# On the Chaotic Behaviour of Buck Converters

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***Abstract*—Power electronic circuits exhibit nonlinear dynamical behaviour due to their inherent inhomogeneity and switching. Among power electronic converters, the DC/DC buck converter is studied with constant-frequency pulse-width modulation feedback control in continuous conduction mode. Phase-space and time-domain plots for several periodic and chaotic orbits are presented. The bifurcation diagram is studied together with periodic orbits and chaotic behaviour of the circuit. Several simulation methods including exact solution and simulation in an EMTP-type program are used and the importance of accurate modeling is justified. Finally, a method for computation of Lyapunov exponents in discontinuous systems is reviewed and implemented.**

***Index Terms*—Bifurcation diagram, buck converter, chaos, Lyapunov exponents, symbolic analysis, transient simulation.**

## I. Introduction

POWER ELECTRONICS has changed the way electric energy is used and processed. DC/DC converters that are used to regulate and step down (buck converters), step up (boost converters), or both step down and step up (buck-boost or Ćuk converters) are among the most-widely used power electronic circuits. One of the most important characteristics of power electronic circuits is their highly nonlinear behaviour. This nonlinearity is due to both nonlinear elements used in these devices (e.g., diodes, BJTs, transformers, and control circuitry employed such as comparators and pulse-width modulators) and the switching operation, which changes the topology of the circuit [1], [2].

The traditional method for dealing with systems with slight nonlinearity is to linearalize the system equations around the operating point. This technique, however, is good only for a small neighbourhood of the operating point, which in turn causes difficulties in simulation of a nonlinear circuit in its entire operating range, and is not suitable for modeling of switched-mode DC/DC converters that are both nonlinear and time-varying dynamical systems. Other efforts for devising conventional linear models for power electronic circuits, such as state-space averaging, can only represent the details of the behaviour of the system to a certain harmonic order [3].

The behaviour of an electrical circuit can be characterized in its steady-state, if any, or in the transient state. In its steady state, an electrical circuit can exhibit one of the following four behaviours [4]: (i) point stability, (ii) cycle stability, (iii) instability (but saturated), and (iv) chaotic stability. In point stability, the circuit currents and voltages settle down to a constant value. In this case the circuit is called stable and representation of the system in phase-space is a single point. Most circuits are designed to operate in this mode. In cycle stability, the circuit states repeat themselves as periodic functions of time with a single period of *T*, period *T* and its multiples, or some disproportionate period. An oscillator circuit is perhaps the most used example of this type of behaviour. In saturated instability, voltages and currents diverge until bounded by an external factor, e.g., limited voltage of the power supply. Some circuits with very specific functions, for example Schmitt triggers, voltage clippers, and flip-flops use this mode of operation. In chaotic stability, the dynamical system is divergent but its trajectory is bounded. This fourth class is called chaotic behaviour and that trajectory is called strange attractor, which arises in many power electronic converters, such as buck converter, boost converter, and the ripple regulator circuit (a buck converter with constant reference voltage instead of a PWM feedback control) [5].

Existence of chaos in power electronic circuits has received great attention during last two decades. Due to their simpler structure, the most studied power electronic circuits are DC/DC converters. Chaotic behaviour of buck converters has been studied in [6]-[9]. A method for controlling chaos in the buck converter based on pole-placement is suggested in [10]. Boost converters are considered in [11]-[13].

This paper presents a study of the chaotic behaviour of the buck converter. In Section II a brief introduction to chaos is presented. Section III discusses the buck converter and its mathematical modeling. Three methods for simulation of the buck converter, consisting of the exact solution, numerical integration, and simulation in PSCAD/EMTDC program are presented in Section IV and results are compared. Lyapunov exponents are defined and calculated in Section V. Some final remarks in Section VI conclude the report.

## II. Review of Chaotic Dynamics

Chaotic operation is the fourth class of stability of a dy-



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namical system. A continuous system governed by a set of at least three first-order, nonlinear, differential equations with no external input (autonomous), or of lower order but with an external input such as time (non-autonomous), can exhibit chaotic behaviour [14].

The signals resulting from a chaotic system, although aperiodic, are bounded. The behaviour of a system is referred to as chaotic if the trajectory of its states possesses three properties. First, it should show high sensitivity to the initial conditions. Even the smallest changes can lead to very large differences in the trajectory, although the chaotic system is governed by a set of completely deterministic equations and even in the absence of noise.

The second property of chaos is the underlying process of folding. While trajectories do not intersect, they are limited to a certain area—the strange attractor.

The third characteristic of chaos is mixing, which means trajectories, regardless of the initial conditions, will eventually reach everywhere in the phase-space. A more formal definition is that for any two open intervals of non-zero length, a value from one interval maps to another point in the other interval after a sufficient number of iterations [15, p. 520], [16].

Two types of diagrams are frequently used in the study of chaotic systems: the phase-space diagram and the bifurcation diagram [17]. The phase-space diagram, which is an $n$-dimensional diagram with $n$ being the number of states of the autonomous system, shows the state trajectory of the system. For a stable system, the phase-space diagram is just a single point. For a periodic system, it is a closed trajectory. For an unbounded unstable system, the phase-space diagram is divergent, while for a chaotic system, although the phase-space diagram is divergent, the trajectory is bounded. Such a trajectory is non-intersecting [16]. Note that, in general, any projection of the strange attractor on a sub-space below its embedding dimension becomes intersecting.

A bifurcation diagram is a visual summary of succession of period-doublings. In a bifurcation diagram, the bifurcation parameter is plotted on the abscissa and the states of the system are plotted on the ordinate. Circuit parameters [8], [11] or feedback loop parameters [17] can be chosen as the bifurcation parameter.

There are several methods to characterize chaos. The largest Lyapunov exponent and the information dimension are among them [16]. The largest Lyapunov exponent and the information dimension for the studied buck converter are 0.64 and 2.21, respectively [18].

## III. The Second-Order Buck Converter

A buck converter is a step-down power electronic converter that converts an unregulated DC voltage to a lower DC voltage regulated by means of closed-loop feedback operation. The circuit diagram of the buck converter is shown in Fig. 1.

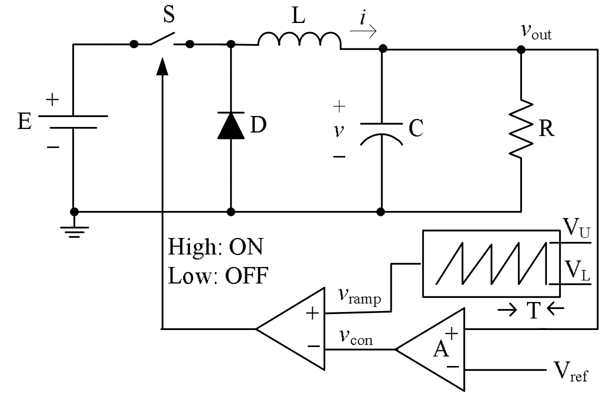


Fig. 1. Schematic of the second-order buck converter with simplified control circuitry.

### *A. Behaviour of the Circuit*

There are two switches in a second-order buck converter. One switch is uncontrolled (diode D) and the other one (S) is controlled by the feedback controller. At any time, only one of these two switches is in the ON state. A capacitor C is connected in parallel with the load to help maintaining a relatively constant load voltage. The series inductor L is used as an energy-storing device. During the ON state of S, energy from the source E is stored in L. When S is open, the inductor delivers the stored energy to the load R.

The feedback loop tries to keep the load voltage, $v_{out}$, constant. The load voltage is measured and passed to the subtractor block to form the error signal, $v_{con}$, which is

$$v_{con} = A\left(v_{out} - V_{ref}\right) \tag{1}$$

where $A$ is the amplification factor. This signal is then compared with a saw-tooth ramp signal with a minimum of $V_L$, maximum of $V_U$ and period of $T$, defined as

$$v_{ramp} = V_L + \left(V_U - V_L\right)\mathrm{mod}(t/T,1) \tag{2}$$

If the magnitude of the saw-tooth signal is greater than that of the error signal $v_{con}$, S is turned ON, otherwise S remains OFF. This means that the switch state changes whenever $v_{con} = v_{ramp}$ is satisfied.

Normally, the load voltage is passed through a low-pass filter (an integrator, which can be realized as a shunt RC circuit) before being fed to the subtractor to reduce its ripple. This filter is neglected here for simplicity.

### *B. Model of the Circuit*

At each instant, the system state is determined by the two state variables $v$ (capacitor voltage) and $i$ (inductor current) as well as the state of switch S. The buck converter can be considered as two circuits multiplexed in time. The differential equations for $v$ and $i$ are

$$\begin{aligned} \frac{dv}{dt} &= -\frac{1}{RC}v(t) + \frac{1}{C}i(t) \\ \frac{di}{dt} &= -\frac{1}{L}v(t) + \frac{\zeta(t)}{L}E \end{aligned} \tag{3}$$

where $\zeta$ is the control signal and is 1 when the switch is ON



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and is 0 when the switch is OFF.

The circuit is simulated using the parameters shown in Table I. Input voltage $E$ is used as the bifurcation parameter and is varied between 15 and 40 V.

TABLE I: BUCK CONVERTER PARAMETERS USED FOR SIMULATION

| | | | | | |
|---|---|---|---|---|---|
| Circuit Parameters | R (Ω) | L (mH) | C (μF) | | |
| | 22 | 20 | 47 | | |
| Controller Parameters | $V_U$ (V) | $V_L$ (V) | T (μs) | $V_{ref}$ (V) | A |
| | 8.2 | 3.8 | 400 | 11.3 | 8.4 |

The differential equations (3) of the circuit are solved in the next section by three methods: the exact closed-form solution, numerical integration, and PSCAD simulation.

## IV. SIMULATION OF THE BUCK CONVERTER

Behaviour of the closed-loop buck converter is analyzed using three methods. First, the piecewise closed-form solution of the system equations is presented [7]. The extreme sensitivity of the circuit is the main incentive for looking for the exact solution of the circuit, so that the round-off error does not propagate from one step to another and the most accurate results can be obtained. The system equations are also solved by numerical integration and by the commercial simulation emtp-type program PSCAD/EMTDC. In all cases, circuit elements are assumed to be ideal.

### *A. Closed-Form Solution*

In this method, (3) is solved for $v(t)$ and $i(t)$ with a constant $\zeta$ and the closed-from solution is obtained. The switching happens whenever the following boundary condition is satisfied

$$v_{con}(t_c) = v_{ramp}(t_c) \tag{4}$$

where $t_c$ is the switching time. Then, the equation is solved using Newton-Raphson method with a maximum allowable error of $10^{-10}$ to find the exact switching time.

For $E$ equal to 24, 28, 32, and 33 V, phase-space plots show period-1, -2, -4, and chaotic behaviour of the system as shown in Fig. 2. Figure 3 shows time-domain waveforms for chaotic operation for $E$ = 33 V. It can be clearly seen that it is possible for $v_{con}$ to skip some cycles (no switching in a cycle) as well as to intersect the ramp voltage more than once in a cycle (multiple switchings in a cycle).

Bifurcation diagram is plotted for input voltage $E$ swept from 15 to 40 V (Fig. 4) and is obtained by recording the voltage at the end of each period. It clearly shows the succession of period doublings.

The separation between period-doubling points decreases with the number of periods. The ratio of successive bifurcation parameters approaches the Feigenbaum number, 4.6692···. This number, which is believed to be transcendental but not yet proved to be so, also arises in many physical systems before they enter the chaotic regime. The abrupt transition from the period-doubling to chaotic region is related to the

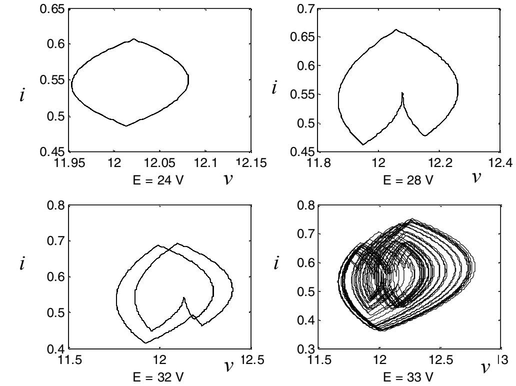


Fig. 2. Phase-space diagram of the buck converter showing period-1 ($E$ = 24 V), period-2 ($E$ = 28 V), period-4 ($E$ = 32 V), and chaotic ($E$ = 33 V) waveforms obtained form the exact solution.

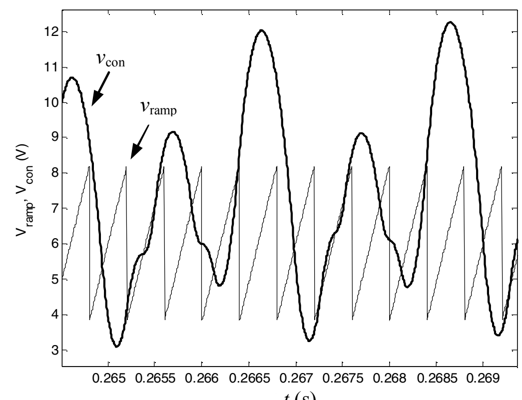


Fig. 3. Chaotic operation of the buck converter obtained by exact solution with $E$ = 33 V (shown are $v_{con}$ and $v_{ramp}$).

sharp, singular points in the phase-space diagram of the converter.

### *B. Numerical Integration*

The equations in (3) are already in the suitable form for computer implementation of numerical integration of the state-space representation. Both Euler's and trapezoidal methods are implemented with a small time-step of 1 μs.

The results are shown in Fig. 5. The discrepancy observed between the results of this method and the exact solution is due to the extensive round-off errors, which are magnified not only by the sensitivity of the circuit to initial conditions, but also by the discretization of time $t_c$, in contrast to the previous approach where Newton-Raphson method is used to find the almost exact $t_c$. That is, a flow has been converted to a map.

### *C. PSCAD/EMTDC Simulation*

The model is also implemented in PSCAD/EMTDC electromagnetic transient simulation program [19]. Figure 6 shows the converter model. The results are used to verify those of numerical integration method and to investigate the effects of limited accuracy used for $t_c$. Taking advantage of the interpolation block in PSCAD [20], $t_c$ is found with an accuracy of 0.01% of time step.

Phase-space diagrams for four input voltages values (24,



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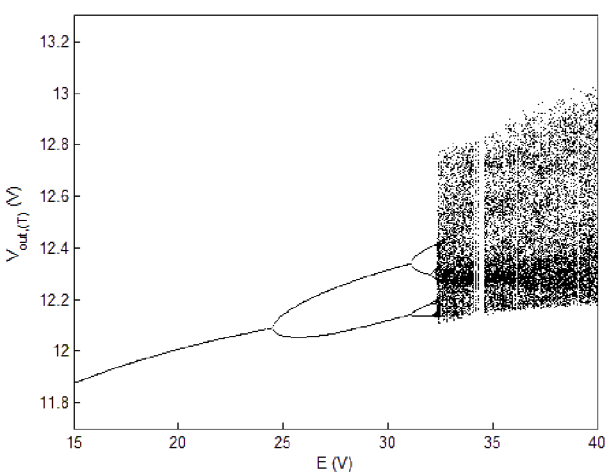


Fig. 4. Bifurcation diagram obtained by sampling the output voltage at the end of each cycle.

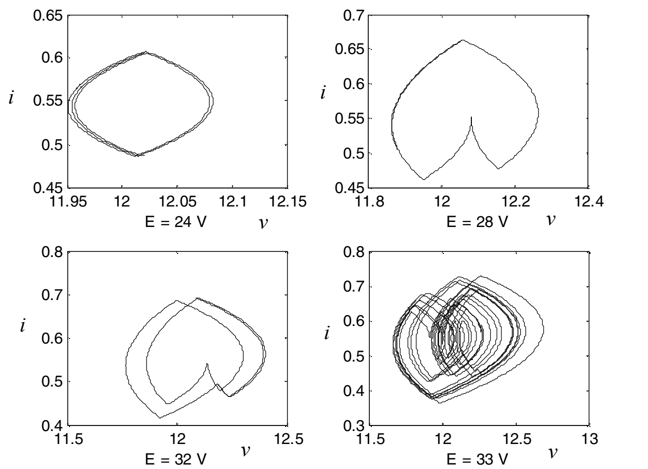


Fig. 5. Phase-space diagram of buck converter showing period-1 ($E$ = 24 V), period-2 ($E$ = 28 V), period-4 ($E$ = 32 V), and chaotic ($E$ = 33 V) waveforms obtained form numerical integration. Note jitters for $E$ = 24 and 32 V.

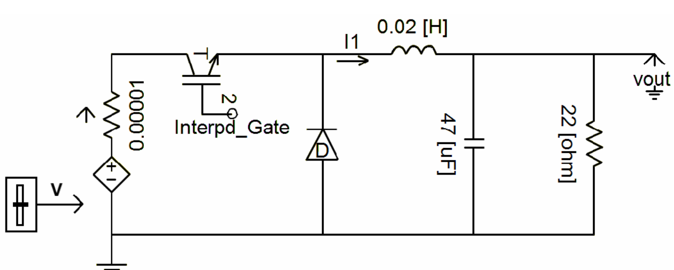


Fig. 6. PSCAD model of the buck converter.

28, 32, and 33V) are shown in Fig. 7, which are quite similar to those of the exact solution in Fig. 2. This is because of the proper selection of time-step as well as approximating the witching instant by interpolation. Figure 8 shows time-domain voltage waveform an input voltage of $E$ = 33 V.

## V. The Largest Lyapunov Exponent

Lyapunov exponent is a quantitative measure of the sensitive dependence of a dynamical system on the initial conditions [21]. It shows the rate of divergence of the system trajectories corresponding to close initial conditions. The number of Lyapunov exponents for a system is equal to the dimension of its phase space. Normally the largest exponent is used, because it determines the horizon of predictability of the system. In this sense, the inverse of the largest Lyapunov exponent is called Lyapunov time, which defines the characteristic folding time of the system.

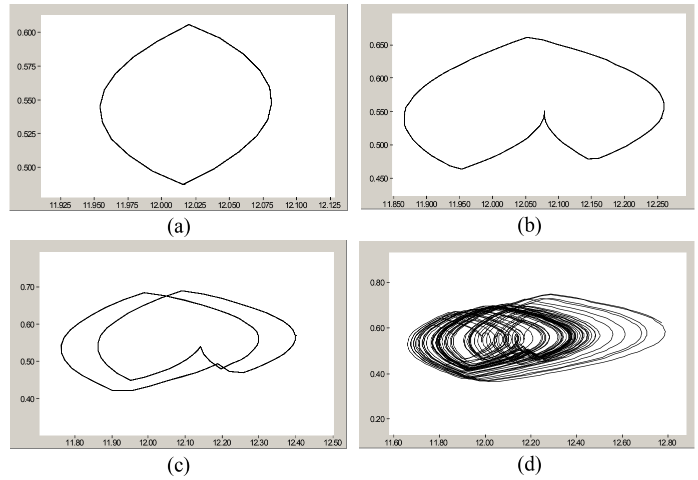


Fig. 7. Phase-space diagrams (output voltage on $x$-axis, inductor current on $y$-axis) of the PSCAD run for (a) $E$ = 24 V showing periodic operation, (b) $E$ = 28 V showing period-2, (c) $E$ = 32 V showing period-4, and (d) $E$ = 33 V showing chaotic operation.

The concept of Lyapunov exponents can be considered as the nonlinear counterpart of eigenvalues for linear systems. As it shows the rate of separation of infinitesimally close trajectories, one can predict the behaviour of the system based on the sign of the Lyapunov exponent.

A negative Lyapunov exponent is characteristic of dissipative (non-conservative) systems, which exhibit point stability. The more negative the exponent, the faster the stability. An exponent of $-\infty$ shows the extremely fast convergence, and hence stability. A Lyapunov exponent of zero is characteristic of a cycle-stable system. In this case, the orbits maintain their separation. A positive Lyapunov exponent, on the other hand, implies that nearby points, no matter how close, will finally diverge to an arbitrary separation. This happens in the case of instable as well as chaotic system. The distinction between these two is made by using the set of Lyapunov exponents.

The largest Lyapunov exponent is defined as

$$\lambda_{\max} = \lim_{\delta\mathbf{x}(0)\to 0}\lim_{t\to\infty}\left(\frac{1}{t}\ln\frac{\|\delta\mathbf{x}(t)\|}{\|\delta\mathbf{x}(0)\|}\right) \tag{5}$$

where $\mathbf{\delta x}(t)$ shows the perturbation of the system.

To overcome the problems in applying the above equation to power electronic circuits [22], an approximate method has been suggested by Müller [23]. This method is used for the buck converter and $\lambda_{\max}$ is calculated from

$$\lambda_{\max} = \frac{1}{(t-t_0)/T}\ln\frac{\|\delta\mathbf{x}(t)\|}{\|\delta\mathbf{x}(t_0)\|} \tag{6}$$



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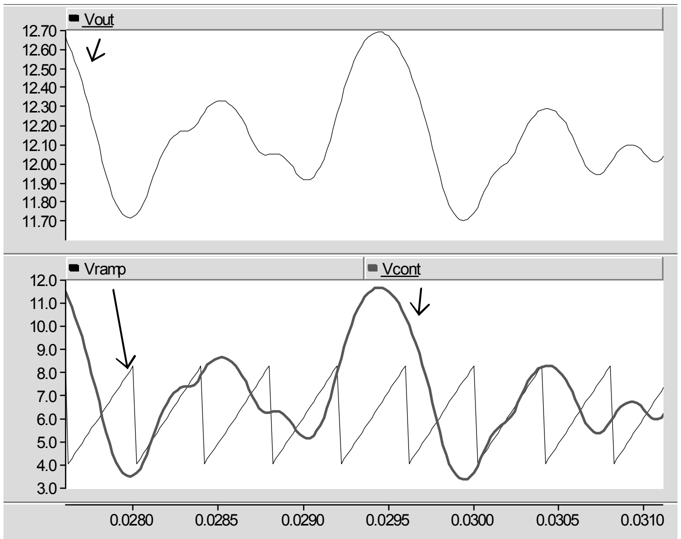

Fig. 8. Plot of output voltage, ramp generator output, and the control voltage vs. time for the chaotic operation with $E = 33$ V.

While for $E = 24$ V, the maximum Lyapunov exponent is $\lambda_{max} = 3\times10^{-4}$ (practically zero) that indicates a stable system, for chaotic region, $E = 33$ V, $\lambda_{max} = 0.68$, which is a positive number, in agreement with [18].

## VI. Conclusions

In this paper, the buck converter and its operation in the chaotic regime is studied using time-domain, phase-space, and bifurcation diagrams, as well as Lyapunov exponents.

The chaotic nature of circuit operation intensifies the need for precise determination of the switching instances. Therefore, three methods (analytical solution, numerical integration, and simulation in the PSCAD/EMTDC program) are used to study the circuit and find the most suitable combination of simplicity of implementation and accuracy of results. Comparing the results, it is found that simulation in PSCAD/EMTDC, being a simulation program primarily developed for study of rapidly changing phenomena, requires less effort, is generally faster, and offers more flexibility in tailoring the model to include complex converter and control circuitry models. This could establish a new and comprehensive platform to study and detect chaos in power electronic circuits.

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