Analytical Formulas for the Optical Gain of Quantum Wells

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Abstract—Analytical expressions for the quantized energy levels in quantum wells, the optical gain, the differential optical gain, and the linewidth enhancement factor are presented based on a simple parabolic-band gain model. Explicit formulas show clearly the dependence of these factors on well width, doping, and photon energy. The optical gain in the form of $g = g_0 \ln(N/N_0)$ is derived using explicit approximations in the Fermi functions, where $g_0$ is the proportionality constant, $N$ is the injected carrier density, and $N_0$ is the transparency carrier density. The approximate formulas are shown to provide not only an efficient way of computing the gain-related parameters but also a convenient way of getting physical insights into the overall interplay of quantum well parameters.

I. INTRODUCTION

Quantum well (QW) lasers have been developed extensively since they have many advantages, such as very low threshold current densities, large modulation bandwidth, narrow static and dynamic linewidth, high output power, and high lasing temperature [1].

The optical gain is a key parameter in the design of multi-quantum well (MQW) lasers [2], [3]. The design optimization necessitates a large degree of numerical computation because there are a large number of laser parameters involved, such as the quantum well/barrier composition, the number of QW’s, the cavity length, and the facet reflectivity. In the dynamic performance of QW lasers, the differential gain and the linewidth enhancement factor are key parameters in addition to the gain [3]-[7]. The efficiency of numerical computation becomes a serious consideration when analyzing 2D or 3D MQW laser structures such as ridge waveguide lasers [8] and MQW distributed feedback (DFB) lasers where the carrier density has a nonuniform distribution in the longitudinal direction [9]. Furthermore, in the design optimization, some of the important physics becomes obscured because of the large number of laser parameters. Therefore, simple analytical formulas are desired that can clearly delineate the relations between design parameters and predict device performance characteristics.

Several efficient analytical models have been reported for the gain-related parameters. McIlroy et al. [10] have derived analytical optimum conditions for the lowest threshold current density and the lowest threshold current by using an empirical logarithmic gain as a function of injection current. Vahala and Zab [11] have derived simple analytical expressions for the Fermi functions at the gain peak and given insightful explanations for the effect of doping on the gain and the noise properties. Westbrook and Adams [12] have derived explicit formulas for the linewidth enhancement factor as a function of QW parameters and shown the optimum strategy for reducing the linewidth enhancement factor.

In strained-layer QW lasers, another complication is the effect of strain on the laser performance. A band structure calculation taking into account strain effects is required in order to calculate the gain [13]. This is time consuming. To overcome this problem, several simple and explicit formulas for the strain effect have been reported. Li et al. [14] have recently developed an efficient approximate $\kappa$-$\rho$ theory for the gain of strained QW’s and shown that the parabolic-band gain model is a good approximation if the change of the effective masses due to strains is taken into account. Ma et al. [15] have reported empirical gain formulas in logarithmic forms as a function of carrier concentration for strained quaternary QW’s. Suemune [4] has derived an analytical expression for the effective mass for strained QW’s and shown that the increase of the differential gain due to the strain effect can be explained by the reduction of the effective mass of the holes. Using a simple parabolic-band gain model, Lau et al. [5] have shown that the enhancement of the differential gain due to the strain effect is substantial only for relatively low carrier densities. A simple evaluation of the effect of strain on the linewidth enhancement factor has been reported by Wartak and Makino [16] using an explicit formula similar to the one of Westbrook and Adams [12].

These analytical approaches provide not only a means for efficient computation but also an overall view of the interplay of material parameters on the laser performance. However, to the best of our knowledge, only approximate maximum gains at the gain peak have been reported until now, and there has been no explicit formulas for the gain spectrum reported. Although Westbrook and Adams [12] have calculated the linewidth enhancement factor that takes into account the dispersion, the differential gain and the differential real refractive index have not been calculated in the context of explicit formulas.

The purpose of this paper is to derive analytical formulas for the optical transition wavelength between the quantized energy levels of electrons and holes in QW’s, the optical gain spectrum, the differential gain, and the linewidth enhancement factor. The analysis is based on a simple parabolic-band gain model with an intraband relaxation broadening [2]. The primary aim of this approach is to obtain explicit physical insights.
into the interplay of QW parameters. Although the accuracy is to be verified, this approach may have overall advantages for the design optimization of QW lasers. Moreover, there are still many unknown parameters to be measured in order to verify the validity of even the detailed models.

The logarithmic gain as a function of carrier density or injection current is derived from first principles, and, therefore, related to physical parameters. The effect of impurity doping on the gain spectrum is interpreted in a simple and insightful manner. The analytical expressions for the differential gain and the linewidth enhancement factor derived by Westbrook and Adams [12] are reformulated in the context of the present approach and compared to numerical calculations.

The paper is organized as follows. In Section II, we derive an analytical expression for the quantized energy levels in quantum well structures. In Section III, using explicit approximations, we derive analytical expressions for the Fermi functions, and then, the optical gain. Section IV presents analytical formulas for the differential gain, the differential refractive-index, and the linewidth enhancement factor. Finally, conclusions are given in Section V.

II. APPROXIMATE QUANTIZED ENERGY LEVELS

A schematic energy band diagram for a typical QW structure is shown in Fig. 1. $\Delta E_c$ and $\Delta E_v$ are the discontinuities of the band edges of conduction and valence bands at the heterojunction, respectively. $E_{cn}$ and $E_{vn}$ are the quantized energy levels in the conduction and valence band, respectively. $E_g$ is the bandgap energy, and $E_{tr}$ is the transition energy between the two quantized energy levels. $E_{fc}$ and $E_{f0}$ are the quasi-Fermi levels for electrons and holes in the well. We assume that the origin of the energy levels is located at the bottom of the conduction-band potential well, except for the quantized levels for holes ($E_{cn}$) that are measured from the top of the valence band down into the valence band.

![Image of a quantum well structure](image)

Fig. 1. Band model for a quantum well structure.

Using the parabolic band model [2], $E_{cn}$ can be obtained by solving the eigenvalue equations

$$
\frac{m_{e0}}{m_{e0}} \sqrt{\frac{\Delta E_c - E_{cn}}{E_{cn}}} = \begin{cases} 
\tan \left( \frac{W \sqrt{2m_{e0}E_{cn}}}{2\hbar} \right) & \text{n : even} \\
-\cot \left( \frac{W \sqrt{2m_{e0}E_{cn}}}{2\hbar} \right) & \text{n : odd}
\end{cases}
$$

(1)

where $\hbar = h/2\pi$ is Planck’s constant, $W$ is the well width, and $m_{e0}$ and $m_{h0}$ are the effective masses of electrons inside of the well, and the barrier, respectively. In the range $E_{cn} \ll \Delta E_c$, $E_{cn}$ can be shown to be approximated as (see Appendix A)

$$
E_{cn} = \frac{\left( \frac{(n+1)\pi}{2} \frac{a_c}{W + \Delta W_c} \right)^2}{1 + \left( \frac{(n+1)\pi}{2} \frac{b_c}{W + \Delta W_c} \right)^2}
$$

(2)

with

$$
\Delta W_c = \frac{a_c}{\sqrt{b_c} \Delta E_c}
$$

(3)

where

$$
a_c = \frac{2\hbar}{\sqrt{2m_{e0}}}, \quad b_c = \frac{m_{e0}}{m_{h0}}
$$

(4)

It should be noted that (2) is an exact solution of (1) when $\Delta E_c$ becomes infinity; the solution is $E_{cn} = \left( \frac{(n+1)\pi}{2a_c}/W \right)^2$. If we neglect the $(\Delta W_c)^3$ term in (2), $W + \Delta W_c$ can be interpreted as an effective well width where $\Delta W_c$ is induced by a finite barrier height $\Delta E_c$. The energy levels $E_{vn}$ for the valence band can be expressed by (2) if the subscript $c$ is replaced by $v$. The wavelength corresponding to a transition between the quantized levels of conduction and valence bands can be expressed as

$$
\lambda_n = 1.24/(E_g + E_{cn} + E_{vn})(\mu m)
$$

(5)

where the unit of the energy levels is electron volts.

In Fig. 2, calculated $\lambda_0$ for the first transition ($n = 0$) between electrons and heavy holes are shown as a function of the well width $W$ for a QW structure consisting of lattice matched In$_{0.53}$Ga$_{0.47}$As and In$_{0.37}$Ga$_{0.63}$As$_{0.25}$A$_{0.75}$P$_{0.45}$ ($\lambda_c = 1.25 \mu m$) barrier. The parameter values used in the calculation are listed in Table I, and $\Delta E_c = 0.4\Delta E_g$ is assumed. The solid, dashed, and chain-dotted curves correspond to numerically calculated values, approximate values calculated by neglecting $(\Delta W_i)^3$ ($i = c, v$) in (2) and the corresponding equation for the valence band (denoted by “approximate 1” in Fig. 2), and approximate values calculated by retaining up to $(\Delta W_i)^3$ ($i = c, v$) terms (denoted by “approximate 2” in Fig. 2).
III. APPROXIMATE OPTICAL GAIN

We take the optical gain model of Asada et al. [2]; it is assumed that all subbands are parabolic and that optical transitions obey \( k \)-selection-rules. The optical gain for a single QW is expressed as [2]

\[
g(\omega) = \omega \sqrt{\frac{\mu}{\varepsilon}} \sum_{n=0}^{\infty} \left( \frac{m_e}{\pi \hbar^2 W} \right) \left( R_{ev}^{2} \right) \left( f_{e} - f_{v} \right) F_{e}(E_{ev}) dE_{ev},
\]

where \( \omega \) is the angular frequency of light, \( \mu \) is the permeability, \( \varepsilon \) is the dielectric constant, \( m_e \) is the reduced effective mass given by \( m_{ew} m_{ew}/(m_{ew} + m_{ew}) \), \( E_{ev} \) is the transition energy, and \( \left( R_{ev}^{2} \right) \) is the matrix element of the dipole element formed by an electron in subband \( n \) in the conduction band and a hole in subband \( n \) in the valence band. In (6), the light-hole band is neglected, and only the transitions between the electrons and the heavy-holes are considered. This is a reasonable approximation for most of lattice matched QW’s [2]. Although this model cannot be applied directly to strained QW’s in which valence band mixing may occur, the effective mass approach has been shown to be a good approximation for some strained QW’s [14]. \( F_{e}(E_{ev}) \) is a function expressing the transition broadening. Asada et al. [2] used the Lorentzian function based on the density matrix formalism as follows:

\[
F_{e}(E_{ev}) = \frac{\hbar}{(E_{ev} - \hbar \omega)^2 + (\frac{\hbar}{\tau_{in}})^2},
\]

where \( \tau_{in} \) is the intraband relaxation time. Different broadening models have been proposed by some authors [17, 18].

\( f_{e} \) and \( f_{v} \) are the Fermi functions given by

\[
f_{e} = \frac{1}{1 + \exp \left( (E_{fn} - E_{ev})/kT \right)}
\]

\[
f_{v} = \frac{1}{1 + \exp \left( (E_{vn} - E_{ev})/kT \right)}
\]

where \( E_{cn} \) and \( E_{vn} \) are the total energies of electrons and holes for subbands \( n \), \( k \) is the Boltzmann constant, and \( T \) is the absolute temperature. \( E_{fc} \) and \( E_{fc} \) are related to the densities of electrons and holes injected into the well as [2]

\[
N = \frac{m_{ew} kT}{(\pi \hbar^2 W) \sum_{n} \ln \left( 1 + \exp \left( (E_{fn} - E_{cn})/kT \right) \right)}
\]

\[
P = \frac{m_{ew} kT}{(\pi \hbar^2 W) \sum_{n} \ln \left( 1 + \exp \left( (E_{fn} - E_{vn})/kT \right) \right)}
\]

A. Approximate Fermi Functions

\( f_{e} \) and \( f_{v} \) for \( n = 0 \) can be expressed as (see Appendix B)

\[
f_{e} = \frac{1}{1 + e^{2\Delta E / kT} e^{-\chi_{e}}}
\]

\[
f_{v} = \frac{1}{1 + e^{-2\Delta E / kT} e^{\chi_{v}}}
\]

where

\[
\chi_{e} = \frac{E_{fc} - E_{e0}}{kT}
\]

\[
\chi_{v} = -\frac{(E_{v0} + E_{v0} + E_{fc})}{kT}
\]

\[
\Delta E = E_{ev} - E_{fr}
\]

\[
E_{fr} = E_{g} + E_{e0} + E_{e0}.
\]

Following Vahala and Zan [11], we can express \( \chi_{e} \) and \( \chi_{v} \) approximately as (see Appendix B)

\[
\chi_{e} = \ln \left( e^{\frac{N_{s}}{P_{s}} - 1} \right)
\]

\[
\chi_{v} = \ln \left( e^{\frac{N_{s}}{P_{s}} - 1} \right)
\]

where \( N_{s} \) and \( P_{s} \) are given as

\[
N_{s} = \frac{m_{ew} kT}{(\pi \hbar^2 W) \sum_{n} \ln \left( 1 + \exp \left( (E_{fn} - E_{cn})/kT \right) \right)}
\]

\[
P_{s} = \frac{m_{ew} kT}{(\pi \hbar^2 W) \sum_{n} \ln \left( 1 + \exp \left( -(E_{fn} - E_{vn} - E_{g})/kT \right) \right)}.
\]

Equation (10) gives analytical expressions for \( f_{e} \) and \( f_{v} \); the photon energy dependence is represented by \( \Delta E \) in (13), and the carrier density dependence is expressed by (14) with (15).
C. Maximum Optical Gain

The Fermi functions at the gain peak wavelength are obtained by setting $\Delta E = 0$ in (10) as

\[ f_{c0} = 1 - \exp \left( \frac{-N}{N_0} \right) \]  
(19a)

\[ f_{v0} = \exp \left( \frac{-N}{P_0} \right) \]  
(19b)

These were originally derived by Vahala and Zaw [11]. In this approximation, the transparency carrier density $N_0$ corresponds to the carrier density satisfying $f_{c0} = f_{v0}$. Vahala and Zaw [11] have given an insightful explanation by using (19) that p-type doping is expected to increase the differential gain, while n-type doping is expected to decrease the transparency carrier density.

$f_{c0} - f_{v0}$ can be expressed in a simpler form as follows. Assuming $f_{c0} - f_{v0} = A \ln \left( N/N_0 \right)$, we choose $A$ such that, at $N = eN_0$, the value of $f_{c0} - f_{v0}$ becomes equal to the value given by (19), i.e., $A = 1 - \exp \left( -eN_0/N_s \right) \exp \left( -eN_0/P_s \right)$. Therefore, we get

\[ g_{\text{max}} = g_0 \ln \left( \frac{N}{N_0} \right), \]  
(20)

where

\[ g_0 = K \left[ 1 - \exp \left( -eN_0/N_s \right) - \exp \left( -eN_0/P_s \right) \right]. \]  
(21)

It has been shown [9] that (20) can predict the saturation of the modulation speed of MQW DFB lasers due to the linear gain saturation under the longitudinal spatial hole-burning.

Calculated maximum gains are shown as a function of the carrier density in Fig. 4. The solid, dashed, and chain-dotted curves correspond to the numerical result [(17) with (8)], (17) with expression (19) by Vahala and Zaw [11], and (20), respectively. In this case, $g_0 = 1575$ cm$^{-1}$ and $N_0 = 1.17 \times 10^{18}$ cm$^{-3}$ are obtained. It has been shown that the form of (20) is also a good approximation for strained QW's [15].

Several authors [10], [19]–[23] have reported the empirical relation $g = \xi \ln \left( J/J_0 \right)$ where $J$ is the injected current density, $J_0$ is the transparency current density, and $\xi$ is a proportionality constant. We can convert the gain of (20) as a function of carrier density to the above form as a function of current density as follows: $J$ can be expressed as $J = AN + BN^2 + CN^3$ where $A$, $B$, and $C$ are recombination coefficients [21]. We can express $J$ approximately as $J = B_{\text{eff}} N^2$ for some range of QW parameters where $B_{\text{eff}}$ is the effective recombination coefficient [2]. Solving this relation with respect to $N$, and substituting the result into (20), we can obtain $g = g_0/2 \ln \left( J/J_0 \right)$ where $J_0 = B_{\text{eff}} N_0^2$. The modal gain of QW laser structures can be expressed as $\Gamma g = \Gamma \xi \ln \left( J/J_0 \right)$ where $\Gamma$ is the optical confinement factor [24]. This form of modal gain is convenient for deriving analytical expressions for the threshold current of MQW lasers [24].

D. Effects of Impurity Doping

The presence of donors or acceptors can be accounted for by replacing $N$ in (9a) (therefore (19a)) by $N + N_d$ and $N$...
in (9b) (therefore (19b)) by \(N + N_a\) where \(N_d\) and \(N_a\) are the donor and acceptor densities, respectively [11]. The value of photon energy \(E_0\) at which the optical gain becomes zero \((f_e = f_v)\) for electron density \(N\) with doping densities \(N_d\) and \(N_a\) is easily obtained from (10) as

\[
E_0 - E_{tr} = \frac{\chi_e + X_v}{2} \left( \frac{1}{\Gamma_e} + \frac{1}{\Gamma_v} \right) \\
= kT \ln \left[ \left( e^{(N+N_d)/N_a} - 1 \right) \left( e^{(N+N_a)/P_v - 1} \right) \right].
\]

The effect of p-type doping on the gain spectra is shown in Fig. 5 for acceptor densities \(N_a = 5 \times 10^{17} \text{ cm}^{-3}\) and \(1 \times 10^{18} \text{ cm}^{-3}\). Similar plots are shown in Fig. 6 for n-type doping with donor densities \(N_d = 5 \times 10^{17} \text{ cm}^{-3}\) and \(1 \times 10^{18} \text{ cm}^{-3}\). The peak gain is determined by \(f_{e0} - f_{v0}\) given by (19); p-type doping effectively decreases \(f_{e0}\) in (19b) since \(N\) is increased to \(N + N_a\), causing an increase in \(f_{e0} - f_{v0}\). However, the increase of (22) is small because of the large value of \(P_v\). Therefore, p-type doping mostly increases the peak value of the gain, as is shown in Fig. 5. On the other hand, n-type doping does not give rise to a significant increase in \(f_{e0} - f_{v0}\) when \(N\) is increased to \(N + N_d\) in (19a). However, \(E_0 - E_{tr}\) increases effectively because of small \(N_a\), as is shown in Fig. 6.

E. Effects of Lorentzian Broadening

We shall examine only the effects of Lorentzian broadening for the sake of brevity, although other broadening models have been proposed [17, 18]. In the case of finite \(\tau_{in}\), the integration in (6) has to be performed. However, the effect of finite \(\tau_{in}\) can be examined separately from the optical energy dependence of the Fermi functions in the following approximate way. For small values of \(\hbar/\tau_{in}\), the Fermi functions \(f_e\) and \(f_v\) are approximately constant over the range of energies for which the broadening Lorentzian is significant. In this approximation, (6) becomes for \(n = 0\)

\[
g(\omega) = K [f_e(\omega) - f_v(\omega)] \left[ \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left( \frac{\hbar \omega - E_{tr}}{\hbar / \tau_{in}} \right) \right]
\]

where \(K\) is defined by (18). It should be noted that (23) reproduces (17) when \(\tau_{in}\) approaches infinity because the second term in the last bracket in (23) approaches 1/2. Therefore, the
Fig. 7. Gain spectra calculated for the same structure as Fig. 3 but with the Lorentzian broadening with finite intraband relaxation time $\tau_{in} = 1 \times 10^{-13}$ s. The solid and dashed curves correspond to numerical and approximate results calculated by (6) with exact Fermi functions and with approximate Fermi functions using (14), respectively.

reduction of the optical gain due to the Lorentzian broadening can be approximated by the term in the last bracket of (23).

Calculated gain spectra are shown in Fig. 7 for the same quantum well structure as Fig. 3 with $\tau_{in} = 1 \times 10^{-13}$ s. The solid and dashed curves correspond to numerical result obtained by (6) and approximate results obtained by (23), respectively. We can see that (23) provides a fairly reasonable approximation. Broadening functions different from the Lorentzian will cause a different reduction of the peak gain as well as a different gain spectrum shape [17], [18]. In this case, it may be difficult to express the broadening effect in an analytical form.

IV. APPROXIMATE DIFFERENTIAL GAIN AND LINEWIDTH ENHANCEMENT FACTOR

Under the assumption of the Lorentzian broadening, the derivatives of the gain $g$ and (real) refractive index $n$ for a quantum well structure are given by [3]

$$\frac{dg(\omega)}{dN} = \frac{K}{\pi} \int_{E_{tr}}^{\infty} \frac{d(f_c - f_w)}{dN} \frac{\hbar}{\tau_{in}} \left(\frac{\hbar}{E_{cv} - \hbar \omega} + \frac{\hbar}{\tau_{in}}\right)^2 dE_{cv}$$

(24)

and

$$\frac{dn(\omega)}{dN} = \frac{K}{2\pi k_0} \int_{E_{tr}}^{\infty} \frac{d(f_c - f_w)}{dN} \frac{E_{cv} - \hbar \omega}{\left(E_{cv} - \hbar \omega\right)^2 + \left(\frac{\hbar}{\tau_{in}}\right)^2} dE_{cv}$$

(25)

where $K$ is given by (18), $k_0$ is the free-space wavenumber, and it has been assumed that all radiative transitions take place between the first conduction sub-band and the first valence band. Using a similar approximation to that in (23), the differential gain in (24) can be approximated as [14]

$$\frac{dg(\omega)}{dN} = K \frac{d[f_c(\omega) - f_w(\omega)]}{dN} \left\{ \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left( \frac{\hbar \omega - E_{tr}}{\hbar / \tau_{in}} \right) \right\}$$

(26)

We can approximate (25) as (see Appendix C).

$$\frac{dn(\omega)}{dN} = K \frac{d[f_c(\omega) - f_w(\omega)]}{dN} \left\{ \frac{1}{2\pi} \log \frac{(E_{tr} - \xi_c)^2 + \frac{\Gamma_o^2}{2}}{(E_{tr} - h \omega)^2 + \left(\frac{\hbar}{\tau_{in}}\right)^2} \right\}$$

$$+ \frac{\xi_c - h \omega}{\Gamma_v} \left( \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \frac{\chi_c}{2} \right)$$

(27)

where

$$\frac{df_i}{dN} = \frac{1}{1 + y_i^2} \frac{dN}{dN} \quad (i = c, v)$$

(28)

$$y_i = \frac{2(\hbar \omega - \xi_i)}{\Gamma_i} \quad (i = c, v)$$

(29)

$$\xi_i \equiv E_{tr} + \Gamma_i \chi_i \quad (i = c, v)$$

(30)

and $\Gamma_i$ is defined by (11). Equation (27) was derived by Westbrook and Adams [12]. However, they used approximations for $\chi_c$ and $\chi_v$ that are different from our (14).

The linewidth enhancement factor $\alpha$ is defined as

$$\alpha = \frac{\frac{dn(\omega)}{dN}}{2\pi k_0 \frac{1}{dN}}$$

(31)

In Fig. 8, calculated differential gains are shown as a function of wavelength for the carrier densities $N = 2 \times 10^{16}$ cm$^{-3}$ and $3 \times 10^{18}$ cm$^{-3}$. The solid and dashed curves correspond to numerical and approximate results, respectively. Similar plots for the differential refractive index are shown in Fig. 9. Calculated linewidth enhancement factors are shown in Fig. 10, as a function of wavelength. The differential gain increases rapidly as the wavelength detuning (to shorter side) from the gain peak increases for relatively low carrier densities, while the differential refractive index does not.
Fig. 8. Dispersion of the differential gain calculated for 80 Å InGaAs well–InGaAsP (λ = 1.25 μm) with carrier density as a parameter. The solid and dashed curves correspond to numerical and approximate results calculated by (24) with exact Fermi functions, and by (26) with (28), respectively.

Fig. 9. Dispersion of the differential refractive index calculated for the same quantum well structure as Fig. 8. The solid and dashed curves correspond to numerical and approximate results calculated by (25) with exact Fermi functions, and by (27) with (28), respectively.

increase rapidly. Therefore, the absolute value of the linewidth enhancement factor decreases.

Important factors determining the differential gain and refractive index are \( d\chi_c/dN \) and \( d\chi_e/dN \) in (28), which are expressed from (14) as

\[
\frac{d\chi_c}{dN} = \frac{1}{N_s(1 - e^{-N/N_s})} \tag{32a}
\]

\[
\frac{d\chi_e}{dN} = \frac{1}{P_e(1 - e^{-N/P_e})}. \tag{32b}
\]

In lattice-matched quantum wells, \( P_e \) is much larger than \( N_s \) because the heavy hole effective mass is much heavier than the electron effective mass. Therefore, the contribution from the valence band, i.e., \( d\chi_e/dN \) is much smaller than \( d\chi_c/dN \) in (26) and (27) [12]. However, in typical strained-layer QW’s, either the heavy hole effective mass is significantly reduced or the light hole dominates the transition [25]; in this situation, the contribution of \( d\chi_c/dN \) becomes significant through the reduction of \( P_e \) in (32b). This will give an enhancement in the gain and the differential gain and a reduction in the linewidth enhancement factor. Kikuchi et al. [26] have given a similar theoretical interpretation as well as an experimental confirmation. Using the above approximate formula, a simple evaluation of the effect of strain on the linewidth enhancement factor has also been reported for strained-layer QW lasers [16].

V. CONCLUSION

Analytical expressions have been derived for the quantized energy levels, the optical gain, the differential optical gain, the differential refractive index, and the linewidth enhancement factor by using explicit approximations in the Fermi functions. The approximate formulas have been shown to provide not only an efficient way of computing the gain-related parameters but also a convenient way of getting physical insights for the overall interplay of QW parameters. Based on the simplified expressions, the role of the effective mass of the hole has been interpreted explicitly in terms of the effect of strain on QW laser performance. The presented analytical approach is expected to show its significant advantage especially in the design optimization of practical MQW DFB lasers, which requires local gain parameters in three dimensions.

APPENDIX A

APPROXIMATE QUANTIZED ENERGY

Setting \( x = E_{cv}/\Delta E_c \) in (1), we obtain for \( x \ll 1 \)

\[
\cot \left( \frac{W\sqrt{\Delta E_c \sqrt{x}}}{a_c} \right) \approx \frac{\sqrt{x(1 + x)}}{\sqrt{h_c}} \tag{A.1}
\]
can be approximated by
\[ W \frac{\Delta E_c}{a_c} \sqrt{x} = \frac{(n + 1)\pi}{2} - \frac{\sqrt{x}}{\sqrt{b_c}} \left( 1 + \frac{1}{2} x^2 \right). \]  
(A.2)

We can rewrite (A.2) as
\[ x + \frac{x^2}{1 + W} + \frac{x^3}{4 \left( 1 + W \Delta W_c \right)} = x_0 \]  
(A.3)

with
\[ x_0 \equiv \frac{1}{\Delta W_c} \left[ \frac{(n + 1)\pi}{2} - \frac{a_c}{W + \Delta W_c} \right]^2, \]  
(A.4)

where \( \Delta W_c \) is defined by (3). Neglecting the term \( x^3 \) and solving (A.3) with respect to \( x \), we obtain
\[ x = \frac{x_0}{1 + \Delta W_c x_0}. \]  
(A.5)

Using (3) and noting (A.4) and \( E_{cv} = x \Delta E_c \), (A.5) yields (2).

**APPENDIX B**

**APPROXIMATE FERMI FUNCTIONS**

\( \varepsilon_{v0} \) and \( \varepsilon_{v0} \) can be expressed as
\[ \varepsilon_{v0} = \frac{\hbar^2 k^2}{2m_{cv}} + E_{v0}, \]  
(B.1)

\[ \varepsilon_{v0} = -\frac{\hbar^2 k^2}{2m_{cw}} - E_{v0} - E_g. \]  
(B.2)

Then, the transition energy is expressed as
\[ E_{cv} = \varepsilon_{v0} - \varepsilon_{v0} \]
\[ = \frac{\hbar^2 k^2}{2m_{cv}} \left( \frac{1}{m_{cv}^2} + \frac{1}{m_{cw}^2} \right) + E_{v0} + E_{v0} + E_g, \]  
(B.3)

where \( k_{c//} = k_{v//} \) is assumed. Using (B.3) in (B.1), we obtain
\[ \varepsilon_{v0} = \frac{\hbar^2 k^2}{m_{cv} k T} \left( E_{cv} - E_{v0} \right) + \frac{E_{v0} - E_{v0}}{k T}. \]  
(B.4)

Similarly, we obtain
\[ \varepsilon_{v0} - \varepsilon_{v0} = -\frac{m_{cw} k T}{m_{cw} k T} \left( E_{cv} - E_{v} \right) - \frac{E_{v0} + E_g + E_{v0}}{k T}. \]  
(B.5)

Using (B.4) and (B.5) in (8), we obtain (10).

For \( E_{cv} = E_{v//} \), (10a) gives
\[ f_{v//} = \frac{1}{1 + e^{-\varepsilon_{v//}}} \]  
(B.6)

(9a) can be rewritten as
\[ N = D_N \sum_n \ln \left[ 1 + \frac{f_{v//}}{1 - f_{v//}} e^{-(E_{cv} - E_{v//})/kT} \right] \]
\[ \approx D_N \frac{f_{v//}}{1 - f_{v//}} \sum_n e^{-(E_{cv} - E_{v//})/kT}, \]  
(B.7)

where
\[ D_N \equiv \frac{m_{cw} k T}{\pi \hbar^2 W}. \]  
(B.8)

We assume
\[ f_{v0} = 1 - e^{-N/N_s}. \]  
(B.9)

Substitution of (B.9) into (B.7) gives
\[ N \approx \frac{D_N N_s}{N_s} \sum_{n} e^{-(E_{cv} - E_{v0})/kT}. \]  
(B.10)

Therefore, we obtain (15a). For the valence band, similarly, we obtain (14b) and (15b).

**APPENDIX C**

**APPROXIMATE DIFFERENTIAL REFRACTIVE INDEX**

Noting
\[ \frac{E_{cv} - \hbar \omega}{(E_{cv} - \hbar \omega)^2 + \left( \frac{\hbar}{\tau_{vn}} \right)^2} \]
\[ \left[ \frac{E_{cv} - \hbar \omega}{(E_{cv} - \hbar \omega)^2 + \left( \frac{\hbar}{\tau_{vn}} \right)^2} - \frac{1}{E_{cv} - \hbar \omega} \right] + \frac{1}{E_{cv} - \hbar \omega} \]  
(C.1)

we can approximate \( dn(\omega)/dN \) as [12]
\[ \frac{dn(\omega)}{dN} \approx -K \frac{d[f_c(\omega) - f_v(\omega)]}{dN} \frac{1}{2\pi} \log \left[ \frac{(E_{v//} - \hbar \omega)^2}{(E_{v//} - \hbar \omega)^2 + \left( \frac{\hbar}{\tau_{vn}} \right)^2} \right] \]
\[ - K \pi \int_{E_{v//}}^{\infty} \frac{d[f_v(E_{cv}) - f_c(E_{cv})]}{dN} \frac{1}{E_{cv} - \hbar \omega} dE_{cv}. \]  
(C.2)

From (10), we get
\[ \frac{d[f_c(E_{cv}) - f_v(E_{cv})]}{dN} \approx \frac{d\chi_c}{dN} \frac{d\chi_v}{dN} \]
\[ \frac{2 + 2 e^{y_i} + e^{-y_i}}{-2 + 2 e^{y_i} + e^{-y_i}} \]
\[ \approx \frac{d\chi_c}{dN} \frac{d\chi_v}{dN} \frac{1}{4 + y_i^2}. \]  
(C.3)

where \( y_i (i = c, v) \) is defined by (29). Using \( \xi_i (i = c, v) \) of (30), we can rewrite (C.3) as
\[ \frac{d[f_c(E_{cv}) - f_v(E_{cv})]}{dN} \approx \Gamma_c^2 \frac{d\chi_c}{dN} \frac{d\chi_v}{dN} \]
\[ \left( E_{cv} - \xi_c^2 + \Gamma_c^2 \right) \]  
(C.4)

Substituting (C.4) into the second term in (C.2), we obtain (27) after some manipulation.
REFERENCES


Toshihiko Makino (M'90), for a biography, see this issue, p. 416.