}^X_{X_i(p_j),1}, \mathcal{P}^Y_{X,X_i(p_j),1}) < \eta_0$, for all $i \in \{0,\dots,N\}$ and $j \in \{1,\dots,\ell\}$;
- for each $k \in \{0,\dots,2^d-1\}$ and for each $z \in \bigcup_{j \in J_k} \Delta_j$, there exists an integer $n \in \{1,\dots,n_0\}$ such that:
$$\left| \log\det P^Y_{X_{n_0 k}(x),n} - \log\det P^Y_{\mathcal{P}^X_{p_j,n_0 k}(z),n} \right| > 3K.$$

Claim 5. *For each $y \in \bigcup_{j=1}^{\ell} D_j$, there exist $k \in \{0,\dots,2^d-1\}$, $j \in J_k$, and $t \in [0,2]$, such that $y = Y_t(w)$, with $w \in \Delta_j$ and $\mathcal{P}^Y_{X,p_j,n_0 k}(w) \in \mathcal{P}^X_{p_j,n_0 k}(\Delta_j)$.*

Proof. Let $y \in \bigcup_{j=1}^{\ell} D_j$. By Claim 4, there exist $j \in \{1,\dots,\ell\}$, $z \in \Delta_j$ and $u \in [0,\frac{3}{2}]$ such that $y = \mathcal{P}^X_{p_j,u}(z)$, and $\mathcal{N}_{X,z}(2\lambda) \subset \Delta_j$. We have $d_{C^0}((\mathcal{P}^X_{p_j,u})^{-1}, (\mathcal{P}^Y_{X,p_j,u})^{-1}) < \eta_0 < \frac{\lambda}{2}$, hence $y = \mathcal{P}^Y_{X,p_j,u}(w) = Y_t(w)$, for some $t \in [0,2]$, and $w \in \Delta_j$ satisfying $\mathcal{N}_{X,w}(\lambda) \subset \Delta_j$. Moreover, X is bounded by C, hence $\mathcal{N}_{X,\mathcal{P}^i_{X,p_j}(w)}((2C)^{-i}\lambda) \subset \mathcal{P}^X_{p_j,i}(\Delta_j)$, for all $i \in \{0,\dots,N-1\}$. For any $i \in \{0,\dots,N-1\}$, by (6.19), and by the fact that $d_{C^0}(\mathcal{P}^X_{X_i(p_j),1}, \mathcal{P}^Y_{X,X_i(p_j),1}) < \eta_0$, we have the estimate

$$\begin{aligned} d(\mathcal{P}^X_{p_j,i+1}(w), \mathcal{P}^Y_{X,p_j,i+1}(w)) &\le d(\mathcal{P}^X_{X_i(p_j),1} \circ \mathcal{P}^X_{p_j,i}(w), \mathcal{P}^X_{X_i(p_j),1} \circ \mathcal{P}^Y_{X,p_j,i}(w)) \\ &\quad + d(\mathcal{P}^X_{X_i(p_j),1} \circ \mathcal{P}^Y_{X,p_j,i}(w), \mathcal{P}^Y_{X,X_i(p_j),1} \circ \mathcal{P}^Y_{X,p_j,i}(w)) \\ &\le 2Cd(\mathcal{P}^X_{p_j,i}(w), \mathcal{P}^Y_{X,p_j,i}(w)) + \eta_0. \end{aligned}$$

Thus, for any $i \in \{0,\dots,N-1\}$, we obtain
$$d(\mathcal{P}^X_{p_j,i}(w), \mathcal{P}^Y_{X,p_j,i}(w)) \le a_{\eta_0}(i) < (2C)^{-N}\lambda.$$

Let $k \in \{0,\dots,2^d-1\}$ be such that $j \in J_k$. We conclude that $\mathcal{P}^Y_{X,p_j,n_0 k}(w) \in \mathcal{N}_{X,\mathcal{P}^Y_{X,p_j,n_0 k}(w)}((2C)^{-n_0 k}\lambda) \subset \mathcal{P}^X_{p_j,n_0 k}(\Delta_j)$, where $Y_t(w) = y$. □

We deduce that for each $y \in D \subset \cup_{j=1}^{\ell} D_j$, there exist $k \in \{0,\dots,2^d-1\}$, $j \in J_k$, $w \in \Delta_j$, $t \in [0,2]$, such that $y = Y_t(w)$, and there exists $n \in \{1,\dots,n_0\}$, such that
$$|\log\det P^Y_{X_{n_0 k}(x),n} - \log\det P^Y_{\mathcal{P}^Y_{X,p_j,n_0 k}(w),n}| > 3K.$$

Since the vector field $X - Y$ has support in $\bigcup_{j=1}^{\ell} \mathcal{U}^X(p_j,U_j,N)$, which is disjoint from the orbit $\mathrm{orb}^X(x)$, we have $X_{n_0 k}(x) = Y_{n_0 k}(x)$. Moreover, there exists $t' \in [n_0 k - 2, n_0 k + 2]$ such that $\mathcal{P}^Y_{X,p_j,n_0 k}(w) = Y_{t'}(y)$. We thus have
$$|\log\det P^Y_{Y_{n_0 k}(x),n} - \log\det P^Y_{Y_{t'}(y),n}| > 3K.$$

We have $P^Y_{Y_{t'}(y),n} = P^Y_{Y_{n_0k}(y),n} \circ P^Y_{Y_{t'}(y),n_0k-t'}$, with $n_0k - t' \in [-2,2]$. Recall that $K > 0$ was chosen such that $K > 2d\log(2C)$. Since Y is close to X, we can assume that Y is bounded by $2C$. We thus get

$$\begin{aligned}
&|\log\det P^Y_{Y_{n_0k}(x),n} - \log\det P^Y_{Y_{n_0k}(y),n}| \\
\geq\ &|\log\det P^Y_{Y_{n_0k}(x),n} - \log\det P^Y_{Y_{t'}(y),n}| - \max_{u'\in[-2,2]}\max_{y'\in M_X} |\log\det P^Y_{y',u'}| \\
>\ &3K - 2d\log(2C) > 2K.
\end{aligned}$$

Besides, $P^Y_{z,n+n_0k} = P^Y_{Y_{n_0k}(z),n} \circ P^Y_{z,n_0k}$, hence of the following two cases holds:

- $|\log\det P^Y_{x,n_0k} - \log\det P^Y_{y,n_0k}| > K$;
- $|\log\det P^Y_{x,n+n_0k} - \log\det P^Y_{y,n+n_0k}| > K$.

In either case, $|\log\det P^Y_{x,n'} - \log\det P^Y_{y,n'}| > K$, for some $n' \in \{1,\dots,N\}$, as required.

By construction, the support of $X - Y$ is contained in at most N iterates of $O - X_1(\overline{O})$ for some trapping region O, and thus, the iterates of $O - X_1(\overline{O})$ for X and Y coincide. This implies that the vector fields X and Y have the same chain recurrent set, and they coincide on this set, which concludes the proof. □

6.1.4. *Proof of Theorem 6.3.* Let $\mathcal{F}$ be a countable and dense subset of M, and let $\mathcal{K} = \{D_n\}_{n\in\mathbb{N}}$ be a countable collection of compact sets D_n, that verifies the following conditions:

– $\operatorname{diam} D_n \to 0$, as $n \to +\infty$;
– for any $n_0 \geq 1$, it holds $\displaystyle\bigcup_{n\geq n_0} D_n = M$.

For each $D \in \mathcal{K}$ we define the following set

$$\mathcal{O}_D = \{X \in \mathfrak{X}^1(M) : \exists \text{ open set } U,\ X_1(\overline{U}) \subset U \text{ and } D \subset (U - X_1(\overline{U}))\}.$$

It is easy to see that $\mathcal{O}_D$ is open. For any point $x \in \mathcal{F}$ we define

$$\mathcal{U}_{x,D} = \{X \in \mathcal{O}_D : x \notin \operatorname{Zero}(X) \text{ and } \operatorname{orb}^X(x) \cap D = \emptyset\}.$$

This set is not open. The next lemma gives a criterion for a vector field X to be in its interior.

Lemma 6.13. *Let $X \in \mathcal{U}_{x,D}$ and let $U \subset M$ be an open subset such that $X_1(\overline{U}) \subset U$ and $D \subset (U - X_1(\overline{U}))$. Assume that $\operatorname{orb}^X(x) \cap (U - X_1(\overline{U})) \neq \emptyset$. Then X belongs to the interior of $\mathcal{U}_{x,D}$, in particular, for any $Y \in \mathfrak{X}^1(M)$ sufficiently close to X it holds that $\operatorname{orb}^Y(x) \cap D = \emptyset$.*

Proof. Observe that the conditions $X_1(\overline{U}) \subset U$ and $D \subset (U - X_1(\overline{U}))$ are open. If $\operatorname{orb}^X(x) \cap U \neq \emptyset$, we can fix $t_1, t_2 \in \mathbb{R}$ such that $\left(\operatorname{orb}^X(x) \cap U - X_1(\overline{U})\right) \subset X_{[t_1,t_2]}(x)$. We can also assume that this property is open, that is, for any C^1 vector field Y sufficiently close to X, it holds

$$\operatorname{orb}^Y(x) \cap (U - Y_1(\overline{U})) \subset Y_{[t_1,t_2]}(x).$$

Since D and $X_{[t_1,t_2]}(x)$ are compact and disjoint, the distance between them is strictly positive. This implies that for any Y sufficiently C^1-close to X it holds that $Y_{[t_1,t_2]}(x)$ does not intersect D. Since for any Y close to X, $U - Y_1(\overline{U})$ is a fundamental domain for the attracting region Y, we conclude that $\operatorname{orb}^Y(x) \cap D = \emptyset$. In particular, X belongs to the interior of $\mathcal{U}_{x,D}$. □

The proof of the following lemma is the same as Lemma 15 in [BCW09].

Lemma 6.14. *The set* $\mathrm{Int}(\mathcal{U}_{x,D}) \cup \mathrm{Int}(\mathcal{O}_D - \mathcal{U}_{x,D})$ *is open and dense inside* $\mathcal{O}_D$.

First, observe that if $X \in \mathcal{U}_{x,D}$ then $D \cup \{x\}$ do not have any singularity of X. In particular, the linear Poincaré flow is well defined for any point $y \in D \cup \{x\}$. For $x \in \mathcal{F}$, $D \in \mathcal{K}$ and any $K \in \mathbb{N}$, we define:

$$\mathcal{V}_{x,D,K} := \left\{X \in \mathrm{Int}(\mathcal{U}_{x,D}) : \forall y \in D,\ \exists n \geq 1,\ |\log\det P^X_{y,n} - \log\det P^X_{x,n}| > K\right\}.$$

Using the fact that D is compact, it is easy to see that $\mathcal{V}_{x,D,K}$ is open inside $\mathrm{Int}(\mathcal{U}_{x,D})$. Proposition 6.12 implies that $\mathcal{V}_{x,D,K}$ is dense in $\mathrm{Int}(\mathcal{U}_{x,D})$. Therefore, the set

$$\mathcal{W}_{x,D,K} = \mathcal{V}_{x,D,K} \cup \mathrm{Int}(\mathcal{O}_D - \mathcal{U}_{x,D}) \cup \mathrm{Int}(\mathfrak{X}^1(M) - \mathcal{O}_D)$$

is open and dense in $\mathfrak{X}^1(M)$. Define the set

$$\mathcal{R}_0 = \bigcap_{x\in\mathcal{F}, D\in\mathcal{K}, K\in\mathbb{N}} \mathcal{W}_{x,D,K}.$$

By Baire's theorem, this set is residual in $\mathfrak{X}^1(M)$. Let $\mathcal{R} = \mathcal{R}_0 \cap \mathcal{R}_*$, where $\mathcal{R}_*$ is the residual subset given by Theorem 6.2.

Let $X \in \mathcal{R}$. Consider $x \in \mathcal{F} - \mathrm{Zero}(X)$ and $y \in M - \mathrm{CR}(X)$ such that $y \notin \mathrm{orb}^X(x)$. Since $y \notin \mathrm{CR}(X)$, by Conley's theory there exists an open set $U \subset M$ such that $X_1(\overline{U}) \subset U$ and $y \in (U - X_1(\overline{U}))$ (see for instance chapter 4 in [AN07]). Observe also that $\mathrm{orb}^X(x) \cap (U - X_1(\overline{U}))$ is either empty or a compact orbit segment. Take $D \in \mathcal{K}$ a compact set that contains y. If its diameter is sufficiently small, we have that $D \subset (U - X_1(\overline{U})$ and $\mathrm{orb}^X(x) \cap D = \emptyset$.

Hence $X \in \mathcal{U}_{x,D}$. Since $X \in \mathcal{R}_0$ and by the definition of $\mathcal{R}_0$, for every $K \in \mathbb{N}$, it holds that $X \in \mathcal{W}_{x,D,K}$. By the definition of $\mathcal{W}_{x,D,K}$ and since $X \in \mathcal{U}_{x,D}$, we have that $X \in \mathcal{V}_{x,D,K}$. Therefore, for any $K \in \mathbb{N}$, there exists $n \geq 1$ such that

$$|\log\det P^X_{x,n} - \log\det P^X_{y,n}| > K.$$

We conclude that X verifies the unbounded normal distortion property. $\square$

6.2. **Collinearity.** Once we have established Theorem 6.3, by combining Proposition 2.5 and some known generic results one obtains the collinearity of the centralizer of a C^1-generic vector field.

Theorem 6.15. *There exists a residual subset of* $\mathcal{R} \subset \mathfrak{X}^1(M)$ *such that if* $X \in \mathcal{R}$ *then the* C^1*-centralizer of* X *is collinear.*

Proof. The result follows directly from Proposition 2.5, and Theorems 6.2 and 6.3. $\square$

6.3. **Quasi-triviality.** By Theorem 6.2, we have that C^1-generically all the singularities are hyperbolic. As a consequence of Theorem 4.3, since C^1-generically the C^1-centralizer is collinear and all the singularities are hyperbolic, we conclude that C^1-generically the C^1-centralizer is quasi-trivial. More precisely, we have

Theorem 6.16. *Let* M *be a compact manifold. there exists a residual subset* $\mathcal{R}_1 \subset \mathfrak{X}^1(M)$ *such that if* $X \in \mathcal{R}_1$*, then any singularity and periodic orbit of* X *is hyperbolic,* $\overline{\mathrm{Per}(X)} = \Omega(X) = \mathcal{CR}(X)$*, and*

$$\mathfrak{C}^1(X) = \{fX : f \in \mathfrak{I}^1(X)\},\ \textit{where}\ \mathfrak{I}^1(X) = \{f \in C^1(M,\mathbb{R}),\ X\cdot f \equiv 0\}.$$

Proof. By Theorem 6.15, there exists a residual subset $\mathcal{R} \subset \mathfrak{X}^1(M)$ whose elements have collinear C^1-centralizer. Moreover, by Theorem 6.2, there exists a residual subset $\mathcal{R}_* \subset \mathfrak{X}^1(M)$ such that for any $X \in \mathcal{R}_*$, any singularity and periodic orbit of X is hyperbolic, and $\overline{\mathrm{Per}(X)} = \Omega(X) = \mathcal{CR}(X)$. Then, $\mathcal{R}_1 := \mathcal{R} \cap \mathcal{R}_*$ is residual, and any $X \in \mathcal{R}_1$ satisfies the hypotheses of Theorem 4.3, which concludes. $\square$

6.4. **Triviality.**

6.4.1. *C^1-generic triviality for systems with a countable number of chain recurrent classes.* We can now conclude the proof of Theorem 6.1. To prove that we need the following lemma.

Lemma 6.17. *there exists a residual subset $\mathcal{R}_{\mathcal{CR}} \subset \mathfrak{X}^1(M)$ such that if $X \in \mathcal{R}_{\mathcal{CR}}$ and $f \in C^0(M)$ is an X-invariant function, then f is constant on chain-recurrent classes.*

Proof. By Theorem 1 in [Cro06], there exists a residual subset $\mathcal{R}_{\mathcal{CR}} \subset \mathfrak{X}^1(M)$ that verifies the following: if $X \in \mathcal{R}_{\mathcal{CR}}$ and $C \subset \mathcal{CR}(X)$ is a chain-recurrent class, then there exists a sequence of periodic orbits $(O(p_n))_{n\in\mathbb{N}}$ that converges to C in the Hausdorff topology.

By this property, for any two points $x, y \in C$, there exist two sequences of points $(q_n)_{n\in\mathbb{N}}$ and $(q'_n)_{n\in\mathbb{N}}$, with $q_n, q'_n \in O(p_n)$, such that $q_n \to x$ and $q'_n \to y$ as $n \to +\infty$. Let f be a continuous function which is X-invariant. By continuity,

$$\lim_{n\to+\infty} f(q_n) = f(x) \text{ and } \lim_{n\to+\infty} f(q'_n) = f(y).$$

However, since f is X-invariant and by our choice of q_n and q'_n, we have that $f(p_n) = f(q_n) = f(q'_n)$, which implies that $f(x) = f(y)$. $\square$

Proof of Theorem 6.1. Take $\mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_{\mathcal{CR}}$, where $\mathcal{R}_1$ is the residual subset given by Theorem 6.16. Using the conclusion of Lemma 6.17 and arguments analogous to the proof of Theorem 5.1 we can easily obtain the conclusion of Theorem 6.1. $\square$

Appendix A. The separating property is not generic

In this section we prove that the separating property is not generic. Let M be a compact, connected Riemannian manifold. Take any Morse function $f \in C^2(M, \mathbb{R})$ and let $X := \nabla f$ be the gradient vector field which is C^1. It holds that X has two hyperbolic singularities, σ_s and σ_u with the following properties:

- σ_s is a hyperbolic sink and σ_u is a hyperbolic source;
- $W^s(\sigma_s) \cap W^u(\sigma_u) \neq \emptyset$;
- for any C^1 vector field Y which is sufficiently C^1-close to X, then $W^s(\sigma_s(Y)) \cap W^u(\sigma_u(Y)) \neq \emptyset$, where $\sigma_*(Y)$ is the continuation of σ_* for the vector field Y, for $* = s, u$.

We claim that X is C^1-robustly not separating. Let U be a compact ball inside $(W^s(\sigma_s) \cap W^u(\sigma_u)) - \{\sigma_s, \sigma_u\}$. Since compact parts of stable and unstable manifolds vary continuously with the vector field, it holds for any Y sufficiently C^1-close to X it holds that $U \subset (W^s(\sigma_s(Y)) \cap W^u(\sigma_u(Y))) - \{\sigma_s, B_u\}$.

Take any $\varepsilon > 0$ and consider the the balls $B(\sigma_s, \frac{\varepsilon}{2})$ and $B(\sigma_u, \frac{\varepsilon}{2})$. Since U is compact, there exists $T_X = T(\varepsilon) > 0$ such that any point $x \in U$ verifies

$$X_{-t}(x) \in B\left(\sigma_u, \frac{\varepsilon}{2}\right) \text{ and } X_t(x) \in B\left(\sigma_s, \frac{\varepsilon}{2}\right), \text{ for all } t \geq T. \tag{A.1}$$

Notice that for any two points $x, y \in B(\sigma_s, \frac{\varepsilon}{2})$ it holds that $d(X_t(x), X_t(y)) < \varepsilon$, for all $t \geq 0$. Similar statement is true for points in $B(\sigma_u, \frac{\varepsilon}{2})$ and the backward orbit.

Since T that verifies (A.1) is fixed, there exists $\delta > 0$ such that for any $x \in U$ and any $y \in B(x, \delta) \subset U$, it holds that

$$d(X_t(x), X_t(y)) < \varepsilon, \text{ for any } t \in \mathbb{R}.$$

In particular X is not separating. Also, observe that this holds for any Y sufficiently C^1-close to X. Thus we conclude that X is C^1-robustly not separating.

Remark A.1. It is easy to see that the same type of example proves that the hypothesis of Proposition 3.2 is not generic. We conclude that the hypotheses of Propositions 2.4 and 3.2 are not generic.

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