Supplemental Note

"Pricing and Matching with Forward-looking Buyers and Sellers"

A. Auxiliary Properties

This section discusses some useful auxiliary properties that will be used for the analysis in this paper.

LEMMA S.1. *If* (ICd') *and* (IRd) *hold, then for any* ϕ*,*

$$\mathsf{E}_{-\phi}\left[p_{\phi}\right] \leq v_{\phi}\mathsf{E}_{-\phi}\left[m_{\phi}\right] - \int_{v'=\underline{v}}^{v_{\phi}} \mathsf{E}_{-\phi}\left[m_{\phi_{v'}}\right] dv' - b\mathsf{E}_{-\phi}\left[\left(s_{\phi}-t_{\phi}\right)\right].$$

Proof of Lemma S.1. Define $u^{d}(\phi, y_{\phi}) \triangleq \frac{\partial}{\partial v_{\phi}} U^{d}(\phi, y_{\phi})$. Applying the envelope theorem, we have:

$$\begin{aligned}
\mathsf{E}_{-\phi}\left[U^{d}\left(\phi, y_{\phi}\right)\right] &= \int_{v'=\underline{v}}^{v_{\phi}} \mathsf{E}_{-\phi}\left[u^{d}\left(\phi_{v'}, y_{\phi_{v'}}\right)\right] dv' + \mathsf{E}_{-\phi}\left[U^{d}\left(\phi_{\underline{v}}, y_{\phi_{\underline{v}}}\right)\right] \\
&= \int_{v'=\underline{v}}^{v_{\phi}} \mathsf{E}_{-\phi}\left[m_{\phi_{v'}}\right] dv' + \mathsf{E}_{-\phi}\left[U^{d}\left(\phi_{\underline{v}}, y_{\phi_{\underline{v}}}\right)\right] \\
&\geq \int_{v'=\underline{v}}^{v_{\phi}} \mathsf{E}_{-\phi}\left[m_{\phi_{v'}}\right] dv', \qquad \text{(S.1)}
\end{aligned}$$

where the first equality follows from Fubini's theorem and the envelope theorem (specifically, Theorem 2 of Milgrom, P, I Segal. 2002. Envelope theorems for arbitrary choice sets. *Econometrica* **70**(2) 583–601), the second equality follows from the definition of $u^{d}(\cdot)$ and $U^{d}(\phi, y_{\phi}) = v_{\phi}m_{\phi} - p_{\phi} - b\left(s_{\phi}-t_{\phi}\right)$, and the inequality follows from (IRd) for $\phi_{\underline{v}}$. Consequently,

$$\begin{aligned}
\mathsf{E}_{-\phi}\left[p_{\phi}\right] &= v_{\phi}\mathsf{E}_{-\phi}\left[m_{\phi}\right] - \mathsf{E}_{-\phi}\left[U^{d}(\phi, y_{\phi})\right] - b\mathsf{E}_{-\phi}\left[\left(s_{\phi}-t_{\phi}\right)\right] \\
&\leq v_{\phi}\mathsf{E}_{-\phi}\left[m_{\phi}\right] - \int_{v'=\underline{v}}^{v_{\phi}} \mathsf{E}_{-\phi}\left[m_{\phi_{v'}}\right] dv' - b\mathsf{E}_{-\phi}\left[\left(s_{\phi}-t_{\phi}\right)\right],
\end{aligned}$$

where the first equality follows from the definition of $U^{d}(\cdot)$, the first inequality follows from (S.1). □

LEMMA S.2. *If* (ICd') *and* (IRd) *hold, then*

$$\mathsf{E}\left[\sum_{\phi\in H^{T}} p_{\phi}\right] \leq \mathsf{E}\left[\sum_{\phi\in H^{T}} V^{d}\left(v_{\phi}\right) m_{\phi} - b\left(s_{\phi}-t_{\phi}\right)\right].$$

Proof of Lemma S.2. We have that

$$\begin{aligned}\mathsf{E}\left[\sum_{\phi\in H^T} p_\phi\right] &= \mathsf{E}\left[\sum_{\phi\in H^T} \mathsf{E}_{-\phi}\left[p_\phi\right]\right] \\ &\leq \mathsf{E}\left[\sum_{\phi\in H^T} v_\phi \mathsf{E}_{-\phi}\left[m_\phi\right] - \int_{v'=\underline{v}}^{v_\phi} \mathsf{E}_{-\phi}\left[m_{\phi_{v'}}\right] dv' - b\mathsf{E}_{-\phi}\left[(s_\phi - t_\phi)\right]\right],\end{aligned}$$

where the inequality follows from Lemma S.1. We now prove that the right hand side of the above is the desired quantity by changing the order of integration:

$$\begin{aligned}&\mathsf{E}\left[\sum_{\phi\in H^T} v_\phi \mathsf{E}_{-\phi}\left[m_\phi\right] - \int_{v'=\underline{v}}^{v_\phi} \mathsf{E}_{-\phi}\left[m_{\phi_{v'}}\right] dv' - b\mathsf{E}_{-\phi}\left[(s_\phi - t_\phi)\right]\right] \\ &= \mathsf{E}\left[\sum_{\phi\in H^T} \mathsf{E}_{v_\phi}\left[v_\phi \mathsf{E}_{-\phi}\left[m_\phi\right] - \int_{v'=\underline{v}}^{v_\phi} \mathsf{E}_{-\phi}\left[m_{\phi_{v'}}\right] dv'\right] - b\mathsf{E}_{-\phi}\left[(s_\phi - t_\phi)\right]\right] \\ &= \mathsf{E}\left[\sum_{\phi\in H^T} \int_{v_\phi=\underline{v}}^{\bar{v}} \left(v_\phi \mathsf{E}_{-\phi}\left[m_\phi\right] - \int_{v'=\underline{v}}^{v_\phi} \mathsf{E}_{-\phi}\left[m_{\phi_{v'}}\right] dv'\right) f^d(v_\phi) dv_\phi - b\mathsf{E}_{-\phi}\left[(s_\phi - t_\phi)\right]\right] \\ &= \mathsf{E}\left[\sum_{\phi\in H^T} \int_{v_\phi=\underline{v}}^{\bar{v}} \left(v_\phi - \frac{\bar{F}^d(v_\phi)}{f^d(v_\phi)}\right) \mathsf{E}_{-\phi}\left[m_\phi\right] f^d(v_\phi) dv_\phi - b\mathsf{E}_{-\phi}\left[(s_\phi - t_\phi)\right]\right] \\ &= \mathsf{E}\left[\sum_{\phi\in H^T} \mathsf{E}_{v_\phi}\left[\left(v_\phi - \frac{\bar{F}^d(v_\phi)}{f^d(v_\phi)}\right) \mathsf{E}_{-\phi}\left[m_\phi\right]\right] - b\mathsf{E}_{-\phi}\left[(s_\phi - t_\phi)\right]\right] \\ &= \mathsf{E}\left[\sum_{\phi\in H^T} \left(v_\phi - \frac{\bar{F}^d(v_\phi)}{f^d(v_\phi)}\right) m_\phi - b\left(s_\phi - t_\phi\right)\right] \\ &= \mathsf{E}\left[\sum_{\phi\in H^T} V^d\left(v_\phi\right) m_\phi - b\left(s_\phi - t_\phi\right)\right],\end{aligned}$$

where the second and the fourth equalities follow from the fact that v_ϕ is independent of t_ϕ, and the third equality follows from an exchange in the order of integration that

$$\begin{aligned}\int_{v_\phi=\underline{v}}^{\bar{v}} \int_{v'=\underline{v}}^{v_\phi} \mathsf{E}_{-\phi}\left[m_{\phi_{v'}}\right] dv' f^d(v_\phi) dv_\phi &= \int_{v'=\underline{v}}^{\bar{v}} \int_{v_\phi=v'}^{\bar{v}} f^d(v_\phi) dv_\phi \mathsf{E}_{-\phi}\left[m_{\phi_{v'}}\right] dv' \\ &= \int_{v'=\underline{v}}^{\bar{v}} \bar{F}^d(v') \mathsf{E}_{-\phi}\left[m_{\phi_{v'}}\right] dv' \\ &= \int_{v'=\underline{v}}^{\bar{v}} \frac{\bar{F}^d(v')}{f^d(v')} \mathsf{E}_{-\phi}\left[m_{\phi_{v'}}\right] f^d(v') dv' \\ &= \int_{v_\phi=\underline{v}}^{\bar{v}} \frac{\bar{F}^d(v_\phi)}{f^d(v_\phi)} \mathsf{E}_{-\phi}\left[m_\phi\right] f^d(v_\phi) dv_\phi. \quad \square\end{aligned}$$

LEMMA S.3. *If* (ICs') *and* (IRs) *hold, for any* ψ,

$$\mathsf{E}_{-\psi}\left[p_{\psi}\right] \geq c_{\psi}\mathsf{E}_{-\psi}\left[m_{\psi}\right] + \int_{c'=c_{\psi}}^{\bar{c}} \mathsf{E}_{-\psi}\left[m_{\psi_{c'}}\right] dc' + h\mathsf{E}_{-\psi}\left[\left(s_{\psi} - t_{\psi}\right)\right].$$

Proof of Lemma S.3. Define $u^{s}(\psi, y) \triangleq \frac{\partial}{\partial c_{\psi}} U^{s}(\psi, y)$. Applying the envelope theorem, we have:

$$\begin{aligned}
\mathsf{E}_{-\psi}\left[U^{s}\left(\psi, y_{\psi}\right)\right] &= \int_{c'=c_{\psi}}^{\bar{c}} \mathsf{E}_{-\psi}\left[u^{s}\left(\psi_{c'}, y_{\psi_{c'}}\right)\right] dc' + \mathsf{E}_{-\psi}\left[U^{s}\left(\psi_{\bar{c}}, y_{\psi\bar{c}}\right)\right] \\
&= \int_{c'=c_{\psi}}^{\bar{c}} \mathsf{E}_{-\psi}\left[m_{\psi_{c'}}\right] dc' + \mathsf{E}_{-\psi}\left[U^{s}\left(\psi_{\bar{c}}, y_{\psi\bar{c}}\right)\right] \\
&\geq \int_{c'=c_{\psi}}^{\bar{c}} \mathsf{E}_{-\psi}\left[m_{\psi_{c'}}\right] dc', \qquad \text{(S.2)}
\end{aligned}$$

where the first equality follows from Fubini's theorem and the envelope theorem (specifically, Milgrom and Segal 2002, Theorem 2), the second equality follows from the definition of $u^{s}(\cdot)$ and $U^{s}(\cdot)$, and the inequality follows from (IRs) for $\psi_{\bar{c}}$. Consequently,

$$\begin{aligned}
\mathsf{E}_{-\psi}\left[p_{\psi}\right] &= c_{\psi}\mathsf{E}_{-\psi}\left[m_{\psi}\right] + \mathsf{E}_{-\psi}\left[U^{s}(\psi, y_{\psi})\right] + h\mathsf{E}_{-\psi}\left[\left(s_{\psi} - t_{\psi}\right)\right] \\
&\geq c_{\psi}\mathsf{E}_{-\psi}\left[m_{\psi}\right] + \int_{c'=c_{\psi}}^{\bar{c}} \mathsf{E}_{-\psi}\left[m_{\psi_{c'}}\right] dc' + h\mathsf{E}_{-\psi}\left[\left(s_{\psi} - t_{\psi}\right)\right],
\end{aligned}$$

where the first equality follows from the definition of $U^{s}(\cdot)$, the first inequality follows from (S.2). □

LEMMA S.4. *If* (ICs') *and* (IRs) *hold, then*

$$\mathsf{E}\left[\sum_{\psi \in H^{T}} p_{\psi}\right] \geq \mathsf{E}\left[\sum_{\psi \in H^{T}} V^{s}\left(c_{\psi}\right) m_{\psi} + h\left(s_{\psi} - t_{\psi}\right)\right].$$

Proof of Lemma S.4. We have that

$$\begin{aligned}
\mathsf{E}\left[\sum_{\psi \in H^{T}} p_{\psi}\right] &= \mathsf{E}\left[\sum_{\psi \in H^{T}} \mathsf{E}_{-\psi}\left[p_{\psi}\right]\right] \\
&\geq \mathsf{E}\left[\sum_{\psi \in H^{T}} c_{\psi}\mathsf{E}_{-\psi}\left[m_{\psi}\right] + \int_{c'=c_{\psi}}^{\bar{c}} \mathsf{E}_{-\psi}\left[m_{\psi_{c'}}\right] dc' + h\mathsf{E}_{-\psi}\left[\left(s_{\psi} - t_{\psi}\right)\right]\right],
\end{aligned}$$

where the inequality follows from Lemma S.3. We now prove that the right hand side of the above is the desired quantity by changing the order of integration:

$$
\begin{aligned}
&\mathsf{E}\left[\sum_{\psi\in H^T} c_\psi \mathsf{E}_{-\psi}\left[m_\psi\right] + \int_{c'=c_\psi}^{\bar{c}} \mathsf{E}_{-\psi}\left[m_{\psi_{c'}}\right] dc' + h\mathsf{E}_{-\psi}\left[(s_\psi - t_\psi)\right]\right] \\
&= \mathsf{E}\left[\sum_{\psi\in H^T} \mathsf{E}_{c_\psi}\left[c_\psi \mathsf{E}_{-\psi}\left[m_\psi\right] + \int_{c'=c_\psi}^{\bar{c}} \mathsf{E}_{-\psi}\left[m_{\psi_{c'}}\right] dc'\right] + h\mathsf{E}_{-\psi}\left[(s_\psi - t_\psi)\right]\right] \\
&= \mathsf{E}\left[\sum_{\psi\in H^T} \int_{c_\psi=\underline{c}}^{\bar{c}} \left(c_\psi \mathsf{E}_{-\psi}\left[m_\psi\right] + \int_{c'=c_\psi}^{\bar{c}} \mathsf{E}_{-\psi}\left[m_{\psi_{c'}}\right] dc'\right) f^s(c_\psi) dc_\psi + h\mathsf{E}_{-\psi}\left[(s_\psi - t_\psi)\right]\right] \\
&= \mathsf{E}\left[\sum_{\psi\in H^T} \int_{c_\psi=\underline{c}}^{\bar{c}} \left(c_\psi + \frac{F^s(c_\psi)}{f^s(c_\psi)}\right) \mathsf{E}_{-\psi}\left[m_\psi\right] f^s(c_\psi) dc_\psi + h\mathsf{E}_{-\psi}\left[(s_\psi - t_\psi)\right]\right] \\
&= \mathsf{E}\left[\sum_{\psi\in H^T} \mathsf{E}_{c_\psi}\left[\left(c_\psi + \frac{F^s(c_\psi)}{f^s(c_\psi)}\right) \mathsf{E}_{-\psi}\left[m_\psi\right]\right] + h\mathsf{E}_{-\psi}\left[(s_\psi - t_\psi)\right]\right] \\
&= \mathsf{E}\left[\sum_{\psi\in H^T} \left(c_\psi + \frac{F^s(c_\psi)}{f^s(c_\psi)}\right) m_\psi + h\left(s_\psi - t_\psi\right)\right] \\
&= \mathsf{E}\left[\sum_{\psi\in H^T} V^s\left(c_\psi\right) m_\psi + h\left(s_\psi - t_\psi\right)\right],
\end{aligned}
$$

where the second and the fourth equalities follow from the fact that c_ψ is independent of t_ψ, and the third equality follows from an exchange in the order of integration that

$$
\begin{aligned}
\int_{c_\psi=\underline{c}}^{\bar{c}} \int_{c'=c_\psi}^{\bar{c}} \mathsf{E}_{-\psi}\left[m_{\psi_{c'}}\right] dc' f^s(c_\psi) dc_\psi &= \int_{c'=\underline{c}}^{\bar{c}} \int_{c_\psi=0}^{c_\psi} f^s(c_\psi) dc_\psi \mathsf{E}_{-\psi}\left[m_{\psi_{c'}}\right] dc' \\
&= \int_{c'=\underline{c}}^{\bar{c}} F^s(c') \mathsf{E}_{-\psi}\left[m_{\psi_{c'}}\right] dc' \\
&= \int_{c'=\underline{c}}^{\bar{c}} \frac{F^s(c')}{f^s(c')} \mathsf{E}_{-\psi}\left[m_{\psi_{c'}}\right] f^s(c') dc' \\
&= \int_{c_\psi=\underline{c}}^{\bar{c}} \frac{F^s(c_\psi)}{f^s(c_\psi)} \mathsf{E}_{-\psi}\left[m_\psi\right] f^s(c_\psi) dc_\psi. \quad \square
\end{aligned}
$$

LEMMA S.5. *μ^* and $\bar{J}^*$ are increasing in λ^d and λ^s, respectively.*

Proof of Lemma S.5. (1) In this part, we prove that μ^* and $\bar{J}^*$ are increasing in λ^d.

Consider any two market conditions with λ_1^d and λ_2^d, where $\lambda_1^d < \lambda_2^d$. The subscript (1 or 2) in all following notation denotes the index of the market environment that the intermediary operates at. Therefore, we need to show $\mu_2^* \geq \mu_1^*$ and $J_2^* \geq J_1^*$.

The proof of $\mu_2^* \geq \mu_1^*$ is as follows.

For $\mu \in \left[0, \min\left\{\lambda_1^d, \lambda^s\right\}\right] \cap \left[0, \min\left\{\lambda_2^d, \lambda^s\right\}\right] = \left[0, \min\left\{\lambda_1^d, \lambda^s\right\}\right]$, we have

$$V_2(\mu) - V_1(\mu) = V^d\left(\bar{F}^{d,-1}\left(\frac{\mu}{\lambda_2^d T}\right)\right) - V^d\left(\bar{F}^{d,-1}\left(\frac{\mu}{\lambda_1^d T}\right)\right) \geq 0,$$

where the inequality follows from Assumption 1. In addition, Assumptions 1 and 2 imply that $V_i(\mu)$ is decreasing in μ. Therefore, $\mu_2^* \geq \mu_1^*$.

The proof of $J_2^* \geq J_1^*$ is as follows.

Given the optimal prices p_1^* and w_1^* in market environment 1 with λ_1^d, we construct prices p_2 and w_2 for market environment 2 with λ_2^d, where $\lambda_2^d \bar{F}^d(p_2) = \lambda_1^d \bar{F}^d(p_1^*)$ and $w_2 = w_1^*$.

Therefore, $\bar{J}_2^* \geq \lambda_2^d T p_2 \bar{F}^d(p_2) - \lambda^s T w_2 F^s(w_2) = \lambda_1^d T p_2 \bar{F}^d(p_1^*) - \lambda^s T w_1^* F^s(w_1^*) \geq \lambda_1^d T p_1^* \bar{F}^d(p_1^*) - \lambda^s T w_1^* F^s(w_1^*) = \bar{J}_1^*$. The first inequality follows from the property that p_2 and w_2 are feasible but not necessarily optimal solutions to the optimization problem (D). The first equality follows from the definitions of p_2 and w_2. The second inequality holds since the condition that $\lambda_1^d < \lambda_2^d$ implies $p_1^* \leq p_2$.

(2) In this part, we prove that μ^* and $\bar{J}^*$ are increasing in λ^s.

Consider any two market conditions with λ_1^s and λ_2^s, where $\lambda_1^s < \lambda_2^s$. The subscript (1 or 2) in all following notation denotes the index of the market environment that the intermediary operates at. Therefore, we need to show $\mu_2^* \geq \mu_1^*$ and $J_2^* \geq J_1^*$.

The proof of $\mu_2^* \geq \mu_1^*$ is as follows.

For $\mu \in \left[0, \min\left\{\lambda^d, \lambda_1^s\right\}\right] \cap \left[0, \min\left\{\lambda^d, \lambda_2^s\right\}\right] = \left[0, \min\left\{\lambda^d, \lambda_1^s\right\}\right]$, we have

$$V_2(\mu) - V_1(\mu) = -V^s\left(F^{s,-1}\left(\frac{\mu}{\lambda_2^s T}\right)\right) + V^s\left(F^{s,-1}\left(\frac{\mu}{\lambda_1^s T}\right)\right) \geq 0,$$

where the inequality follows from Assumption 2. In addition, Assumptions 1 and 2 imply that $V_i(\mu)$ is decreasing in μ. Therefore, $\mu_2^* \geq \mu_1^*$.

The proof of $J_2^* \geq J_1^*$ is as follows.

Given the optimal prices p_1^* and w_1^* in market environment 1 with λ_1^s, we construct prices p_2 and w_2 for market environment 2 with λ_2^s, where $p_2 = p_1^*$ and $\lambda_2^s F^s(w_2) = \lambda_1^s F^s(w_1^*)$.

Therefore, $\bar{J}_2^* \geq \lambda^d T p_2 \bar{F}^d(p_2) - \lambda_2^s T w_2 F^s(w_2) = \lambda^d T \bar{p}_1 \bar{F}^d(p_1^*) - \lambda_1^s T w_2^* F^s(w_1^*) \geq \lambda^d T p_1^* \bar{F}^d(p_1^*) - \lambda_1^s T w_1^* F^s(w_1^*) = \bar{J}_1^*$. The first inequality follows from the property that p_2 and w_2 are feasible but not necessarily optimal solutions to the optimization problem (D). The first equality follows from the definitions of p_2 and w_2. The second inequality holds since the condition that $\lambda_1^d > \lambda_2^d$ implies $w_1^* \leq w_2$. Therefore, $\bar{J}^*$ is increasing in λ^s. $\square$

B. Proofs for §5

Proof of Proposition 1. Note that $\bar{F}^d(\cdot)$ has the inverse $\bar{F}^{d,-1}(\cdot)$ and $F^s(\cdot)$ has the inverse $F^{s,-1}(\cdot)$. Then we can prove the result in the quantile space. Define $q_t^d \triangleq \bar{F}^d\left(\hat{\pi}_t^d\right)$ and $R(q_t^d) \triangleq \hat{\pi}_t^d \bar{F}^d\left(\hat{\pi}_t^d\right)$. Then

$$\begin{aligned} R'\left(q_t^d\right) &= \frac{\mathrm{d}\hat{\pi}_t^d}{\mathrm{d}q_t^d}\bar{F}^d\left(\hat{\pi}_t^d\right) - \hat{\pi}_t^d f^d\left(\hat{\pi}_t^d\right)\frac{\mathrm{d}\hat{\pi}_t^d}{\mathrm{d}q_t^d} = -\left(\hat{\pi}_t^d - \frac{\bar{F}^d\left(\hat{\pi}_t^d\right)}{f^d\left(\hat{\pi}_t^d\right)}\right) f^d\left(\hat{\pi}_t^d\right)\frac{\mathrm{d}\hat{\pi}_t^d}{\mathrm{d}q_t^d} \\ &= \left(\hat{\pi}_t^d - \frac{\bar{F}^d\left(\hat{\pi}_t^d\right)}{f^d\left(\hat{\pi}_t^d\right)}\right)\frac{\mathrm{d}\bar{F}^d\left(\hat{\pi}_t^d\right)}{\mathrm{d}q_t^d} = \hat{\pi}_t^d - \frac{\bar{F}^d\left(\hat{\pi}_t^d\right)}{f^d\left(\hat{\pi}_t^d\right)}. \end{aligned} \tag{S.3}$$

Hence, Assumption 1 implies that $R(q)$ is concave in q, since $\hat{\pi}_t^d$ decreases in q_t^d. Similarly, define $q_t^s \triangleq F^s\left(\hat{\pi}_t^s\right)$ and $C(q_t^s) \triangleq \hat{\pi}_t^s F^s\left(\hat{\pi}_t^s\right)$. Analogously, we can show that Assumption 2 implies that $C(q)$ is convex in q.

Optimization problem (D) is equivalent to the following optimization problem:

$$\begin{aligned} \max_{\{q_t^d, q_t^s \in [0,1]: \forall t \in [0,T]\}} & \int_0^T \lambda^d R(q_t^d)dt - \int_0^T \lambda^s C(q_t^s)dt \\ \text{s.t.} \quad & \lambda^d q_t^d = \lambda^s q_t^s, \quad \forall\, t \in [0,T]. \end{aligned}$$

The optimization problem above is equivalent to the following optimization problem

$$\begin{aligned} \max_{\{q^d, q^s \in [0,1]\}} & \; T\left(\lambda^d R(q^d) - \lambda^s C(q^s)\right) \\ \text{s.t.} \quad & \lambda^d T q^d = \lambda^s T q^s. \end{aligned} \tag{S.4}$$

To compute the optimal solution to the optimization problem (S.4), denoted as $(q^{*,d}, q^{*,s})$, we define a new variable $\mu \triangleq \lambda^d T q^d$. Now, we can write the optimization problem (S.4) in the following tractable form with respect to μ_t:

$$\begin{aligned} \max \quad & T\left(\lambda^d R\left(\frac{\mu}{\lambda^d T}\right) - \lambda^s C\left(\frac{\mu}{\lambda^s T}\right)\right) \\ \text{s.t.} \quad & \mu \in \left[0, \min\left\{\lambda^d T, \lambda^s T\right\}\right]. \end{aligned} \tag{S.5}$$

In the optimization problem (S.5), for the objective function, we have

$$\frac{d}{d\mu} T\left(\lambda^d R\left(\frac{\mu}{\lambda^d T}\right) - \lambda^s C\left(\frac{\mu}{\lambda^s T}\right)\right) = R'\left(\frac{\mu}{\lambda^d T}\right) - C'\left(\frac{\mu}{\lambda^s T}\right) = V(\mu).$$

Recall that $R(q)$ is concave in q and $C(q)$ is convex in q. Hence,

$$\frac{d^2}{d\mu^2}T\left(\lambda^d R\left(\frac{\mu}{\lambda^d T}\right)-\lambda^s C\left(\frac{\mu}{\lambda^s T}\right)\right)\leq 0.$$

In addition, we notice that

$$\frac{d}{d\mu}T\left(\lambda^d R\left(\frac{\mu}{\lambda^d T}\right)-\lambda^s C\left(\frac{\mu}{\lambda^s T}\right)\right)\Big|_{\mu=0}=R'(0)-C'(0)=\bar{v}-\underline{c}\geq 0.$$

Therefore, the optimal solution to the optimization problem (S.5), denoted as μ^*, is given by the following equation:

$$\begin{aligned}\mu^* &= \max\left\{\mu\in\left[0,\min\left\{\lambda^d T,\lambda^s T\right\}\right]:\frac{d}{d\mu}T\left(\lambda^d R\left(\frac{\mu}{\lambda^d T}\right)-\lambda^s C\left(\frac{\mu}{\lambda^s T}\right)\right)\geq 0\right\}\\ &= \max\left\{\mu\in\left[0,\min\left\{\lambda^d T,\lambda^s T\right\}\right]:V(\mu)\geq 0\right\}.\end{aligned}$$

Therefore, the optimal value of the optimization problems (D) and (S.4) is $\bar{J}^*=(p^*-w^*)\mu^*$.

Next, we prove $p^*\geq w^*$. This result immediately follows from properties that

$$p^*\geq p^*-\frac{\bar{F}^d(p^*)}{f^d(p^*)}\geq w^*+\frac{F^s(w^*)}{f^s(w^*)}\geq w^*,$$

where the second inequality follows from Equation (2). □

Proof of Lemma 2. Consider the optimal solution of optimization problem (B), $\left\{x^*_{\phi,\psi}:\phi,\psi\in H^T\right\}$. Define $m^*_\phi\triangleq\sum_{\psi\in H^T}x^*_{\phi,\psi}$ and $m^*_\psi\triangleq\sum_{\phi\in H^T}x^*_{\phi,\psi}$. Hence, $m^*_\phi,m^*_\psi\in\{0,1\}$. We have

$$\begin{aligned}\mathsf{E}\left[\bar{J}\left(H^T\right)\right] &= \mathsf{E}\left[\sum_{\phi,\psi\in H^T}\left(V^d(v_\phi)-V^s(c_\psi)-b(t_\psi-t_\phi)^+-h(t_\phi-t_\psi)^+\right)x^*_{\phi\psi}\right]\\ &\leq \mathsf{E}\left[\sum_{\phi,\psi\in H^T}\left(V^d(v_\phi)-V^s(c_\psi)\right)x^*_{\phi\psi}\right]\\ &= \mathsf{E}\left[\sum_{\phi,\psi\in H^T}\left(V^d(v_\phi)-\eta-V^s(c_\psi)+\eta\right)x^*_{\phi\psi}\right]\\ &= \mathsf{E}\left[\sum_{\phi\in H^T}\left(V^d(v_\phi)-\eta\right)m^*_\phi-\sum_{\psi\in H^T}\left(V^s(c_\psi)-\eta\right)m^*_\psi\right]\\ &= \mathsf{E}\left[\sum_{\phi\in H^T}\left(v_\phi-\frac{\bar{F}^d(v_\phi)}{f^d(v_\phi)}-\eta\right)m^*_\phi-\sum_{\psi\in H^T}\left(c_\psi+\frac{F^s(c_\psi)}{f^s(c_\psi)}-\eta\right)m^*_\psi\right]\end{aligned}$$

$$\leq \mathsf{E}\left[\sum_{\phi\in H^T}\left(v_\phi - \frac{\bar{F}^d(v_\phi)}{f^d(v_\phi)} - \eta\right)^+ + \sum_{\psi\in H^T}\left(\eta - c_\psi - \frac{F^s(c_\psi)}{f^s(c_\psi)}\right)^+\right]$$
$$\triangleq \bar{J}^{*,\eta}.$$

The first inequality follows from the property that $b, h \geq 0$. The second equality holds for any $\eta \in \mathbb{R}$. The fourth equality follows from the definitions of $V^d(v_\phi)$ and $V^s(c_\psi)$. The second inequality follows from the property that $m^*_\phi, m^*_\psi \in \{0,1\}$.

Define

$$g^d(\eta) \triangleq \max\left\{v \in [\underline{v}, \bar{v}] : v - \frac{\bar{F}^d(v)}{f^d(v)} \leq \eta\right\}.$$

Following from the property that $\frac{d}{dv}\left(\bar{F}^d(v)(v-\eta)\right) = -f^d(v)\left(v - \frac{\bar{F}^d(v)}{f^d(v)} - \eta\right)$ and Assumption 1, we have

$$g^d(\eta) \in \arg\max_{v\in[\underline{v},\bar{v}]} \bar{F}^d(v)(v-\eta).$$

Define

$$g^s(\eta) \triangleq \max\left\{c \in [\underline{c}, \bar{c}] : c + \frac{F^s(c)}{f^s(v)} \leq \eta\right\}.$$

Following from the property that $\frac{d}{dv}\left(F^s(c)(c-\eta)\right) = f^s(c)\left(c + \frac{F^s(c)}{f^s(c)} - \eta\right)$ and Assumption 2, we have

$$g^s(\eta) \in \arg\min_{c\in[\underline{c},\bar{c}]} F^s(c)(c-\eta).$$

We have

$$\begin{aligned}
\bar{J}^{*,\eta} &= \lambda^d T\ \mathsf{E}\left[\left(v_\phi - \frac{\bar{F}^d(v_\phi)}{f^d(v_\phi)} - \eta\right)^+\right] + \lambda^s T\ \mathsf{E}\left[\left(\eta - c_\psi - \frac{F^s(c_\psi)}{f^s(c_\psi)}\right)^+\right] \\
&= \lambda^d T \int_{v=g^d(\eta)}^{\bar{v}} \left(v - \frac{\bar{F}^d(v)}{f^d(v)} - \eta\right) f^d(v)dv - \lambda^s T \int_{c=\underline{c}}^{g^s(\eta)} \left(c + \frac{F^s(c)}{f^s(c)} - \eta\right) f^s(c)dc \\
&= \lambda^d T \bar{F}^d\left(g^d(\eta)\right)\left(g^d(\eta) - \eta\right) - \lambda^s T F^s\left(g^s(\eta)\right)\left(g^s(\eta) - \eta\right) \qquad \text{(S.6)} \\
&= \max_{v\in[\underline{v},\bar{v}], c\in[\underline{c},\bar{c}]} T\left(\lambda^d \bar{F}^d(v)\left(v-\eta\right) - \lambda^s F^s(c)\left(c-\eta\right)\right). \qquad \text{(S.7)}
\end{aligned}$$

The first equality is due to Wald's identity. The second equality follows from the definitions of $g^d(\eta)$ and $g^s(\eta)$. The third equality is due to $\int_{v=g^d(\eta)}^{\bar{v}}(v - \frac{\bar{F}^d(v)}{f^d(v)})f^d(v)dv = \int_{v=g^d(\eta)}^{\bar{v}}(vf^d(v) - \bar{F}^d(v))dv = \int_{v=g^d(\eta)}^{\bar{v}} vf^d(v)dv - v\bar{F}^d(v))|_{v=g^d(\eta)}^{\bar{v}} - \int_{v=g^d(\eta)}^{\bar{v}} vf^d(v)dv = g^d(\eta)\bar{F}^d(g^d(\eta))$ by integration by parts, and analogously, $\int_{c=\underline{c}}^{g^s(\eta)}(c + \frac{F^s(c)}{f^s(c)})f^s(c)dc = g^s(\eta)F^s(g^s(\eta))$. The fifth equality follows from properties that $g^d(\eta) \in \arg\max_{v\in[\underline{v},\bar{v}]} \bar{F}^d(v)(v-\eta)$ and $g^s(\eta) \in \arg\min_{c\in[\underline{c},\bar{c}]} F^s(c)(c-\eta)$.

By setting $q^d \triangleq \bar{F}^d(v)$, $R(q^d) \triangleq v\bar{F}^d(v)$, $q^s \triangleq F^s(c)$, and $C(q^s) \triangleq cF^s(c)$, Equation (S.7) can be written in the following way:

$$\max_{q^d, q^s \in [0,1]} T\left(\lambda^d \left(R(q^d) - \eta q^d\right) - \lambda^s \left(C(q^s) - \eta q^s\right)\right). \tag{S.8}$$

Optimization problem (S.8) is a Lagrangian relaxation of the optimization problem (S.4). Recall from the proof of Proposition 1 that the objective function in (S.8) is concave in (q^d, q^s). In addition, the solution $(q^d, q^s) = \left(\frac{\lambda^s}{\lambda^d+\lambda^s}, \frac{\lambda^d}{\lambda^d+\lambda^s}\right)$ satisfies conditions that $q^d, q^s \in (0,1)$ and $\lambda^d T q^d = \lambda^s T q^s$. Therefore, Slater's theorem (strong duality theorem) implies

$$\min_{\eta \in \mathbb{R}} \bar{J}^{*,\eta} = \bar{J}^*.$$

Therefore,

$$\mathsf{E}\left[\bar{J}\left(H^T\right)\right] \leq \bar{J}^*.$$

□

Proof of Theorem 1. (i) Under policy π^{WFP}, given that all other buyers $\hat{\phi} \neq \phi$ and all sellers behave myopically, buyer ϕ's best response stopping rule $\tau_\phi^{\pi^{\mathrm{WFP}}, M^{\mathrm{g}}}$ and purchasing rule $a_\phi^{\pi^{\mathrm{WFP}}, M^{\mathrm{g}}}$ can be calculated by solving the following optimization problem:

$$\sup_{\substack{\tau_\phi \in [t_\phi, T] \\ a_\phi \in \{0,1\}}} \mathsf{E}\left[U^d\left(\phi, y_\phi\right) \middle| \pi_{t_\phi}^{\mathrm{WFP},d}, \left(I_{t_\phi -}\right)^+, \phi\right].$$

Denote by y_ϕ^m buyer ϕ's myopic policy, where $\tau_\phi^m = t_\phi$ and $a_\phi^m = \mathbf{1}\{v_\phi \geq p^*\}$. Consider any y_ϕ with $\tau_\phi \in [t_\phi, T]$ and $a_\phi \in \{0,1\}$. We have

$$\begin{aligned}
&\mathsf{E}\left[U^d\left(\phi, y_\phi\right) \middle| \pi_{t_\phi}^{\mathrm{WFP},d}, \left(I_{t_\phi -}\right)^+, \phi\right] = \mathsf{E}\left[v_\phi m_\phi - p_\phi - b\left(s_\phi - t_\phi\right) \middle| \pi_{t_\phi}^{\mathrm{WFP},d}, \left(I_{t_\phi -}\right)^+, \phi\right] \\
&= \mathsf{E}\left[\left(v_\phi - \pi_{\tau_\phi}^{\mathrm{WFP},d}\right) m_\phi - b\left(s_\phi - t_\phi\right) \middle| \pi_{t_\phi}^{\mathrm{WFP},d}, \left(I_{t_\phi -}\right)^+, \phi\right] = \mathsf{E}\left[\left(v_\phi - p^*\right) m_\phi - b\left(\tau_\phi - t_\phi\right) \middle| \pi_{t_\phi}^{\mathrm{WFP},d}, \left(I_{t_\phi -}\right)^+, \phi\right] \\
&\leq \mathsf{E}\left[\left(v_\phi - p^*\right)^+ m_\phi - b\left(\tau_\phi - t_\phi\right) \middle| \pi_{t_\phi}^{\mathrm{WFP},d}, \left(I_{t_\phi -}\right)^+, \phi\right] \leq \mathsf{E}\left[\left(v_\phi - p^*\right)^+ m_\phi \middle| \pi_{t_\phi}^{\mathrm{WFP},d}, \left(I_{t_\phi -}\right)^+, \phi\right] \\
&= \mathsf{E}\left[U^d\left(\phi, y_\phi^m\right) \middle| \pi_{t_\phi}^{\mathrm{WFP},d}, \left(I_{t_\phi -}\right)^+, \phi\right],
\end{aligned}$$

where the second inequality follows from the greedy matching policy that if $\tau_{\hat{\phi}} > \tau_\phi$ and $a_{\hat{\phi}} = a_\phi = 1$, then $m_{\hat{\phi}} \leq m_\phi$, and the property that $\tau_\phi \geq t_\phi$.

Therefore, buyer ϕ's best response is $\tau_\phi^{\pi^{\mathrm{WFP}},M^{\mathrm{g}}} = t_\phi$ and $a_\phi^{\pi^{\mathrm{WFP}},M^{\mathrm{g}}} = \mathbf{1}\{v_\phi \geq p^*\}$.

(ii) The proof is analogous to part (i), expect that the expectations are not conditional on $(I_{t-})^+$. $\square$

Proof of Proposition 2. (a) We denote $N_t^d \triangleq \sum_{\phi\in H^t} \mathbf{1}\{v_\phi \geq p^*\}$ and $N_t^s \triangleq \sum_{\psi\in H^t} \mathbf{1}\{c_\psi \leq w^*\}$. Hence, N_t^d is a Poisson random variable with parameter $\lambda^d t \bar{F}^d(p^*) = \mu^* \frac{t}{T}$, and N_t^s is a Poisson random variable with parameter $\lambda^s t F^s(w^*) = \mu^* \frac{t}{T}$.

First, we analyze the price process on the demand side. For the unmatched supply-demand quantity $I_{t-} = N_{t-}^s - N_{t-}^d$, we have $\mathsf{E}[I_{t-}] = \mathsf{E}\left[N_{t-}^s\right] - \mathsf{E}\left[N_{t-}^d\right] = 0$ and $\mathrm{Var}[I_{t-}] = \mathrm{Var}\left[N_{t-}^s\right] + \mathrm{Var}\left[N_{t-}^d\right] = \frac{2\mu^* t}{T}$. Therefore, for any $k \in [0,1)$, we have

$$\begin{aligned}
\Pr\left(I_{t-} > -\frac{1}{2}\mu^* \min\left\{k, 1-\frac{t}{T}\right\} \middle| I_{t-} \leq 0\right) &= 1 - \Pr\left(I_{t-} \leq -\frac{1}{2}\mu^* \min\left\{k, 1-\frac{t}{T}\right\} \middle| I_{t-} \leq 0\right) \\
&= 1 - \frac{\Pr\left(I_{t-} \leq -\frac{1}{2}\mu^* \min\left\{k, 1-\frac{t}{T}\right\}\right)}{\Pr(I_{t-} \leq 0)} \\
&\geq 1 - 2\Pr\left(I_{t-} \leq -\frac{1}{2}\mu^* \min\left\{k, 1-\frac{t}{T}\right\}\right) \\
&\geq 1 - \frac{16t}{\mu^* \min\left\{k^2, \left(1-\frac{t}{T}\right)^2\right\} T} \\
&\geq 1 - \frac{16}{\mu^* \min\left\{k^2, \left(1-\frac{t}{T}\right)^2\right\}}.
\end{aligned}$$

The first inequality follows from the symmetry property that for any $n \in \mathbb{N}$, $\mathbb{P}(I_{t-} = n) = \mathbb{P}(I_{t-} = -n)$. Thus, $\mathbb{P}(I_{t-} \leq 0) \geq \frac{1}{2}$. The second inequality follows from Chebyshev's inequality. The third inequality follows from the property that $t \leq T$.

Consider any $k \in [0,1)$. Define $\Delta N_t^s \triangleq N^s_{\min\{t+kT,T\}} - N_{t-}^s$. Hence, $\mathsf{E}[\Delta N_t^s] = \mathrm{Var}[\Delta N_t^s] = \mu^* \min\left\{k, 1-\frac{t}{T}\right\}$. Therefore, we have

$$\begin{aligned}
\Pr\left(\Delta N_t^s \geq \frac{1}{2}\mu^* \min\left\{k, 1-\frac{t}{T}\right\}\right) &\geq 1 - \frac{\mu^* \min\left\{k, 1-\frac{t}{T}\right\}}{\left(\mu^* \min\left\{k, 1-\frac{t}{T}\right\} - \frac{1}{2}\mu^* \min\left\{k, 1-\frac{t}{T}\right\}\right)^2} \\
&\geq 1 - \frac{\mu^*}{\left(\mu^* \min\left\{k, 1-\frac{t}{T}\right\} - \frac{1}{2}\mu^* \min\left\{k, 1-\frac{t}{T}\right\}\right)^2} \\
&= 1 - \frac{4}{\mu^* \min\left\{k^2, \left(1-\frac{t}{T}\right)^2\right\}}.
\end{aligned}$$

The first inequality follows from Chebyshev's inequality. The second inequality follows from the property that $\min\left\{k, 1-\frac{t}{T}\right\} \leq 1$.

For any $t \in [0,T)$, define $\mathcal{A}_t \triangleq \left\{I_{t-} \in \left(-\frac{1}{2}\mu^* \min\left\{k, 1-\frac{t}{T}\right\}, 0\right] \text{ and } \Delta N_t^s \geq \frac{1}{2}\mu^* \min\left\{k, 1-\frac{t}{T}\right\}\right\}$

and $\mathcal{A}_t^c \triangleq \left\{ I_{t-} \leq -\frac{1}{2}\mu^* \min\left\{k, 1-\frac{t}{T}\right\} \text{ or } \Delta N_t^s < \frac{1}{2}\mu^* \min\left\{k, 1-\frac{t}{T}\right\}\right\}$. Hence,

$$\begin{aligned}
\Pr\left(\mathcal{A}_t^c | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0\right) &= 1 - \Pr\left(\mathcal{A}_t | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0\right) \\
&= 1 - \Pr\left(I_{t-} > -\frac{1}{2}\mu^* \min\left\{k, 1-\frac{t}{T}\right\} \middle| I_{t-} \leq 0 \right) \\
&\quad \cdot \Pr\left(\Delta N_t^s \geq \frac{1}{2}\mu^* \min\left\{k, 1-\frac{t}{T}\right\} \right) \\
&\leq 1 - \left(1 - \frac{16}{\mu^* \min\left\{k^2, \left(1-\frac{t}{T}\right)^2\right\}}\right)\left(1 - \frac{4}{\mu^* \min\left\{k^2, \left(1-\frac{t}{T}\right)^2\right\}}\right) \\
&\leq \frac{20}{\mu^* \min\left\{k^2, \left(1-\frac{t}{T}\right)^2\right\}}.
\end{aligned}$$

In addition, we notice that $\Pr\left(\mathcal{A}^c | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0\right) \leq 1$. Therefore, $\Pr\left(\mathcal{A}^c | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0\right) \leq \min\left\{ \frac{20}{\mu^* \min\left\{k^2, \left(1-\frac{t}{T}\right)^2\right\}}, 1 \right\}$.

Therefore, for any $t \in [0, T)$, we have

$$\begin{aligned}
&\mathsf{E}\left[s_\phi - t_\phi | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0\right] \\
=\ &\mathsf{E}\left[s_\phi - t_\phi | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0, \mathcal{A}_t\right] \cdot \Pr\left(\mathcal{A}_t | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0\right) \\
&\quad + \mathsf{E}\left[s_\phi - t_\phi | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0, \mathcal{A}_t^c\right] \cdot \Pr\left(\mathcal{A}_t^c | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0\right) \\
\leq\ &\min\left\{kT, T-t\right\} \cdot 1 + T \cdot \Pr\left(\mathcal{A}^c | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0\right) \\
\leq\ &\min\left\{kT, T-t\right\} + T \min\left\{ \frac{20}{\mu^* \min\left\{k^2, \left(1-\frac{t}{T}\right)^2\right\}}, 1 \right\}.
\end{aligned}$$

For any $t \in [0, T)$, we have

$$\begin{aligned}
\Pr\left(m_\phi = 1 | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0\right) &\geq \Pr\left(\mathcal{A}_t | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0\right) \\
&= 1 - \Pr\left(\mathcal{A}_t^c | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0\right) \\
&\geq 1 - \min\left\{ \frac{20}{\mu^* \min\left\{k^2, \left(1-\frac{t}{T}\right)^2\right\}}, 1 \right\}.
\end{aligned}$$

Therefore,

$$\frac{\mathsf{E}\left[s_\phi - t_\phi | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0\right]}{\Pr\left(m_\phi = 1 | t_\phi = t, v_\phi \geq p^*, I_{t-} \leq 0\right)} \leq \frac{\min\left\{kT, T-t\right\} + T \min\left\{ \frac{20}{\mu^* \min\left\{k^2, (1-\frac{t}{T})^2\right\}}, 1 \right\}}{1 - \min\left\{ \frac{20}{\mu^* \min\left\{k^2, (1-\frac{t}{T})^2\right\}}, 1 \right\}}.$$

In addition, for any $I_{t-}>0$ and buyer ϕ with $t_\phi=t$ and $v_\phi\geq p^*$, we have $s_\phi=t_\phi$ and $m_\phi=1$. Hence, for any $I_{t-}>0$,

$$\frac{\mathsf{E}\left[s_\phi-t_\phi|t_\phi=t,v_\phi\geq p^*,I_{t-}\right]}{\Pr\left(m_\phi=1|t_\phi=t,v_\phi\geq p^*,I_{t-}\right)}=\frac{0}{1}=0.$$

Therefore, for any $t\in[0,T)$,

$$\begin{aligned}\mathsf{E}_{I_{t-}}\left[p^*-\pi_t^{\mathrm{WFP},d}\right]&=\mathsf{E}_{I_{t-}}\left[p^*-\pi_t^{\mathrm{WFP},d}|I_{t-}\leq 0\right]\cdot\mathbb{P}\left(I_{t-}\leq 0\right)\\&\quad+\mathsf{E}_{I_{t-}}\left[p^*-\pi_t^{\mathrm{WFP},d}|I_{t-}>0\right]\cdot\mathbb{P}\left(I_{t-}>0\right)\\&\leq b\frac{\min\left\{kT,T-t\right\}+T\min\left\{\frac{20}{\mu^*\min\{k^2,(1-\frac{t}{T})^2\}},1\right\}}{1-\min\left\{\frac{20}{\mu^*\min\{k^2,(1-\frac{t}{T})^2\}},1\right\}}.\end{aligned}$$

As a special case, when $b=0$, we immediately have $\pi_t^{\mathrm{WFP},d}=p^*$.

By doing the similar analysis for the price dynamics on the supply side, $\{\pi^{\mathrm{WFP},s}\}$, except that the probability and expectation are no loner conditional on I_{t-}, for any $t\in[0,T)$, we have

$$\pi_t^{\mathrm{WFP},s}-w^*\leq h\frac{\min\left\{kT,T-t\right\}+T\min\left\{\frac{20}{\mu^*\min\{k^2,(1-\frac{t}{T})^2\}},1\right\}}{1-\min\left\{\frac{20}{\mu^*\min\{k^2,(1-\frac{t}{T})^2\}},1\right\}}.$$

As a special case, when $h=0$, we immediately have $\pi_t^{\mathrm{WFP},s}=w^*$.

(b) Now, we prove the asymptotic result. Following from Equations (1) and (2) and Lemma S.5 that μ^* is increasing in λ^d and λ^s, respectively, we have $\mu^{*,(n)}\geq n^{\underline{\alpha}}\mu^*$. By setting $k^{(n)}=\frac{k}{n^{\underline{\alpha}/3}}$, we have that for any $t\in[0,T)$,

$$\begin{aligned}\limsup_{n\to\infty}\mathsf{E}_{I_{t-}}\left[p^{*,(n)}-\pi_t^{\mathrm{WFP},d,(n)}\right]&\leq\frac{bT}{n^{\underline{\alpha}/3}}\left(k+\frac{20}{\mu^*k^2}\right)=O\left(\frac{1}{n^{\underline{\alpha}/3}}\right),\\\limsup_{n\to\infty}w^{*,(n)}-\pi_t^{\mathrm{WFP},s,(n)}&\leq\frac{hT}{n^{\underline{\alpha}/3}}\left(k+\frac{20}{\mu^*k^2}\right)=O\left(\frac{1}{n^{\underline{\alpha}/3}}\right).\quad\square\end{aligned}$$

Proof of Theorem 2. The first inequality immediately follows from Lemma 2. Now, we prove the second inequality. Under the waiting adjusted FP policy π^{WFP}, we denote by $N_t^d\triangleq\sum_{\phi\in H^t}\mathbf{1}\left\{v_\phi\geq p^*\right\}$ the number of buyers who arrive no later than time t and request to buy the product, and $N_t^s\triangleq\sum_{\psi\in H^t}\mathbf{1}\left\{c_\psi\leq w^*\right\}$ the number of sellers who arrive no later than time t and request to sell the product. Hence, N_t^d is a Poisson random variable with parameter $\lambda^d t\bar{F}^d\left(p^*\right)=\mu^*\frac{t}{T}$, and N_t^s is a Poisson random variable with parameter $\lambda^s tF^s\left(w^*\right)=\mu^*\frac{t}{T}$. We denote by N_t the Poisson random variable with parameter $\mu^*\frac{t}{T}$.

Therefore, under the waiting adjusted FP policy π^{WFP} and the greedy policy M^{g}, we have

$$
\begin{aligned}
J^{\pi^{\mathrm{WFP}},M^{\mathrm{g}}} &= \mathsf{E}\left[\sum_{\phi\in H^T}\pi^{\mathrm{WFP},d}_{t_\phi}\mathbf{1}\left\{v_\phi\geq p^*\right\}\mathbf{1}\left\{m_\phi=1\right\}-\sum_{\psi\in H^T}\pi^{\mathrm{WFP},s}_{t_\psi}\mathbf{1}\left\{c_\psi\leq w^*\right\}\mathbf{1}\left\{m_\psi=1\right\}\right]\\
&= (p^*-w^*)\,\mathsf{E}\left[\min\left\{N_T^d,N_T^s\right\}\right]\\
&\quad -\mathsf{E}\Bigg[b\sum_{\phi\in H^T}\frac{\mathsf{E}\left[s_\phi-t_\phi|t_\phi,v_\phi\geq p^*,\left(I_{t_\phi-}\right)^+\right]}{\Pr\left(m_\phi=1|t_\phi,v_\phi\geq p^*,\left(I_{t_\phi-}\right)^+\right)}\mathbf{1}\left\{v_\phi\geq p^*\right\}\mathbf{1}\left\{m_\phi=1\right\}\\
&\quad +h\sum_{\psi\in H^T}\frac{\mathsf{E}\left[s_\psi-t_\psi|t_\psi,c_\psi\leq w^*\right]}{\Pr\left(m_\psi=1|t_\psi,c_\psi\leq w^*\right)}\mathbf{1}\left\{c_\psi\leq w^*\right\}\mathbf{1}\left\{m_\psi=1\right\}\Bigg]\\
&= (p^*-w^*)\,\mathsf{E}\left[\min\left\{N_T^d,N_T^s\right\}\right]\\
&\quad -\mathsf{E}\Bigg[b\sum_{\phi\in H^T}\frac{\mathsf{E}\left[s_\phi-t_\phi|t_\phi,v_\phi\geq p^*,\left(I_{t_\phi-}\right)^+\right]}{\Pr\left(m_\phi=1|t_\phi,v_\phi\geq p^*,\left(I_{t_\phi-}\right)^+\right)}\mathbf{1}\left\{v_\phi\geq p^*\right\}\mathsf{E}\left[\mathbf{1}\left\{m_\phi=1\right\}|t_\phi,v_\phi\geq p^*,\left(I_{t_\phi-}\right)^+\right]\\
&\quad +h\sum_{\psi\in H^T}\frac{\mathsf{E}\left[s_\psi-t_\psi|t_\psi,c_\psi\leq w^*\right]}{\Pr\left(m_\psi=1|t_\psi,c_\psi\leq w^*\right)}\mathbf{1}\left\{c_\psi\leq w^*\right\}\mathsf{E}\left[\mathbf{1}\left\{m_\psi=1\right\}|t_\psi,c_\psi\leq w^*\right]\Bigg]\\
&= (p^*-w^*)\,\mathsf{E}\left[\min\left\{N_T^d,N_T^s\right\}\right]-\mathsf{E}\Bigg[b\sum_{\phi\in H^T}\mathsf{E}\left[s_\phi-t_\phi|t_\phi,v_\phi\geq p^*,\left(I_{t_\phi-}\right)^+\right]\mathbf{1}\left\{v_\phi\geq p^*\right\}\\
&\quad +h\sum_{\psi\in H^T}\mathsf{E}\left[s_\psi-t_\psi|t_\psi,c_\psi\leq w^*\right]\mathbf{1}\left\{c_\psi\leq w^*\right\}\Bigg]\\
&= (p^*-w^*)\,\mathsf{E}\left[\min\left\{N_T^d,N_T^s\right\}\right]-\mathsf{E}\left[b\sum_{\phi:v_\phi\geq p^*}(s_\phi-t_\phi)+h\sum_{\psi:c_\psi\leq w^*}(s_\psi-t_\psi)\right]\\
&= (p^*-w^*)\,\mathsf{E}\left[\min\left\{N_T^d,N_T^s\right\}\right]-\mathsf{E}\left[\int_{t=0}^{T}b\sum_{\phi:v_\phi\geq p^*}\mathbf{1}\left\{t\in[t_\phi,s_\phi]\right\}+h\sum_{\psi:c_\psi\leq w^*}\mathbf{1}\left\{t\in[t_\psi,s_\psi]\right\}dt\right]\\
&= (p^*-w^*)\,\mathsf{E}\left[\min\left\{N_T^d,N_T^s\right\}\right]-\mathsf{E}\left[\int_{t=0}^{T}b\left(N_t^d-N_t^s\right)^+h\left(N_t^s-N_t^d\right)^+dt\right]\\
&= (p^*-w^*)\,\mathsf{E}\left[\min\left\{N_T^d,N_T^s\right\}\right]-\int_{t=0}^{T}\mathsf{E}\left[b\left(N_t^d-N_t^s\right)^+h\left(N_t^s-N_t^d\right)^+\right]dt\\
&\geq (p^*-w^*)\,\mathsf{E}\left[\mu^*-\left(\mu^*-N_T^d\right)^+-\left(\mu^*-N_T^s\right)^+\right]\\
&\quad -\int_{t=0}^{T}\mathsf{E}\left[b\left(N_t^d-\mu^*\frac{t}{T}\right)^+b\left(\mu^*\frac{t}{T}-N_t^s\right)^+h\left(N_t^s-\mu^*\frac{t}{T}\right)^+h\left(\mu^*\frac{t}{T}-N_t^d\right)^+\right]dt\\
&= (p^*-w^*)\,\mathsf{E}\left[\mu^*-2\left(\mu^*-N_T\right)^+\right]-(b+h)\int_{t=0}^{T}\mathsf{E}\left[\left(N_t-\mu^*\frac{t}{T}\right)^+\left(\mu^*\frac{t}{T}-N_t\right)^+\right]dt\\
&= (p^*-w^*)\,\mathsf{E}\left[\mu^*-2\left(N_T-\mu^*\right)^+2\left(N_T-\mu^*\right)\right]\\
&\quad -(b+h)\int_{t=0}^{T}\mathsf{E}\left[2\left(N_t-\mu^*\frac{t}{T}\right)^+-\left(N_t-\mu^*\frac{t}{T}\right)\right]dt
\end{aligned}
$$

$$
\begin{aligned}
&= (p^* - w^*)\,\mathsf{E}\left[\mu^* - 2\left(N_T - \mu^*\right)^+\right] - (b+h)\int_{t=0}^{T}\mathsf{E}\left[2\left(N_t - \mu^*\frac{t}{T}\right)^+\right]dt \\
&\geq (p^* - w^*)\,\mu^* - (p^* - w^*)\,\sqrt{\mu^*} - (b+h)\int_{t=0}^{T}\sqrt{\mu^*\frac{t}{T}}dt \\
&= (p^* - w^*)\,\mu^* - \left(p^* - w^* + \frac{2}{3}(b+h)\,T\right)\sqrt{\mu^*}.
\end{aligned}
$$

Here the second equality follows from the definition of the waiting adjusted FP policy. The seventh equality follows from the definition of the greedy matching policy. The first inequality follows from the property that $\min\{X,Y\} \geq a-(a-X)^+-(a-Y)^+$ and the property that $(X+Y)^+ \leq X^+ + Y^+$. The ninth and tenth equalities follow from the property that $X^+ = X + (-X)^+$. The eleventh equality is due to $\mathsf{E}[N_t] = \mu^*\frac{t}{T}$. The second inequality follows from Gallego and van Ryzin (1994) Equation (18) that if X is a random variable with mean μ and standard deviation σ, then

$$
\mathsf{E}\left[(X-a)^+\right] \leq \frac{\sqrt{\sigma^2 + (a-\mu)^2} - (a-\mu)}{2}.
$$

Therefore,

$$
\begin{aligned}
\frac{J^{\pi^{\mathrm{WFP}},M^{\mathrm{g}}}}{\bar{J}^*} &\geq \frac{(p^* - w^*)\,\mu^* - \left(p^* - w^* + \frac{2}{3}(b+h)\,T\right)\sqrt{\mu^*}}{(p^* - w^*)\,\mu^*} \\
&= 1 - \left(1 + \frac{2}{3}\frac{(b+h)\,T}{p^* - w^*}\right)\frac{1}{\sqrt{\mu^*}}. \quad \square
\end{aligned}
$$

Next, we do the asymptotic analysis. First, following from Equations (1) and (2) and Lemma S.5 that μ^* is increasing in λ^d and λ^s, respectively, we have $\mu^{*,(n)} \geq n^{\underline{\alpha}}\mu^*$.

Second, we have $\mu^{*,(n)} \leq \min\left\{\lambda^{d,(n)}T, \lambda^{s,(n)}T\right\} = \min\left\{\lambda^d n^{\alpha_d}T, \lambda^s n^{\alpha_s}T\right\} \leq n^{\underline{\alpha}}\max\left\{\lambda^d, \lambda^s\right\}T$.

Third, following from Equations (1), (2), and (3) and Lemma S.5 that $\bar{J}^*$ is increasing in λ^d and λ^s, respectively, we have $\bar{J}^{*,(n)} \geq n^{\underline{\alpha}}\bar{J}^*$.

Therefore,

$$
\begin{aligned}
\left(1 + \frac{2}{3}\frac{(b+h)\,T}{p^{*,(n)} - w^{*,(n)}}\right)\frac{1}{\sqrt{\mu^{*,(n)}}} &= \left(1 + \frac{2}{3}(b+h)\,T\frac{\mu^{*,(n)}}{\bar{J}^{*,(n)}}\right)\frac{1}{\sqrt{\mu^{*,(n)}}} \\
&\leq \left(1 + \frac{2}{3}(b+h)\,T\frac{n^{\underline{\alpha}}\max\left\{\lambda^d, \lambda^s\right\}T}{n^{\underline{\alpha}}\bar{J}^*}\right)\frac{1}{\sqrt{n^{\underline{\alpha}}\mu^*}} \\
&= O\left(\frac{1}{\sqrt{n^{\underline{\alpha}}}}\right),
\end{aligned}
$$

where the first equality follows from Equation (3), the first inequality follows from three properties

that we prove in this corollary above. □

C. Proofs for §6

Proof of Theorem 3. (1) First, we prove that p^* is increasing in λ^d.

Consider any two market conditions with λ_1^d and λ_2^d, where $\lambda_1^d < \lambda_2^d$. The subscript (1 or 2) in all following notation denotes the index of the market environment that the intermediary operates at.

First, we prove that $p_2^* \geq p_1^*$.

For $i \in \{1,2\}$, define $W_i(\nu) \triangleq V_i(\nu\lambda_i^d T)$ and $\nu_i^* \triangleq \mu_i^*/\lambda_i^d T$.

For $\nu \in \left[0, \min\left\{1, \frac{\lambda^s}{\lambda_1^d}\right\}\right] \cap \left[0, \min\left\{1, \frac{\lambda^s}{\lambda_2^d}\right\}\right] = \left[0, \min\left\{1, \frac{\lambda^s}{\lambda_2^d}\right\}\right]$, we have

$$W_2(\nu) - W_1(\nu) = -V^s\left(F^{s,-1}\left(\nu\frac{\lambda_2^d}{\lambda^s}\right)\right) + V^s\left(F^{s,-1}\left(\nu\frac{\lambda_1^d}{\lambda^s}\right)\right) \leq 0,$$

where the inequality follows from Assumption 2. In addition, Assumptions 1 and 2 imply that $W_i(\nu)$ is decreasing in ν. Therefore, $\nu_2^* \leq \nu_1^*$. Therefore, the demand-supply balancing condition (1) implies that $p_2^* = \bar{F}^{d,-1}(\nu_2^*) \geq \bar{F}^{d,-1}(\nu_1^*) = p_1^*$.

Second, the property that $\mu_2^* \geq \mu_1^*$ directly follows from Lemma S.5.

Third, we prove that $w_2^* \geq w_1^*$.

The demand-supply balancing condition (1) and the above property that $\mu_2^* \geq \mu_1^*$ jointly imply that $w_2^* = F^{s,-1}\left(\frac{\mu_2^*}{\lambda^s T}\right) \geq F^{s,-1}\left(\frac{\mu_1^*}{\lambda^s T}\right) = w_1^*$.

Finally, the property that $\bar{J}_2^* \geq \bar{J}_1^*$ directly follows from Lemma S.5.

(2) Consider any two market conditions with λ_1^s and λ_2^s, where $\lambda_1^s < \lambda_2^s$. The subscript (1 or 2) in all following notation denotes the index of the market environment that the intermediary operates at.

First, we prove that $w_2^* \leq w_1^*$.

For $i \in \{1,2\}$, define $W_i(\nu) \triangleq V_i(\nu\lambda_i^s T)$ and $\nu_i^* \triangleq \mu_i^*/\lambda_i^s T$.

For $\nu \in \left[0, \min\left\{1, \frac{\lambda^d}{\lambda_1^s}\right\}\right] \cap \left[0, \min\left\{1, \frac{\lambda^d}{\lambda_2^s}\right\}\right] = \left[0, \min\left\{1, \frac{\lambda^d}{\lambda_2^s}\right\}\right]$, we have

$$W_2(\nu) - W_1(\nu) = V^d\left(\bar{F}^{d,-1}\left(\nu\frac{\lambda_2^s}{\lambda^d}\right)\right) - V^d\left(\bar{F}^{d,-1}\left(\nu\frac{\lambda_1^s}{\lambda^d}\right)\right) \leq 0,$$

where the inequality follows from Assumption 1. In addition, Assumptions 1 and 2 imply that $W_i(\nu)$ is decreasing in ν. Therefore, $\nu_2^* \leq \nu_1^*$. Therefore, the demand-supply balancing condition (1) implies that $w_2^* = F^{s,-1}(\nu_2^*) \leq F^{s,-1}(\nu_1^*) = w_1^*$.

Second, the property that $\mu_2^* \geq \mu_1^*$ directly follows from Lemma S.5.

Third, we prove that $p_2^* \le p_1^*$.

The demand-supply balancing condition (1) and the above property that $\mu_2^* \ge \mu_1^*$ jointly imply that $p_2^* = \bar{F}^{d,-1}\left(\frac{\mu_2^*}{\lambda^d T}\right) \le \bar{F}^{d,-1}\left(\frac{\mu_1^*}{\lambda^d T}\right) = p_1^*$.

Finally, the property that $\bar{J}_2^* \ge \bar{J}_1^*$ directly follows from Lemma S.5. □

D. Proofs for Appendix B

Proof of Theorem 4. Following from Equation 1, we have $\mu^* = \frac{V+\theta^d-p^*}{2\theta^d} = \frac{w^*-(C-\theta^s)}{2\theta^s}$. Hence, $p^* = V + \theta^d - 2\theta^d\mu^*$ and $w^* = C - \theta^s + 2\theta^s\mu^*$.

Because buyer valuation and seller cost are uniformly distributed, for $\mu \le [0,1]$, we have

$$V(\mu) = V^d\left(\bar{F}^{d,-1}(\mu)\right) - V^s\left(F^{s,-1}(\mu)\right) = V^d\left(V+\theta^d-2\theta^d\mu\right) - V^s\left(C-\theta^s+2\theta^s\mu\right)$$
$$= \left(V+\theta^d-2\theta^d\mu-2\theta^d\mu\right) - \left(C-\theta^s+2\theta^s\mu+2\theta^s\mu\right) = V-C-(4\mu-1)\left(\theta^d+\theta^s\right).$$

Hence, following from Equation (2), we have

$$\mu^* = \min\left\{\frac{1}{4}\left(\frac{V-C}{\theta^d+\theta^s}+1\right), 1\right\}.$$

Therefore, μ^* is decreasing in $\theta^d+\theta^s$.

Next, we analyze the effects of θ^d and θ^s on p^*. We have

$$p^* = V+\theta^d-2\theta^d\mu^* = \begin{cases} V-\theta^d & \text{if } \theta^d+\theta^s \le \frac{V-C}{3} \\ V-\frac{\theta^d}{2}\left(\frac{V-C}{\theta^d+\theta^s}-1\right) & \text{if } \theta^d+\theta^s > \frac{V-C}{3} \end{cases}.$$

Now, we analyze the monotonicity property of $\theta^d\left(\frac{V-C}{\theta^d+\theta^s}-1\right)$ w.r.t. θ^d. We have $\frac{\partial}{\partial\theta^d}\theta^d\left(\frac{V-C}{\theta^d+\theta^s}-1\right) = \frac{(V-C)\theta^s}{\left(\theta^d+\theta^s\right)^2} - 1$. Hence, $\frac{\partial}{\partial\theta^d}\theta^d\left(\frac{V-C}{\theta^d+\theta^s}-1\right) \ge 0$ if $\theta^d < \left(\sqrt{(V-C)\theta^s}-\theta^s\right)^+$ and $\frac{\partial}{\partial\theta^d}\theta^d\left(\frac{V-C}{\theta^d+\theta^s}-1\right) \le 0$ if $\theta^d \ge \left(\sqrt{(V-C)\theta^s}-\theta^s\right)^+$. Hence, $\theta^d\left(\frac{V-C}{\theta^d+\theta^s}-1\right)$ is increasing in $\theta^d \in \left[0, \left(\sqrt{(V-C)\theta^s}-\theta^s\right)^+\right]$ and decreasing in $\theta^d \ge \left(\sqrt{(V-C)\theta^s}-\theta^s\right)^+$.

Therefore, p^* is decreasing in $\theta^d \in \left[0, \max\left\{\left(\frac{V-C}{3}-\theta^s\right)^+, \left(\sqrt{(V-C)\theta^s}-\theta^s\right)^+\right\}\right]$ and increasing in $\theta^d \ge \max\left\{\left(\frac{V-C}{3}-\theta^s\right)^+, \left(\sqrt{(V-C)\theta^s}-\theta^s\right)^+\right\}$.

In addition, because μ^* is decreasing in θ^s, p^* is increasing in θ^s.

Next, we analyze the effects of θ^d and θ^s on w^*. We have

$$w^* = C - \theta^s + 2\theta^s \mu^* = \begin{cases} C + \theta^s & \text{if } \theta^d + \theta^s \le \frac{V-C}{3} \\ C + \frac{\theta^s}{2}\left(\frac{V-C}{\theta^d+\theta^s} - 1\right) & \text{if } \theta^d + \theta^s > \frac{V-C}{3} \end{cases}.$$

Analogous to the analysis for p^*, we have that w^* is increasing in $\theta^s \in \left[0, \max\left\{\left(\frac{V-C}{3} - \theta^d\right)^+, \left(\sqrt{(V-C)\,\theta^d} - \theta^d\right)^+\right\}\right]$ and decreasing in $\theta^s \ge \max\left\{\left(\frac{V-C}{3} - \theta^d\right)^+, \left(\sqrt{(V-C)\,\theta^d} - \theta^d\right)^+\right\}$.

In addition, because μ^* is decreasing in θ^d, w^* is decreasing in θ^d.

Next, we analyze the effects of θ^d and θ^s on $\bar{J}^*$. We have

$$\begin{aligned} \bar{J}^* &= (p^* - w^*)\,\mu^* = \left(V - C - (2\mu^* - 1)\left(\theta^d + \theta^s\right)\right)\mu^* \\ &= \begin{cases} V - C - \theta^d - \theta^s & \text{if } \theta^d + \theta^s \le \frac{V-C}{3} \\ \frac{\left(V - C + \theta^d + \theta^s\right)^2}{8\left(\theta^d + \theta^s\right)} & \text{if } \theta^d + \theta^s > \frac{V-C}{3} \end{cases}. \end{aligned}$$

Now, we analyze the monotonicity property of the function $\frac{(V-C+x)^2}{x}$ w.r.t. $x \in \mathbb{R}_+$. We have $\frac{d}{dx}\frac{(V-C+x)^2}{x} = -\frac{(V-C)^2}{x^2} + 1$. Hence, $\frac{d}{dx}\frac{(V-C+x)^2}{x} \le 0$ if $x \in [0, V-C]$ and $\frac{d}{dx}\frac{(V-C+x)^2}{x} \ge 0$ if $x \ge V-C$. Hence, $\frac{(V-C+x)^2}{x}$ is decreasing in $x \in [0, V-C]$ and increasing in $x \ge V-C$.

Therefore, $\bar{J}^*$ is decreasing in $\theta^d + \theta^s \in [0, V-C]$ and increasing in $\theta^d + \theta^s \ge V-C$. □

Proof of Theorem 5. Consider any two market conditions with θ_1 and θ_2, where $\theta_1 < \theta_2$. The subscript (1 or 2) in all following notation denotes the index of the market environment that the intermediary operates at.

(1) First, we prove that $p_2^* \ge p_1^*$.

For $i \in \{1, 2\}$, define $W_i(p) \triangleq V_i\left(\lambda^d T \bar{F}_i^d(p)\right)$ and $p_i^* \triangleq \bar{F}_i^{d,-1}\left(\frac{\mu_i^*}{\lambda^d T}\right)$.

For $p \in \left[\bar{F}_1^{d,-1}\left(\min\left\{\frac{\lambda^s}{\lambda^d}, 1\right\}\right), \bar{v}\right] \cap \left[\bar{F}_2^{d,-1}\left(\min\left\{\frac{\lambda^s}{\lambda^d}, 1\right\}\right), \bar{v}\right] = \left[\bar{F}_2^{d,-1}\left(\min\left\{\frac{\lambda^s}{\lambda^d}, 1\right\}\right), \bar{v}\right]$, we have $\frac{f_2^d(p)}{\bar{F}_2^d(p)} = \frac{\theta_1}{\theta_2}\frac{f_1^d(\theta_1 p/\theta_2)}{\bar{F}_1^d(\theta_1 p/\theta_2)} \le \frac{f_1^d(p)}{\bar{F}_1^d(p)}$, where the inequality follows from the condition $\theta_1 < \theta_2$ and the property that $\frac{p f_1^d(p)}{\bar{F}_1^d(p)}$ is increasing in p. In addition, we have $\bar{F}_2^d(p) = \bar{F}_1^d\left(\frac{\theta_1}{\theta_2}p\right) \ge \bar{F}_1^d(p)$, where the inequality follows from the condition $\theta_1 < \theta_2$ and the property that $\bar{F}_1^d(p)$ is decreasing in p.

Therefore, we have

$$W_2(p) - W_1(p) = \left(V_2^d(p) - V_1^d(p)\right) - \left(V^s\left(F^{s,-1}\left(\frac{\lambda^d}{\lambda^s}\bar{F}_2^d(p)\right)\right) - V^s\left(F^{s,-1}\left(\frac{\lambda^d}{\lambda^s}\bar{F}_1^d(p)\right)\right)\right) \le 0,$$

where the inequality follows from the above two properties and Assumption 2. In addition, Assumptions 1 and 2 imply that $W_i(p)$ is increasing in p. Therefore, $p_2^* \ge p_1^*$.

Second, we prove that $\mu_2^* \ge \mu_1^*$.

The property that Assumptions 1 and 2 imply that $V_i(\mu)$ is decreasing in μ, and the definition of μ_i^* jointly imply that $V_1(\mu_1^*) \geq 0$. Hence, $V_1^d\left(\bar{F}_1^{d,-1}\left(\frac{\mu_1^*}{\lambda^d T}\right)\right) \geq 0$. Thus, for $\mu \in [0, \mu_1^*]$, we have $V_1^d\left(\bar{F}_1^{d,-1}\left(\frac{\mu}{\lambda^d T}\right)\right) \geq 0$, which follows from the property that $V_i(\mu)$ is decreasing in μ.

Therefore, for $\mu \in [0, \mu_1^*]$, we have

$$\begin{aligned}
V_2(\mu) - V_1(\mu) &= V_2^d\left(\bar{F}_2^{d,-1}\left(\frac{\mu}{\lambda^d T}\right)\right) - V_1^d\left(\bar{F}_1^{d,-1}\left(\frac{\mu}{\lambda^d T}\right)\right) \\
&= \left(\bar{F}_2^{d,-1}\left(\frac{\mu}{\lambda^d T}\right) - \frac{\mu/\lambda^d T}{f_2^d(\bar{F}_2^{d,-1}(\mu/\lambda^d T))}\right) - \left(\bar{F}_1^{d,-1}\left(\frac{\mu}{\lambda^d T}\right) - \frac{\mu/\lambda^d T}{f_1^d(\bar{F}_1^{d,-1}(\mu/\lambda^d T))}\right) \\
&= \frac{\theta_2}{\theta_1}\left(\bar{F}_1^{d,-1}\left(\frac{\mu}{\lambda^d T}\right) - \frac{\mu/\lambda^d T}{f_1^d(\bar{F}_1^{d,-1}(\mu/\lambda^d T))}\right) - \left(\bar{F}_1^{d,-1}\left(\frac{\mu}{\lambda^d T}\right) - \frac{\mu/\lambda^d T}{f_1^d(\bar{F}_1^{d,-1}(\mu/\lambda^d T))}\right) \\
&= \left(\frac{\theta_2}{\theta_1} - 1\right)\left(\bar{F}_1^{d,-1}\left(\frac{\mu}{\lambda^d T}\right) - \frac{\mu/\lambda^d T}{f_1^d(\bar{F}_1^{d,-1}(\mu/\lambda^d T))}\right) \\
&= \left(\frac{\theta_2}{\theta_1} - 1\right) V_1^d\left(\bar{F}_1^{d,-1}\left(\frac{\mu}{\lambda^d T}\right)\right) \geq 0,
\end{aligned}$$

where the inequality follows from the condition that $\theta_1 < \theta_2$, and the property established above that $V_1^d\left(\bar{F}_1^{d,-1}\left(\frac{\mu}{\lambda^d T}\right)\right) \geq 0$. Therefore, $\mu_2^* \geq \mu_1^*$.

Third, we prove that $w_2^* \geq w_1^*$.

The demand-supply balancing condition (1) and the property above that $\mu_2^* \geq \mu_1^*$ jointly imply that $w_2^* = F^{s,-1}\left(\frac{\mu_2^*}{\lambda^s T}\right) \geq F^{s,-1}\left(\frac{\mu_1^*}{\lambda^s T}\right) = w_1^*$.

Finally, we prove that $\bar{J}_2^* \geq \bar{J}_1^*$.

Given the optimal prices p_1^* and w_1^* in market environment 1 with θ_1, we construct prices p_2 and w_2 for market environment 2 with θ_2, where $p_2 = p_1^* \frac{\theta_2}{\theta_1}$ and $w_2 = w_1^*$. Hence, $\bar{F}_2^d(p_2) = \bar{F}_1^d(p_1^*)$.

Therefore, $\bar{J}_2^* \geq \lambda^d T p_2 \bar{F}_2^d(p_2) - \lambda^s T w_2 F^s(w_2) = \lambda^d T p_1^* \frac{\theta_2}{\theta_1} \bar{F}_1^d(p_1^*) - \lambda^s T w_1^* F^s(w_1^*) \geq \lambda^d T p_1^* \bar{F}_1^d(p_1^*) - \lambda^s T w_1^* F^s(w_1^*) = \bar{J}_1^*$. The first inequality follows from the property that p_2 and w_2 are feasible but not necessarily optimal solutions to the optimization problem (D). The first equality follows from the definitions of p_2 and w_2. The second inequality follows from the condition that $\theta_1 < \theta_2$. Therefore, $\bar{J}_2^* \geq \bar{J}_1^*$.

(2) First, we prove that $w_2^* \geq w_1^*$.

For $i \in \{1, 2\}$, define $W_i(w) \triangleq V_i(\lambda^d T F_i^s(w))$ and $w_i^* \triangleq F_i^{s,-1}\left(\frac{\mu_i^*}{\lambda^s T}\right)$.

For $w \in \left[0, F_1^{s,-1}\left(\min\left\{\frac{\lambda^d}{\lambda^s}\right\}\right)\right] \cap \left[0, F_2^{s,-1}\left(\min\left\{\frac{\lambda^d}{\lambda^s}\right\}\right)\right] = \left[0, F_2^{s,-1}\left(\min\left\{\frac{\lambda^d}{\lambda^s}\right\}\right)\right]$, we have $\frac{f_2^s(w)}{F_2^s(w)} = \frac{\theta_1}{\theta_2}\frac{f_1^s(\theta_1 w/\theta_2)}{F_1^s(\theta_1 w/\theta_2)} \geq \frac{f_1^s(w)}{F_1^s(w)}$, where the inequality follows from the condition $\theta_1 < \theta_2$ and the property that $\frac{w f_1^s(w)}{F_1^s(w)}$ is decreasing in w. In addition, we have $F_2^s(w) = F_1^s\left(\frac{\theta_1}{\theta_2} w\right) \leq F_1^s(w)$, where the inequality follows from the condition $\theta_1 < \theta_2$ and the property that $F_1^s(w)$ is increasing in w.

Therefore, we have

$$W_2(p) - W_1(p) = \left(V^d\left(\bar{F}^{d,-1}\left(\frac{\lambda^d}{\lambda^s}F_2^s(w)\right)\right) - V^d\left(\bar{F}^{d,-1}\left(\frac{\lambda^d}{\lambda^s}F_1^s(w)\right)\right)\right) - \left(V_2^s(w) - V_1^s(w)\right) \geq 0,$$

where the inequality follows from the two properties above and Assumption 1. In addition, Assumptions 1 and 2 imply that $W_i(w)$ is decreasing in w. Therefore, $w_2^* \geq w_1^*$.

Second, we prove that $\mu_2^* \leq \mu_1^*$.

For $\mu \in \left[0, \min\left\{\lambda^d T, \lambda^s T\right\}\right]$, we have

$$\begin{aligned}
V_2(\mu) - V_1(\mu) &= -V_2^s\left(F_2^{s,-1}\left(\frac{\mu}{\lambda^s T}\right)\right) + V_1^s\left(F_1^{s,-1}\left(\frac{\mu}{\lambda^s T}\right)\right) \\
&= -\left(F_2^{s,-1}\left(\frac{\mu}{\lambda^s T}\right) + \frac{\mu/\lambda^s T}{f_2^s(F_2^{s,-1}(\mu/\lambda^s T))}\right) + \left(F_1^{s,-1}\left(\frac{\mu}{\lambda^s T}\right) + \frac{\mu/\lambda^s T}{f_1^s(F_1^{s,-1}(\mu/\lambda^s T))}\right) \\
&= -\frac{\theta_2}{\theta_1}\left(F_1^{s,-1}\left(\frac{\mu}{\lambda^s T}\right) + \frac{\mu/\lambda^s T}{f_1^s(F_1^{s,-1}(\mu/\lambda^s T))}\right) + \left(F_1^{s,-1}\left(\frac{\mu}{\lambda^s T}\right) + \frac{\mu/\lambda^s T}{f_1^s(F_1^{s,-1}(\mu/\lambda^s T))}\right) \\
&= -\left(\frac{\theta_2}{\theta_1} - 1\right)\left(F_1^{s,-1}\left(\frac{\mu}{\lambda^s T}\right) + \frac{\mu/\lambda^s T}{f_1^s(F_1^{s,-1}(\mu/\lambda^s T))}\right) \\
&= -\left(\frac{\theta_2}{\theta_1} - 1\right) V_1^s\left(F_1^{s,-1}\left(\frac{\mu}{\lambda^s T}\right)\right) \leq 0,
\end{aligned}$$

where the inequality follows from the condition that $\theta_1 < \theta_2$. Therefore, $\mu_2^* \leq \mu_1^*$.

Third, we prove that $p_2^* \geq p_1^*$.

The demand-supply balancing condition (1) and the property above that $\mu_2^* \leq \mu_1^*$ jointly imply that $p_2^* = \bar{F}^{d,-1}\left(\frac{\mu_2^*}{\lambda^d T}\right) \geq \bar{F}^{d,-1}\left(\frac{\mu_1^*}{\lambda^d T}\right) = p_1^*$.

Finally, we prove that $\bar{J}_2^* \leq \bar{J}_1^*$.

Given the optimal prices p_2^* and w_2^* in market environment 2 with θ_2, we construct prices p_1 and w_1 for market environment 1 with θ_1, where $p_1 = p_2^*$ and $w_1 = w_2^* \frac{\theta_1}{\theta_2}$. Hence, $F_2^s(w_2) = F_1^s(w_1^*)$.

Therefore, $\bar{J}_1^* \geq \lambda^d T p_1 \bar{F}^d(p_1) - \lambda^s T w_1 F_1^s(w_1) = \lambda^d T p_2^* \bar{F}^d(p_2^*) - \lambda^s T w_2^* \frac{\theta_1}{\theta_2} F_2^s(w_2^*) \leq \lambda^d T p_2^* \bar{F}^d(p_2^*) - \lambda^s T w_2^* F_2^s(w_2^*) = \bar{J}_2^*$. The first inequality follows from the property that p_1 and w_1 are feasible but not necessarily optimal solutions to the optimization problem (D). The first equality follows from the definitions of p_1 and w_1. The second inequality follows from the condition that $\theta_1 < \theta_2$. Therefore, $\bar{J}_2^* \leq \bar{J}_1^*$. $\square$