Multi-Product Price and Assortment Competition

Awi Federgruen
Graduate School of Business, Columbia University, New York, New York 10027, af7@columbia.edu

Ming Hu
Rotman School of Management, University of Toronto, Toronto, Ontario M5S 3E6, Canada ming.hu@rotman.utoronto.ca

We address a generic price competition model in an industry with an arbitrary number of competitors, each offering all or a subset of a given line of N products. The products are *substitutes* in the sense that the demand volume of each product weakly increases whenever the price of another product increases. The cost structure is linear, with arbitrary cost rates. Our demand model is the unique *regular* extension of a set of demand functions that are affine in a limited polyhedral subset of the price space. A set of demand functions is regular if it satisfies the following conditions: Under any given price vector, when some product is priced out of the market, i.e., has zero demand, any *increase* of its price has no impact on the demand volumes. Depending on the set of prices selected by the competing firms, a different product assortment is offered in the market. We characterize the equilibrium prices, product assortment, and sales volumes in the price competition model, under this demand model. Under minimal conditions, we show that a pure Nash equilibrium always exists; while multiple price equilibria may arise, they are equivalent in the sense of generating an identical product assortment and sales volumes.

1. Introduction and Summary

We address a generic price competition model in an industry with an arbitrary number of competitors, each offering all or a subset of a given line of N products. The products are *substitutes* in the sense that the demand volume of each product weakly increases whenever the price of another product increases. The cost structure is linear, with arbitrary cost rates.

Along with variants of the MultiNomial Logit (MNL) model (e.g., mixed or nested MNL models), the most frequently used demand model in operations management, marketing, and industrial organization studies employs *affine* demand functions. However, the affine structure cannot be assumed to prevail on the complete price space: After all, outside of a special polyhedron P, the affine demand functions predict negative demand volumes for at least some of the products. The straightforward extension of the affine demand functions, beyond P, consists of truncation at zero, but this approach has been shown to result in absurd pricing mechanisms for multi-product firms; see Soon et al. (2009) and Farahat and Perakis (2010).

Shubik and Levitan (1980) stipulated, instead, that the extension of the affine demand functions (beyond P) must satisfy an intuitive regularity condition: Under any given price vector, when some product is priced out of the market, i.e., experiences zero demand, any *increase* of its price has no impact on the demand volumes. Almost 30 years

later, Soon et al. (2009) showed that, innocuous as the regularity condition appears to be, there is a unique extension of the affine demand functions which satisfies this regularity condition: The extension specifies the demand values as the unique solution of a linear complementarity problem.

This demand model has many advantages. First, it is compact and characterized by a single $N \times N$ matrix R of price sensitivity coefficients along with a single intercept vector for the affine part of the demand functions. Second, depending on what prices are selected, a different subset of all potential products is offered in the market. Thus, the model specifies a product assortment, along with specifically associated demand volumes. This is in sharp contrast to all other commonly used demand models. For example, under the various variants of the MNL model, all products attain some market share, irrespective of their absolute and relative price levels.

This paper seeks to characterize the equilibrium behavior of the price competition model under this class of demand functions, both in terms of equilibrium prices and equilibrium sales volumes. Important for its own sake, this characterization is a necessary building block to understand the equilibrium behavior in multi-echelon supply chain networks, where firms compete at each of the network's echelons. (This sequential oligopoly topic is pursued in Federgruen and Hu 2013.) Beyond obtaining a full characterization of the equilibrium behavior, we show how the equilibria can be computed with only a few matrix multiplications

and inverses, in some cases combined with the solution of a single linear program (LP) in N variables and $2N$ constraints. This, in turn, allows us to answer other questions of managerial importance, for example to derive a complete *cost pass-through rate* table, showing how each product's retail price responds to a marginal change in any product's cost rate (e.g., wholesale price).

We build on the work of Farahat and Perakis (2010), who established that a unique price equilibrium exists in the retailer competition model, assuming that the matrix R is symmetric and that the vector of cost rates lies in the interior of the polyhedron P. Under their conditions, this unique price equilibrium can be computed in closed form and is an affine function of the vector of cost rates. Moreover, under this equilibrium, all potential products are offered in the market.

We generalize the results in Farahat and Perakis (2010) in two ways. First and foremost, we provide a full characterization of the equilibrium behavior under arbitrary cost rate vectors, rather than those that reside in the interior of the special polyhedron P. Second, we relax the symmetry assumption for the matrix R. The first generalization is important for the following reasons:

(a) Managerially, the cost rate restriction in Farahat and Perakis (2010) implies that when all firms price all of their products at the level of their marginal cost, there remains positive consumer demand for each of these products. There is no empirical evidence that this applies to most industries. In fact, there is reason to believe that when even the most brand/feature attractive products in the market are offered at marginal cost, this is likely to push less attractive substitutes out of the market. (Such substitutes may only preserve a share in the market, when offering a clear price advantage, thus appealing to the most price-sensitive customers.) Putting it differently, were the restriction in Farahat and Perakis (2010) without loss of generality, this would imply that, irrespective of the model primitives, all retailers, in equilibrium, select a *maximally available product assortment*, defying what we observe in practice, and stripping the extended affine model from the ability to explain less than maximally available product assortments. We show that the equilibrium cost rates or wholesale prices that arise in the above two- or multi-echelon oligopoly often violate the above cost rate restriction; see Example 1. More fundamentally, regardless of where the equilibrium wholesale price vector ultimately resides, to characterize the equilibrium among the retailers' suppliers one must identify the retailers' equilibrium responses to an *arbitrary* wholesale price vector.

(b) Qualitatively, we show that the equilibrium behavior, under a general cost rate vector, may differ from that in Farahat and Perakis (2010) in the following fundamental ways:

• In general, there may be multiple, sometimes infinitely many Nash equilibria.

• Nevertheless there is always a component-wise smallest price equilibrium, and all equilibria are equivalent

in the sense of generating identical sales volumes for all products and identical profit levels for all firms.

In the above sequential oligopoly models, establishing equivalency among all equilibria in the retailer competition model is vital to characterize the equilibrium behavior among the firms competing at higher echelons: Were there multiple equilibria with nonidentical sales volumes, the entire notion of sequential competition in supply chain networks would be ill defined.

• In equilibrium, some of the products may not be selected by some or all of the retailers, i.e., as can be expected, the equilibrium product assortment in the market varies as a function of the model parameters, in particular the products' cost rates. (Note, in view of the above equivalency result, that the product assortment is identical among all equilibria, as well.)

• The equilibrium sales functions fail to be affine functions of the cost rates. However, they may be viewed as the (unique) regular extension of a set of affine functions, with an easily computable price sensitivity matrix and intercept vector, analogous to the structure of the retailer demand functions.

• This characterization of the equilibrium sales functions has two important ramifications: First, it allows for the facile computation of the equilibrium sales and profit levels as the unique solution of a linear complementarity problem, requiring no more than the solution of a simple LP, with N variables and $2N$ constraints. Moreover, the equilibrium sales functions, being the regular extension of a set of affine functions, provide the basis for equilibrium characterizations at higher echelons in a supply chain network, see Federgruen and Hu (2013).

The importance of our second generalization stems from the fact that, in empirical studies, the estimated affine demand functions, typically, have price sensitivity matrices that are far from symmetric, see, e.g., Dubé and Manchanda (2005), Vilcassim et al. (1999) and Li et al. (2015). While most qualitative properties of the equilibria carry over to general asymmetric R-matrices, some do not. For example, we prove that the above cost pass-through rates are nonnegative throughout, when R is at least intra-firm symmetric. (Intrafirm symmetry means that the cross-product price sensitivities are symmetric, among products sold by the same retailer, but not necessarily among products sold via different retailers.) We show that under a general asymmetric R matrix, some of the cross-product cost pass-through rates may be negative, i.e., reduction of the wholesale price of some product does not necessarily result in a reduction of retail prices for all products. Negative cross-product cost pass-through rates have been reported by Dubé and Gupta (2008) among others.

As mentioned, traditional demand models for oligopolies with differentiated products invoke a set of demand functions under which all potential products capture part of the market, irrespective of what prices are selected by the competing firms. Recently, several papers have focused on

the fact that retailers compete not only in terms of their price choices for a given assortment of products, but also in terms of the selected assortment, itself. Some papers, e.g., Rusmevichientong et al. (2010, 2014), develop efficient algorithms to select an optimal assortment of products for a single (monopolistic) firm when the underlying demand functions are specified by an MNL model or a mixture of MNL models, respectively. See also the references mentioned in Besbes and Sauré (2014). The latter is, to our knowledge, the only existing paper addressing a joint price and assortment competition oligopoly model. The underlying consumer choice model is MNL with all products' utility functions sharing a common marginal price sensitivity value, and consumers reacting to the selected product assortments. The authors show that a unique equilibrium always exists, with the property that every firm selects an identical profit margin for all of its products. In our model, assortment choices are implied by price selections allowing for general firm and product-dependent price sensitivities and explaining general profit margins.

Methodologically, we note that our analysis of the equilibrium behavior is based on a new approach. This applies, in particular, to our identification of conditions under which the equilibrium is unique and our proof that, in full generality, when multiple equilibria exist, they are all equivalent.

All vectors in this paper are column vectors and are represented by lowercase symbols. All matrices are denoted by capital letters. $\mathbb{R}_+ \equiv \{r \in \mathbb{R} \mid r \geqslant 0\}$. The cardinality of a set $\mathcal{S}$ is denoted by $|\mathcal{S}|$. $\mathcal{N}$ is the set of all products. For a vector a and an index set $\mathcal{S}$, $a_{\mathcal{S}}$ denotes the subvector with entries specified by $\mathcal{S}$. Similarly, for a matrix M and index sets $\mathcal{S}, \mathcal{T} \in \mathcal{N}$, $M_{\mathcal{S}, \mathcal{T}}$ denotes the submatrix of M with rows specified by the set $\mathcal{S}$ and columns by the set $\mathcal{T}$. The transpose of a matrix M (vector a) is denoted by M^T (a^T). For notational simplicity, 0 may denote a scalar, a vector or a matrix of any dimension with all entries being zeros; I is an identity matrix of appropriate dimension. The matrix inequality $X = (x_{i,j}) \geqslant 0$ means that $x_{i,j} \geqslant 0$ for all i, j. For any polyhedral subset $\Pi \subseteq \mathbb{R}^N$, Π^o denotes its interior and $\partial \Pi$ its boundary.

We use some properties of square matrices of special structure.

DEFINITION 1 (Z-MATRIX). A square matrix whose off-diagonal entries are nonpositive is called a Z-matrix.

DEFINITION 2 (P-MATRIX). A square matrix whose principal minors are all positive (nonnegative) is called a P-matrix (P_0-matrix).

DEFINITION 3 (ZP-MATRIX). A matrix that is both a Z-matrix and a P-matrix is called a ZP-matrix.

It is well known that all positive definite matrices are P-matrices, see, e.g., Cottle et al. (1992, Chapter 3). However, the class of P-matrices is significantly broader.

The remainder of the paper is organized as follows. Section 2 presents the model and derives various properties

of consumer demand functions. We characterize the equilibrium behavior of the retailer competition game in §3. Section 4 concludes the paper. All proofs are relegated to online appendices (available as supplemental material at http://dx.doi.org/10.1287/opre.2015.1380).

2. The Model

Consider a market with a set $\mathcal{I}$ of competing retailers. Each retailer $i \in \mathcal{I}$ has the option of bringing one or several products to the market. For any retailer $i \in \mathcal{I}$, let (i, k) denote his k-th product and $\mathcal{N}(i)$ the set of *all* products potentially sold by retailer i. For all $i \in \mathcal{I}$ and $k \in \mathcal{N}(i)$, let: $p_{ik} =$ the retail price charged by retailer i for product k, $w_{ik} =$ the procurement (purchase and/or manufacturing) cost rate for product (i, k), $d_{ik} =$ the consumer demand for product (i, k). Let p, w and d denote the corresponding vectors.

The demand value of each product may depend on the prices of all products offered in the market. As in the majority of price competition models, we assume that this dependence is, in principle, described by general affine functions. As mentioned in the Introduction, this affine structure prevails in many theoretical models and empirical studies. In matrix notation, this gives rise to a system of demand equations:

$$q(p) = a - Rp, \tag{1}$$

where $a \in \mathbb{R}_+^N$ and $R \in \mathbb{R}^{N \times N}$. The assumption $a \geqslant 0$ means that all products are relevant choices, attracting nonnegative demand at least when they are offered for free.

As noted by many authors, starting with Shubik and Levitan (1980), the affine structure in Equation (1) cannot be used for arbitrary price vectors $p \geqslant 0$. After all, unless the price vector p is chosen in the polyhedron

$$P \equiv \{p \geqslant 0 \mid q(p) = a - Rp \geqslant 0\},$$

some of the components of the q-vector are negative. We call P the *effective retail price polyhedron*. (Note that $P \neq \varnothing$, since $p = 0 \in P$, as $a \geqslant 0$.) To get around this difficulty, some authors, for example Allon and Federgruen (2007), have replaced the right-hand side (RHS) of (1) by its positive part: $d(p) = [a - Rp]^+$, effectively truncating the demand functions at zero. This works well in single-product competition models. However, as pointed out by Soon et al. (2009) and Farahat and Perakis (2010), the truncation procedure fails when firms sell multiple products. Consider, for example, a monopoly with two products: products 1 and 2 with a demand function $d_1(p_1, p_2) = \max\{a_1 - b_{1,1}p_1 + b_{1,2}p_2, 0\}$ and $d_2(p_1, p_2) = \max\{a_2 - b_{2,2}p_2 + b_{2,1}p_1, 0\}$, where $a_i, b_{i,j} > 0$ for $i, j = 1, 2$. Setting the price of product 1 above its marginal cost, while increasing the price of product 2 to infinity, leads to zero profits for product 2, but an infinite profit for product 1.

Farahat and Perakis (2010) suggest that the extension of the affine demand functions in the full price space be

derived from an underlying consumer model, with a representative consumer choosing the demand quantities that optimize the following quadratic program (QP).

Quadratic utility maximization. For any price vector $p \geqslant 0$, the demand function $d(p)$ is implicitly defined by the solution to the following utility maximization problem of a representative consumer:

$$(QP) \quad \max_{d \geqslant 0} \left\{ (R^{-1}a - p)^T d - \frac{1}{2} d^T R^{-1} d \right\}, \tag{2}$$

where $a \in \mathbb{R}_+^N$ and $R \in \mathbb{R}^{N \times N}$ with positive diagonal elements.

To ensure that this quadratic optimization problem is strictly concave and has a unique optimum, one makes the following assumption:

ASSUMPTION (P). (QP *is well defined*). *The matrix R is positive definite.*

We only consider substitutable products, adding the assumption:

ASSUMPTION (Z). (*Substitutes*). *The matrix R is a Z-matrix.*

Assumption (Z) states that, at least in terms of the raw demand functions $q(\cdot)$, a price increase of any product results in an increase of the demand volumes of all other products.

The unique unconstrained optimum of the quadratic utility maximization problem (QP) has $d = a - Rp = q(p)$, however only in case the matrix R is symmetric. If R is asymmetric, the unconstrained optimum is given by $d = ((R^{-1} + (R^{-1})^T)/2)^{-1}(R^{-1}a - p)$. Indeed, when the matrix R is asymmetric, as is typically the case in empirical studies, see, e.g., Dubé and Manchanda (2005) and Vilcassim et al. (1999), it does not appear that the set of affine demand functions $q(p)$ can be derived from an underlying utility maximization problem.

Thus, the model in Farahat and Perakis (2010) is intrinsically confined to settings with a symmetric matrix R; moreover, their equilibrium analysis is based on this assumption. Instead, Shubik and Levitan (1980, Appendix B, Problem 1) stipulated that the extension of the affine demand functions in the full price space must satisfy the following innocuous regularity condition:

DEFINITION 4 (REGULARITY). A demand function $D(p)$: $\mathbb{R}_+^N \to \mathbb{R}_+^N$ is said to be *regular* if, for any product l and any price vector p, $D_l(p) = 0$ implies that $D(p + \Delta \cdot e_l) = D(p)$ for any $\Delta > 0$, where e_l denotes the l-th unit vector.

The regularity property states that if, under a given price vector p, a particular product l is priced out of the market, i.e., has zero demand, any increase in its price has no impact on the demand volumes. Soon et al. (2009) showed that there exists one and only one such regular extension.

PROPOSITION 1 (DEMAND SYSTEM). *Under Assumptions* (P) *and* (Z), *the following specifications of the demand function $d(p)$ are equivalent:*

(a) $d(p) = a - Rp$ *for $p \in P$; $d(p)$ is a regular function.*

(b) *There exists a unique vector of price corrections, t, such that*

$$\begin{aligned} & d(p) = q(p - t) = a - R(p - t) \geqslant 0 \quad and \\ & t^T[a - R(p - t)] = 0, \quad t \geqslant 0. \end{aligned} \tag{3}$$

In the special case where the matrix R is symmetric, the following specification is equivalent to (a) *and* (b): (c) $d(p)$ *is the unique solution to the utility maximization problem* (2).

A minimum set of conditions for the matrix R (Assumptions (P) *and* (Z)).

Given the existence of a unique regular extension of the affine functions under Assumptions (P) and (Z), we develop the equilibrium analysis on the basis of these two properties only.[1] (In other words, unlike Farahat and Perakis (2010), we adopt neither the QP utility maximization specification, nor the related symmetry assumption for the matrix R, as additional assumptions, beyond Assumptions (P) and (Z).) Verification of (Z) is immediate. There are many sufficient conditions and numerical procedures to check (P), see Lemma B.1 in the Online Appendix.

Part (b) of Proposition 1 shows that the demand vector $d(p)$ is obtained by applying the affine transformation $q(\cdot)$, not necessarily to the "raw" price vector p, but, more generally, to the projection of the price vector onto the polyhedron P defined as follows:

DEFINITION 5 (PROJECTION ONTO P). For any $p \in \mathbb{R}_+^N$, the projection $\Omega(p)$ of p onto P is defined as the vector $p' = p - t$, with t the unique solution to (3).

LEMMA 1. (a) *For any $p \in \mathbb{R}_+^N$, $\Omega(p) \in P$.* (b) *If $p \in P$, then $\Omega(p) = p$.*

If a price vector $p \in \mathbb{R}_+^N \backslash P$, the correction vector t must have at least one positive entry. By complementarity, those products whose indices have positive entries in t must have zero demands. The price vector $\Omega(p) = p - t \in P$ is such that those products with zero demands are just priced out of the market, i.e., any unilateral price reduction of such a product results in it being carried with a positive demand volume.

Yet another interpretation of the demand specification in (3) views the vector $t \geqslant 0$ as the unique *price correction* vector that solves the Linear Complementarity Problem (LCP) associated with the vector $q(p)$ and the matrix R, referred to as LCP$(q(p), R)$.

DEFINITION 6 (LINEAR COMPLEMENTARITY PROBLEM). For any vector $q \in \mathbb{R}^N$ and matrix $M \in \mathbb{R}^{N \times N}$, the LCP$(q, M)$ is to find a vector $t \in \mathbb{R}^N$ such that

$$t \geqslant 0, \quad q + Mt \geqslant 0 \quad and \quad t^T(q + Mt) = 0. \tag{4}$$

If $M = R$ and since R is positive definite, the existence of a unique solution t to the LCP$(q(p), R)$ follows from the general theory of LCPs. Since R is a Z-matrix, the solution t may be found by optimizing a linear objective over the polyhedron described by the inequalities in (4), see Mangasarian (1976).

3. The Retailer Competition Model

Fix an arbitrary cost rate vector $w \geqslant 0$ and let: $\pi_{ik}(p) =$ the profit earned by retailer i from the sales of product $k = (p_{ik} - w_{ik})d_{ik}(p)$, $\pi_i(p) = \sum_{k \in \mathcal{N}(i)} \pi_{ik}(p) =$ the aggregate profit earned by retailer i.

Some have argued that, for practical purposes, the equilibrium analyses may be confined to the case wherein $w \in P$, or even $w \in P^o$ (see, e.g., Farahat and Perakis 2010). However, there are many settings wherein the wholesale price vector w is selected outside of P. For example, if the wholesale prices are achieved as a price equilibrium among competing suppliers, in a price competition game preceding the retailer competition game, an equilibrium $w^* \notin P$ may easily occur. A simple duopoly example, with each firm selling a single product, is given as part of Example 1, below. (This example assumes that each of the two retailers has a dedicated supplier; this is one of the simplest channel structures, first studied by McGuire and Staelin 2008.) Moreover, to analyze the above type of sequential oligopoly, it is necessary to characterize the retailers' equilibrium responses to an arbitrary choice of wholesale prices. We therefore characterize the equilibrium behavior in the retailer competition model, under an arbitrary wholesale price vector $w \in \mathbb{R}^N_+$.

We show that, while there may be infinitely many equilibria, there exists *exactly one* equilibrium $p^* \in P$. All other equilibria $p \notin P$ are such that $\Omega(p) = p^*$. Moreover, as stated, all equilibria are equivalent. Since, by its definition, $\Omega(p) \leqslant p$, p^* is the component-wise smallest equilibrium.

THEOREM 1. (a) *Within P, there exists exactly one equilibrium p^* in the retailer competition game.* (b) *Any equilibrium $p^o \notin P$, has $\Omega(p^o) = p^*$ such that p^* is the component-wise smallest equilibrium.* (c) *All equilibria are equivalent.*

Soon et al. (2009, Theorem 15) showed that a Nash equilibrium exists. Part (a) adds to this finding that a Nash equilibrium can be found in P and that, on P, it is the only Nash equilibrium. Parts (b) and (c), for an arbitrary wholesale price vector, are completely new, to our knowledge.

For the results in Theorem 1 to be truly useful, one must provide a simple characterization of the component-wise smallest equilibrium p^* in terms of the model primitives, i.e., the matrix R, the intercept vector a, and the wholesale price vector w. Such a characterization has numerous benefits: It allows the model to be used in various counterfactual studies, e.g., to evaluate the impact of horizontal and vertical mergers. Such studies are mandated and routinely undertaken by the Department of Justice (DOJ) and the Federal Trade Commission (FTC) to evaluate merger proposals, see, e.g., Farrell and Shapiro (2010a, b). It also permits an explicit representation of various comparative statics, for example cost pass-through rates, i.e., the marginal impacts of changes in wholesale prices on retail prices, see Proposition 4 and the subsequent discussion, below. As a last example, an analytical characterization is essential to

studying the competition among firms at higher echelons of the supply chain network. (This topic is pursued in Federgruen and Hu 2013.)

We now derive the desired analytical characterization of the component-wise smallest equilibrium p^*. In fact, we show how this unique equilibrium within P can be computed with a few matrix computations, possibly in conjunction with the solution of a single LP with N variables and $2N$ constraints. Its associated product assortment, sales volumes, and profit levels are, in fact, unique among all price equilibria.

Recall that for $p \in P$, $d(p) = q(p) = a - Rp$. Thus, a price vector $p \in P$ which is an equilibrium in the full price game, may satisfy the First Order Conditions (FOC):

$$0 = \frac{\partial \pi_i(p)}{\partial p_{ik}} = q_{ik}(p) - R_{ik, ik}(p_{ik} - w_{ik})$$
$$- \sum_{k' \neq k} R_{ik', ik}(p_{ik'} - w_{ik'}).$$

Substituting the affine demand function $q_{ik}(p)$ and rearranging terms, we get

$$2R_{ik, ik}p_{ik} + \sum_{k' \neq k}(R_{ik', ik} + R_{ik, ik'})p_{ik'} + \sum_{i' \neq i} R_{ik, i'k'}p_{i'k'}$$
$$= a_{ik} + R_{ik, ik}w_{ik} + \sum_{k' \neq k} R_{ik', ik}w_{ik'}. \quad (5)$$

It is convenient to write this system of N linear equations in N unknowns in matrix form as: $[R + T(R)]p = a + T(R)w$, or equivalently,

$$[R + T(R)](p - w) = a - Rw = q(w), \quad (6)$$

where

$$T(R) \equiv \begin{pmatrix} R^T_{\mathcal{N}(1), \mathcal{N}(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R^T_{\mathcal{N}(|\mathcal{F}|), \mathcal{N}(|\mathcal{F}|)} \end{pmatrix} \in \mathbb{R}^{N \times N}.$$

We now show that this system of linear equations has a unique solution $p^*(w)$, and that this solution is indeed in P if the wholesale price vector w is selected within a specific polyhedron W, defined in the following proposition. Like P, we show that the polyhedron W is always non-empty. If the matrix R is symmetric, we show that W contains the polyhedron P. In fact, a weaker condition suffices here, i.e., the cross-price sensitivity coefficients are symmetric only for products sold by the same retailer:

ASSUMPTION (IS). *The matrix R is intrafirm symmetric, i.e., $R_{ik, ik'} = R_{ik', ik}$ for all $i \in \mathcal{F}$ and $k, k' \in \mathcal{N}(i)$.*

Clearly, intrafirm symmetry, i.e., Assumption (IS) holds, trivially, in the special case wherein each firm sells a single product. Also, the matrix R is intrafirm symmetric (IS) if and only if $T(R)$ is symmetric.

PROPOSITION 2. (a) *The FOC* (6) *has a unique solution*

$$p^*(w) \equiv w + [R + T(R)]^{-1}q(w). \tag{7}$$

(b) $p^*(w) \in P$ *if and only if* $w \in W \equiv \{w \geqslant 0 \mid \Psi(R) \cdot q(w) = \Psi(R)a - \Psi(R)Rw \geqslant 0\}$, *where*

$$\Psi(R) \equiv T(R)[R + T(R)]^{-1}.$$

If $w \in W$, $q(p^*(w)) = \Psi(R)q(w)$.

(c) $W \neq \varnothing$, *since* $w^o = R^{-1}a \in W$.

(d) *The matrix* $\Psi(R)R$ *is positive definite.*

(e) *Under Assumption* (IS), *i.e., when the matrix* R *is intrafirm symmetric, then* $\Psi(R) \geqslant 0$ *and* $P \subseteq W$.

We first characterize the equilibrium behavior for any wholesale price vector $w \in W$.

THEOREM 2 (CHARACTERIZATION OF PRICE EQUILIBRIA, WHEN $w \in W$). (a) *If* $w \in W$, $p^*(w)$ *is the unique price equilibrium in* P. *Any equilibrium* p^o *outside of* P *has* $\Omega(p^o) = p^*(w)$ *and is equivalent to* $p^*(w)$.

(b) *If* $w \in W^o$, $p^*(w) \in P^o$ *is the unique price equilibrium and all products are part of the equilibrium assortment.*

Thus, if $w \in W$, $p^*(w)$ is the unique price equilibrium in P with all equilibria outside of P equivalent to it. In view of Theorem 1, our challenge remains to identify a price equilibrium in P, when $w \notin W$. (By Theorem 1, such an equilibrium is again unique within P.) The vector $p^*(w)$ no longer qualifies, since, when $w \notin W$, $p^*(w)$ does not reside in P. The proof of Theorem 2 is invalid as it assumes that $d(p^*(w)) = q(p^*(w))$. However, if $w \notin W$, we now show, under a mild condition, that $p^*(w')$ *is* the *unique* price equilibrium in P, where $w' = \Theta(w)$, i.e., w' is the projection of w onto the polyhedron W. Recall that W is specified by the induced affine demand inequalities

$$Q(w) \equiv \Psi(R)q(w) \equiv b - Sw \geqslant 0,$$

$$\text{where } b \equiv \Psi(R)a \text{ and } S \equiv \Psi(R)R.$$

As with the definition of the projection mapping $\Omega(\cdot)$ on the polyhedron P, $w' = \Theta(w)$ means that $w' = w - t$, with t the unique solution to the LCP($Q(w), S$):

$$t \geqslant 0, \quad Q(w - t) \equiv b - S(w - t) \geqslant 0 \quad \text{and} \tag{8}$$
$$t^T Q(w - t) = 0.$$

This LCP has a unique solution t, since $S = \Psi(R)R$ is positive definite, see Proposition 2(d). The existence of a unique solution to the LCP (8) follows from Theorem 3.3.7 in Cottle et al. (1992).

Thus, $Q(w') \geqslant 0$. However, to guarantee that $w' = \Theta(w) \in W$, and hence $p^*(w') \in P$, see Proposition 2(b), we need, in addition, that $w' = \Theta(w) \geqslant 0$, the necessary and sufficient condition for which was identified by Soon et al. (2009, Lemma 6 and Theorem 4):

ASSUMPTION (NPW) (NONNEGATIVE PROJECTION ONTO W).

$$S_{\tilde{N}, \tilde{N}}^{-1} b_{\tilde{N}} \geqslant 0, \quad \textit{for all } \tilde{N} \subseteq \mathcal{N}. \tag{NPW}$$

While directly verifiable from the model primitives, Assumption (NPW) is not quite intuitive and its numerical verification requires computing inverses of $2^N - 1$ matrices. We therefore provide a pair of nested, very general, and easily verified sufficient conditions for Assumption (NPW): Assumption (IS), i.e., intrafirm symmetry of the matrix R, is one such condition. A more general condition is:

ASSUMPTION (WRS) (A WHOLESALE MARKET OF RELEVANT SUBSTITUTES). S *is a* Z-*matrix and* $b \geqslant 0$, *i.e., in the wholesale market, the products act as relevant substitutes.*

PROPOSITION 3 (SUFFICIENT CONDITIONS FOR (NPW)). (IS) $\Rightarrow$ (WRS) $\Rightarrow$ (NPW).

The requirement that the matrix S be a Z-matrix appears innocuous and intuitive. It merely states that an increase in the wholesale price of one of the products does not result in a decrease of the (equilibrium) sales volumes of any other products. In other words, the assumption that S is a Z-matrix means that the products act as substitutes in the wholesale market, just as they do in the retailers' market. However, while highly intuitive, the induced raw suppliers' demand functions $Q(\cdot)$ may violate this property in very special cases, see, e.g., Example C.1 in Online Appendix C. The second part of Assumption (WRS), i.e., $b \geqslant 0$, means that these substitutes are relevant in the wholesale market.

We are now ready to complete the equilibrium characterization by considering $w \notin W$.

THEOREM 3 (CHARACTERIZATION OF PRICE EQUILIBRIA, WHEN $w \notin W$). *Suppose Assumption* (NPW) *is satisfied. Fix* $w \notin W$. $p^*(w')$ *is the unique price equilibrium in* P, *with* $w' = \Theta(w)$, *under which only a subset of the full product set is sold in the market. Any equilibrium* p^o *outside of* P *has* $\Omega(p^o) = p^*(w')$ *and is equivalent to* $p^*(w') \in \partial P$. *In particular, all price equilibria induced by the wholesale price vector* w *generate the vector of sales volumes* $Q(w') = \Psi(R)q(w') = q(p^*(w'))$.

The equilibrium characterizations in Theorems 1–3 are summarized in Table 1. As we move from left to right, the conditions for the matrix R are relaxed; the last column merely assumes that the R matrix is a ZP-matrix. As mentioned in the Introduction, our work builds on the seminal results in Farahat and Perakis (2010), which established the equilibrium characterization UF in the upper left cell of Table 1, when $w \in P^o$. (Recall from Proposition 2 that $P^o \subseteq W^o$ when R is symmetric.) To our knowledge, with the exception of the Soon et al. (2009) existence result, no characterizations have been obtained in the literature for the remaining cells of the table or for $w \in W^o \backslash P^o$, even when R is symmetric.[2]

Table 1. Equilibrium behavior and product assortments.

	R is symmetric	R is intrafirm symmetric (IS)	A wholesale market of relevant substitutes (WRS)[a]	Products fail to be relevant substitutes in wholesale market
$w \in W^o$	UF $[p^*(w)]$	UF $[p^*(w)]$	UF $[p^*(w)]$	UF $[p^*(w)]$
$w \in \partial W$	EP $[p^*(w)]$	EP $[p^*(w)]$	EP $[p^*(w)]$	EP $[p^*(w)]$
$w \notin W$	EP $[p^*(w')]$	EP $[p^*(w')]$	EP $[p^*(w')]$	EP

Notes. "UF": A *u*nique Nash equilibrium, with *f*ull product assortment. "EP": A component-wise smallest equilibrium, possibly with other *e*quivalent equilibria; A *p*artial product assortment. The unique and component-wise smallest equilibrium is displayed in those square brackets.
[a] This condition can be generalized to condition (NPW) above.

EXAMPLE 1. Consider a duopoly in which each retailer $i = 1, 2$ carries a single product $i = 1, 2$. Let

$$a = (1,1)^T \quad \text{and} \quad R = \begin{pmatrix} 1 & -\gamma_1 \\ -\gamma_2 & 1 \end{pmatrix},$$

with $\gamma_1, \gamma_2 \in [0, 1)$. We have

$$\Psi(R) = T(R)[R + T(R)]^{-1} = \frac{1}{4 - \gamma_1\gamma_2} \begin{pmatrix} 2 & \gamma_1 \\ \gamma_2 & 2 \end{pmatrix}.$$

Since $T(R) = I$ is symmetric, $P \subseteq W$, by Proposition 2(e). We illustrate the polyhedra, for the asymmetric case $\gamma_1 = 0.7$ and $\gamma_2 = 0.3$ in Figure 1.

Below, we illustrate how the equilibrium behavior depends on the region in which w resides, with three possibilities: (a) $w \in W^o$, (b) $w \in W(\mathrm{I})$ or $W(\mathrm{II})$ and (c) $w \in W(\mathrm{III})$. First, we show that, contrary to common belief, $w \notin P$ may easily arise. Consider the basic channel structure in McGuire and Staelin (2008) where retailer 1 (2) uniquely procures from a dedicated supplier 1 (2), operating with a marginal cost rate vector $c^o = (1, 1.5)^T$. Assume that the market operates as a sequential oligopoly: First the two suppliers, noncooperatively, select their wholesale prices, accounting for the retailers' equilibrium price responses. Then, the retailers follow and select their prices. Following the results in Federgruen and Hu (2013), one

can show that the vector $w^o = (1.55, 1.52)^T \in W^o$ arises as part of the unique supply-chain-wide equilibrium. Note that $a - Rw^o = (0.5139, -0.0569)^T$, i.e., $w^o \notin P$. See Online Appendix D for the auxiliary calculations to verify this result.
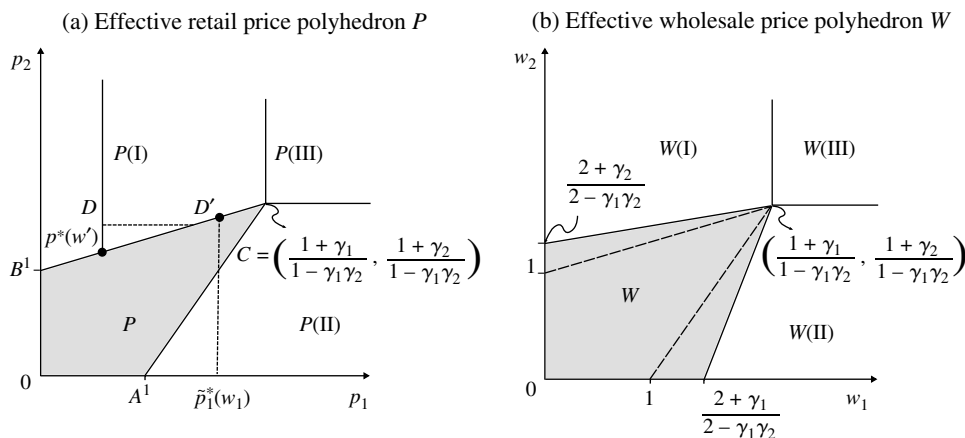
When $w \in W$, the following price vector is the unique equilibrium in P, see (7):

$$p_1^*(w) = \frac{2 + \gamma_1}{4 - \gamma_1\gamma_2} + \frac{2}{4 - \gamma_1\gamma_2} w_1 + \frac{\gamma_1}{4 - \gamma_1\gamma_2} w_2,$$

$$p_2^*(w) = \frac{2 + \gamma_2}{4 - \gamma_1\gamma_2} + \frac{\gamma_2}{4 - \gamma_1\gamma_2} w_1 + \frac{2}{4 - \gamma_1\gamma_2} w_2. \tag{9}$$

As long as $w \in W^o$, by Theorem 2(b), $p^*(w)$ is the unique equilibrium in the entire price space. Equation (9) confirms Proposition 4(b) below, in particular that an increase (decrease) in the wholesale price of one of the two products results in an increase (decrease) of all equilibrium retail prices. Depending on the intensity of the competition, characterized by the magnitude of γ_1 and γ_2, 50%–67% of a wholesale price cut is passed on to the consumers of that product, consistent with the lower bounds in Proposition 4(b). In addition, the cut in the wholesale price also results in a reduction of the retail price of the other product by anywhere from 0%–$33\frac{1}{3}$%. These results show that both the lower and upper bounds in Proposition 4 are tight when $\gamma_1\gamma_2 = 0$.

Figure 1. Effective price polyhedra.

(a) Effective retail price polyhedron P (b) Effective wholesale price polyhedron W

If $w \notin W^o$, the unique equilibrium in P is given by (9) with w replaced by $w' = \Theta(w)$:

$$
\begin{aligned}
p_1^*(w') &= \frac{2+\gamma_1}{4-\gamma_1\gamma_2} + \frac{2}{4-\gamma_1\gamma_2}w_1' + \frac{\gamma_1}{4-\gamma_1\gamma_2}w_2', \\
p_2^*(w') &= \frac{2+\gamma_2}{4-\gamma_1\gamma_2} + \frac{\gamma_2}{4-\gamma_1\gamma_2}w_1' + \frac{2}{4-\gamma_1\gamma_2}w_2'.
\end{aligned}
\tag{10}
$$

Next, consider $w \in W(\mathrm{I})$ (The case $w \in W(\mathrm{II})$ is analogous.): In this area of the plane, $w_1' = w_1$ (i.e., $t_1 = 0$) and $w_2' = w_2 - t_2$, such that $0 = [Q(w-t)]_2 = [\Psi(R)a]_2 - [\Psi(R)R(w-t)]_2$, from which we get $w_2' = w_2 - t_2 = ((2+\gamma_2)+\gamma_2 w_1)/(2-\gamma_1\gamma_2)$. Substituting into (10), we get

$$
\begin{aligned}
p_1^*(w') &= \frac{1+\gamma_1}{2-\gamma_1\gamma_2} + \frac{1}{2-\gamma_1\gamma_2}w_1, \\
p_2^*(w') &= \frac{2+\gamma_2}{2-\gamma_1\gamma_2} + \frac{\gamma_2}{2-\gamma_1\gamma_2}w_1.
\end{aligned}
\tag{11}
$$

Thus, when $w \in W(\mathrm{I})$, the wholesale price w_2 is so high that retailer 2 cannot respond effectively, and retailer 1 remains as a monopolist. His retail price may be expressed as the affine function (11) of w_1 only. Note that when $w \in W(\mathrm{I})$, the equilibrium price of product 1 is *more* sensitive to changes in the wholesale price w_1, than when $w \in W$ because $2/(4-\gamma_1\gamma_2) \leqslant 1/(2-\gamma_1\gamma_2)$.

When $w \in W(\mathrm{I})$, the unique retail price equilibrium in P is on the edge BC between $P(\mathrm{I})$ and P. In Online Appendix D, we verify that when $\gamma_1, \gamma_2 > 0$, $p^*(w')$ is the *unique* equilibrium, altogether. When $\gamma_1 = 0$ or $\gamma_2 = 0$, $p_1^*(w') = \tilde{p}_1^*(w_1) = (1+\gamma_1)/(2(1-\gamma_1\gamma_2)) + \frac{1}{2}w_1$ and all points on the vertical half line above $p^*(w')$ are equilibria.

Finally, consider the case wherein $w \in W(\mathrm{III})$. In Online Appendix D, we verify that all points in $W(\mathrm{III}) = P(\mathrm{III})$ are Nash equilibria, with the point $C \in P$ as the component-wise smallest equilibrium; in all, neither product is sold.

We conclude that the set of price equilibria may be the single vector $p^*(\Theta(w)) = p^*(w')$, a half line or a full quadrant of the plane. In addition, our example shows that, while any equilibrium $p^o \notin P$ has $\Omega(p^o) = p^*(w')$, see Theorem 1, not all points in $\{p \geqslant 0 \mid \Omega(p) = p^*(w')\}$ need to be equilibria.

Beyond a full characterization of the equilibria in the retailer competition model, and beyond a simple scheme to compute the (component-wise smallest) price equilibrium, we illustrate how this analytical characterization can be used to characterize how the retail price equilibrium responds to changes in the wholesale prices. In the marketing literature, this is referred to as the cost pass-through problem, see, e.g., Besanko et al. (2005) and Moorthy (2005). The following proposition provides a closed-form expression for the matrix of all direct and cross-product pass-through rates, along with even simpler upper and lower bound matrices.[3]

PROPOSITION 4 (COST PASS-THROUGHS). *Consider $w \in W^o$. (a) $\partial p^*(w)/\partial w = [R + T(R)]^{-1}T(R)$. (b) Under Assumption* (IS), *i.e., if R is intra-firm symmetric,*

$$
\frac{I}{2} \leqslant \frac{\partial p^*(w)}{\partial w} = [R+T(R)]^{-1}T(R) \leqslant \frac{R^{-1}T(R)}{2};
$$

For a monopoly, the lower bound is tight.

One important qualitative implication of these results is that direct cost pass-through rates, i.e., the diagonal elements of the matrix $\partial p^*(w)/\partial w$ are at least 50%, and all cross-product pass-through rates at least non-negative, as long as the matrix R is intrafirm symmetric. However, when R is a general asymmetric matrix, some of the pass-through rates may be negative; we exhibit this phenomenon as part of Example 2 below. Negative direct and cross-product pass-through rates have been reported in several marketing studies, e.g., Dubé and Gupta (2008).

The following proposition provides analytical expressions for the sensitivity of the above cost pass-through rates to marginal changes in the elements of the R-matrix:

PROPOSITION 5 (SENSITIVITY OF COST PASS-THROUGH RATES WITH RESPECT TO R). *For any entry (l, l') and $\delta > 0$, let $\tilde{R} \equiv R + \delta E_{l,l'}$, where $E_{l,l'}$ is a matrix with only entry (l, l') equal to 1 and the rest equal to 0. Assume $\tilde{R}$ satisfies Assumptions* (Z) *and* (P). *Consider $w \in W^o \cap \tilde{W}^o$, where W^o ($\tilde{W}^o$) denotes the interior of the effective wholesale price polyhedron under R ($\tilde{R}$).*

(a) *The difference of the cost pass-through matrix before and after perturbation is a rational function of δ:*

$$
\frac{\partial \tilde{p}^*(w)}{\partial w} - \frac{\partial p^*(w)}{\partial w}
$$

$$
= \begin{cases}
\Gamma(R, \delta)T(R) + \delta \Upsilon_{l'l}(\Xi(R) + \Gamma(R, \delta)) \\
\quad \text{if products } l, l' \text{ are sold by the same retailer,} \\[2mm]
-\dfrac{\delta \Xi_{\mathcal{N}, l'} \Xi_{l, \mathcal{N}}}{1 + \Xi_{ll'}\delta}T(R) \quad \text{otherwise,}
\end{cases}
$$

where $\Xi(R) = \Xi = [R + T(R)]^{-1}$,

$\Gamma(R, \delta)$

$$
= \frac{-\delta \begin{pmatrix} \Xi_{\mathcal{N}, l} & \Xi_{\mathcal{N}, l'} \end{pmatrix} \begin{pmatrix} -\Xi_{ll}\delta & 1+\Xi_{l'l}\delta \\ 1+\Xi_{ll'}\delta & -\Xi_{l'l'}\delta \end{pmatrix} \begin{pmatrix} \Xi_{l, \mathcal{N}} \\ \Xi_{l', \mathcal{N}} \end{pmatrix}}{(1+\Xi_{ll'}\delta)(1+\Xi_{l'l}\delta) - \Xi_{ll}\Xi_{l'l'}\delta^2}
$$

and $\Upsilon_{l'l}(M)$ is a matrix whose l-th column equals the l'-th column of M and whose remaining entries equal to 0.

(b) *If products l and l' are not sold by the same retailer and R is intrafirm symmetric* (IS), *the cost pass-through matrix remains nonnegative after perturbation.*

Thus, when a pair of products (l, l') is sold by different retailers, the impact of a perturbation in the cross-price sensitivity coefficient $R_{ll'}$ by a quantity δ, is given by the ratio of two affine functions in δ. Moreover, part (b)

shows that all cost pass-through rates remain non-negative, when the matrix R is intrafirm symmetric, regardless of the size of the perturbation in $R_{ll'}$. When the pair of products (l, l') is sold by the same retailer, the proposition shows that the change in any of the cost pass-through rates, and hence the rate itself, is given by the ratio of a cubic and a quadratic function in δ. For such pairs of products, it is possible to obtain negative cost pass-through rates due to the (intrafirm) asymmetry resulting from the perturbation. The sign of the cost pass-through rate may change only at a root of the cubic function in the numerator or the quadratic function in the denominator, all of which can be found in closed form.

Farahat and Perakis (2009) have used the analytical characterization of the competitive equilibrium to derive bounds for the so-called efficiency ratio, i.e., the ratio of $\pi^d(w)$, the aggregate equilibrium profits and $\pi^c(w)$, those arising in a centralized system wherein all retailers merge into one. As mentioned, their analysis is confined to the case wherein the matrix R is symmetric and $w \in P^o$. The lower and upper bounds for the efficiency ratio are given by the smallest and largest eigenvalue of a matrix related to R.

Below, we generalize those bounds to allow for a general asymmetric R-matrix. We start with the case wherein $w \in W \cap P$, so that the full product assortment is sold on the market in the centralized and decentralized system, see Proposition 2(c). It follows from Proposition 2(a) that

$$
\begin{aligned}
\pi^d(w) &= (p^*(w) - w)^T q(p^*(w)) \\
&= q(w)^T [R + T(R)]^{-T} T(R) [R + T(R)]^{-1} q(w) \\
&= q(w)^T \Phi(R) q(w) = q(w)^T \left[\frac{\Phi(R) + \Phi(R)^T}{2} \right] q(w),
\end{aligned}
$$

where $\Phi(R) \equiv [R + T(R)]^{-T} T(R) [R + T(R)]^{-1}$, easily verified to be positive definite, since $T(R)$ is, itself a direct consequence of R being positive definite, see the proof of Theorem 1. Thus, the symmetrized matrix $[(\Phi(R) + \Phi(R)^T)/2]$ is positive definite and symmetric. A similar expression may be obtained for $\pi^c(w)$, merely replacing $T(R)$ by R^T, since in the centralized system all products are sold by the same retailer. Thus, let $\Pi(R) = [R + R^T]^{-T} R^T [R + R^T]^{-1}$. Then $\pi^c(w) = q(w)^T [(\Pi(R) + \Pi(R)^T)/2] q(w)$.

PROPOSITION 6 (EFFICIENCY BOUNDS). *Let L be the Cholesky factorization of $(\Phi(R) + \Phi(R)^T)/2$, i.e., $LL^T = (\Phi(R) + \Phi(R)^T)/2$ and let*

$$
G \equiv L^{-1} \left[\frac{\Pi(R) + \Pi(R)^T}{2} \right] L^{-T}
$$

with the smallest (largest) eigenvalue $\lambda_{\min}$ ($\lambda_{\max}$). For any $w \in W \cap P$,

$$
\frac{1}{\lambda_{\max}} \leqslant \frac{\pi^d(w)}{\pi^c(w)} \leqslant \frac{1}{\lambda_{\min}}.
$$

When $w \notin (W \cap P)$, the equilibrium in the centralized or decentralized system, or in both, fails to be given by the simple expression for $p^*(w)$ in (7). Moreover, the associated product assortments are likely to be different from each other. In this case, whether R is symmetric or asymmetric, it does not appear possible to bound the efficiency ratio by eigenvalues of a matrix that is closely related to the original matrix R, see Proposition 2. However, we have shown that the exact expressions for $\pi^c(w)$ and $\pi^d(w)$ may be computed with a very modest effort that is quite comparable to the effort to compute the matrix G and its eigenvalues. Thus, in the most general case, for an arbitrary matrix R and arbitrary wholesale price vector w, the exact efficiency ratio is easily computed, as we have done in the examples below.

3.1. The Impact of Asymmetry in the Price Sensitivity Matrix R

As mentioned in the Introduction, the price sensitivity matrix R, typically, fails to be symmetric. Table 1 shows that the equilibrium characterizations obtained under symmetry, i.e., its first column, can be extended to general asymmetric structures. The question remains: How much of an impact does asymmetry have on various equilibrium performance measures, i.e., how different are these measures compared with a model in which R is replaced by its symmetrized version $\tilde{R} = (R + R^T)/2$? We explore this with three examples in which a symmetric matrix R is perturbed into an asymmetric one $\tilde{R} = R + \delta U$, where $\delta > 0$ and U is a matrix with upper triangular (lower triangular) off-diagonal elements equal to $+1$ (-1) and all diagonal elements equal to 0. (See Online Appendix E for an additional such example.)

For each example, we exhibit, all as a function of δ, the profits for each product and retailer, the aggregate profits in the decentralized, and centralized system as well as the exact efficiency ratio. (Throughout these examples, the bounds for the efficiency ratio in Proposition 6 tend to be loose, with the lower bound decreasing, and the gap between the bounds increasing with δ; the patterns for the exact efficiency ratio are different, though.)

EXAMPLE 2 (2 FIRMS, 3 PRODUCTS). There are three products {A, B, C}. Product A is carried by retailer 1 and products B and C by retailer 2. The raw demand functions are specified by:
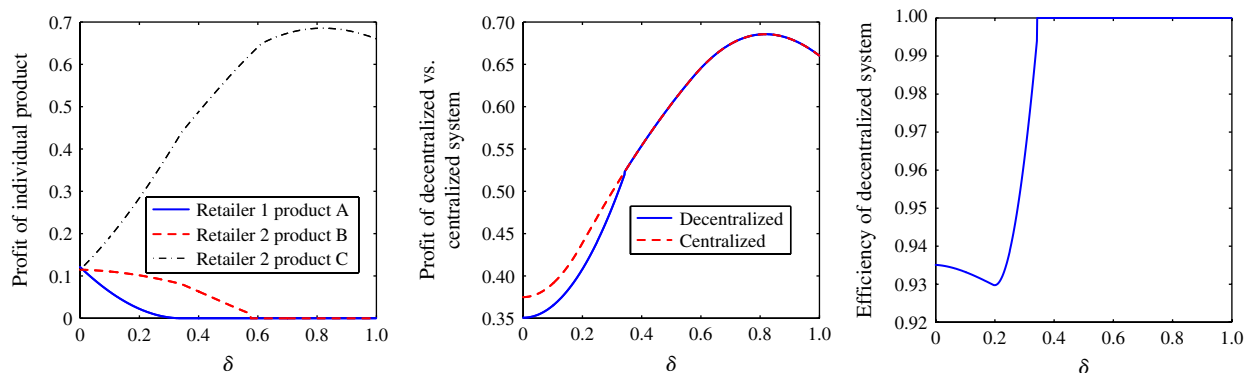
$$
a = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} \quad \text{and} \quad \tilde{R} = \begin{pmatrix} 4 & -1+\delta & -1+\delta \\ -1-\delta & 4 & -1+\delta \\ -1-\delta & -1-\delta & 4 \end{pmatrix}.
$$

The symmetrized matrix $(\tilde{R} + \tilde{R}^T)/2$ is always

$$
R = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}.
$$

As δ increases, the matrix R becomes increasingly asymmetric. Consider the wholesale price vector $w^o = (2, 2, 2)^T$.

Figure 2. (Color online) A 2-firm, 3-product example.

In the symmetric case ($\delta = 0$), $w^o \in P^o$, so that $p^*(w^o)$ arises as the unique equilibrium, and all three products are sold in the market. Figure 2 displays the profits for the three products, and those of the two retailers, as δ increases from 0 to 1, the maximum degree of asymmetry before the products cease to be substitutes, i.e., before $\tilde{R}$ ceases to be a Z-matrix. Equilibrium profits for products A and B decrease with δ, while those for product C increase, faster than the decline of product B's profits, until $\delta = 0.822$. For $\delta > 0.822$, retailer 2's aggregate profits decline with δ. (See Online Appendix F for the supporting calculations.)

Not only does δ, the degree of asymmetry, have a significant impact on prices, sales volumes, and profits, it affects the market structure and product assortment as well. In this case, there are two critical values for δ: $\delta_1 = 0.3423$ and $\delta_2 = 0.5758$. As long as $\delta < \delta_1$, both retailers maintain a market share in equilibrium and all three products are sold in the market. For $\delta_1 < \delta < \delta_2$, retailer 1 is driven out of the market and retailer 2 operates as a monopolist selling products B and C. When $\delta > \delta_2$, retailer 2 drops product B from the assortment, selling C as the exclusive product.

In Figures 2(b) and 2(c), we display the aggregate profits in the centralized and decentralized solution, as well as the efficiency ratio. In the centralized solution, there are also two critical threshold values $\delta_3 = 0.1940$ and $\delta_4 = \delta_1 = 0.3423$, with the same progression of the product assortment from {A, B, C} to {B, C} to {C}. The efficiency ratio decreases when $\delta \leqslant \delta_3$, then increases in $[\delta_3, \delta_1]$. When $\delta \geqslant \delta_1$, retailer 2 is a monopolist, both in the centralized and competitive setting, and the efficiency ratio is 1.

Finally, we exhibit the cost pass-through rate matrix that applies when $\delta < \delta_1 = 0.3423$, so that $w^o \in W^o$ and all three products are sold in equilibrium. It follows from Proposition 4(a) that

$$\frac{\partial p^*(w)}{\partial w}$$
$$= [\tilde{R} + T(\tilde{R})]^{-1} T(\tilde{R})$$

$$= \begin{pmatrix} \dfrac{12}{\delta^2 + 23} & \dfrac{13 - \delta}{\delta^2 + 23} - \dfrac{1}{2} & \dfrac{1}{2} - \dfrac{2\delta + 10}{\delta^2 + 23} \\ \dfrac{2(\delta + 1)}{\delta^2 + 23} & \dfrac{2\delta + 6}{\delta^2 + 23} - \dfrac{\delta}{20} + \dfrac{1}{4} & \boxed{-\dfrac{\delta}{20} + \dfrac{6 - 2\delta}{\delta^2 + 23} - \dfrac{1}{4}} \\ \dfrac{2(\delta + 1)}{\delta^2 + 23} & \dfrac{\delta}{20} + \dfrac{2\delta + 6}{\delta^2 + 23} - \dfrac{1}{4} & \dfrac{\delta}{20} + \dfrac{6 - 2\delta}{\delta^2 + 23} + \dfrac{1}{4} \end{pmatrix}.$$

One easily verifies that $\partial p_2^*(w)/\partial w_3 < 0$ for $0.0789 < \delta < 0.3423$. Thus, negative cost pass-through rates may occur even under a modest degree of asymmetry. Every entry in the cost pass-through matrix is a ratio of two cubic functions in δ. It is easily verified that, in full generality, every element in the cost pass-through matrix associated with $\tilde{R} = R + \delta U$ can be written as the ratio of two polynomials of degree N. This implies that the sign of a cost pass-through rate may change at most $2N$ times at the roots of the two polynomials of degree N.

To assess the robustness of the above patterns, we have repeated these analyses by varying the wholesale price vector w. To this end, we have varied each product's wholesale price from 0.4 to 2 in increments of 0.4, giving rise to 125 wholesale price vectors. Most patterns are indeed robust: The profits for products A and B always decrease with δ, while those for product C always increase. In 57% (31%) of the scenarios, the maximum profit decline for product A (B) is as large as 100%, the same as in Figure 2. For product C, the profit may sometimes increase by more than tenfold! As to the aggregate profits in the decentralized and centralized system, while they may exhibit various monotonicity patterns, the maximum downward deviation may be as large as 70% and the maximum increase, due to asymmetry, above 100%. The maximum positive or negative change in the efficiency ratio may be as large as 8 percentage points. In Online Appendix G, we display histograms of the largest deviations in all 6 of the above measures across all values $\delta \in [0, 1)$.

EXAMPLE 3 (3 FIRMS, 3 PRODUCTS). The next example, taken from Table 3 in Vilcassim et al. (1999), has three retailers, each offering a single product. The raw demand functions are specified by:

$$a = \begin{pmatrix} 1,390,909.00 \\ 777,338.00 \\ 418,007.00 \end{pmatrix} \quad \text{and}$$

$$R = \begin{pmatrix} 554,006.82 & -154,623.86 & 0 \\ 0 & 297,354.55 & -16,136.71 \\ 0 & -154,623.86 & 64,546.59 \end{pmatrix},$$

where the $(1,3)$ entry of matrix R is converted from 8,391.06 to 0, following the treatment by Farahat and Perakis (2009, Section 4). Symmetrizing this asymmetric matrix R and parameterizing it in the same way the R matrix in Example 2 was parameterized, we obtain the family of matrices:

$$\tilde{R} = \begin{pmatrix} 554,006.82 & -77,311.93(1-\delta) & 0 \\ -77,311.93(1+\delta) & 297,354.55 & -85,380.285(1-\delta) \\ 0 & -85,380.285(1+\delta) & 64,546.59 \end{pmatrix}.$$

The reported average retail prices are given by $p = (2.69, 1.99, 2.61)^T$. No wholesale prices are reported in Vilcassim et al. (1999). We arbitrarily select $w = (2, 1.5, 2)^T$.

Figure 3 displays the equilibrium profits for the three retailers, the aggregate of these profits, the aggregate profits in the centralized solution, and the efficiency ratio. In this case, the degree of asymmetry δ does not affect the market structure or product assortment: Over the complete range $\delta \in [0, 1]$, all three retailers maintain a positive market share. The impact of δ on the profit values is significant but far less than in the previous example. At the same time, the efficiency ratio is considerably lower, continuously decreasing from 77.5% when $\delta = 0$ to 71.2% when $\delta = 0.8$ and then increasing to 71.6% when $\delta = 1$.

As in the previous example, we have tested the robustness of the above findings by varying the wholesale price

for products 1 and 3 from 0.4 to 2 in increments of 0.4, and the wholesale price for product 2 from 0.3 to 1.5 in increments of 0.3. This again gives rise to 125 scenarios: Uniformly, the profits for retailers 1 and 2 decrease with δ, and those of retailer 3 increase with δ. In Online Appendix H, we again exhibit the histograms for the maximum deviation, due to asymmetry, i.e., among all $\delta \in [0, 1)$ for all of the above 6 performance measures. The maximum deviation percentages in each of the retailers' profits are, uniformly, in the double digits.

EXAMPLE 4 (10 PRODUCTS). This example demonstrates how easily the above analyses scale up for systems with larger numbers of products N. It also exhibits the impact of different market structures, i.e., when the same set of 10 products with the same raw demand functions are sold by 2, 4 or 10 independent retailers. The raw demand functions are specified by:

$$a = \begin{pmatrix} 5 \\ 5 \\ \vdots \\ 5 \end{pmatrix} \quad \text{and}$$

$$\tilde{R} = \begin{pmatrix} 4 & -0.2(1+\delta) & \cdots & -0.2(1+\delta) \\ -0.2(1-\delta) & 4 & \cdots & -0.2(1+\delta) \\ \vdots & \vdots & \ddots & \vdots \\ -0.2(1-\delta) & \cdots & -0.2(1-\delta) & 4 \end{pmatrix}.$$

Consider the whole price vector $w^o = (2, 2, \ldots, 2)^T$. We compare the following three market structures: (a) 2 Firms: Retailer 1 carries the first product, and Retailer 2 carries the remaining 9 products; (b) 4 Firms: Retailer 1 carries the first product, and each of the other three retailers carries the next set of 3 products; (c) 10 Firms: Each of the ten retailers carries 1 product. See Figure 4 for a display of the various performance measures under the three market structures.

Figure 3. (Color online) A 3-firm, 3-product example from Vilcassim et al. (1999).
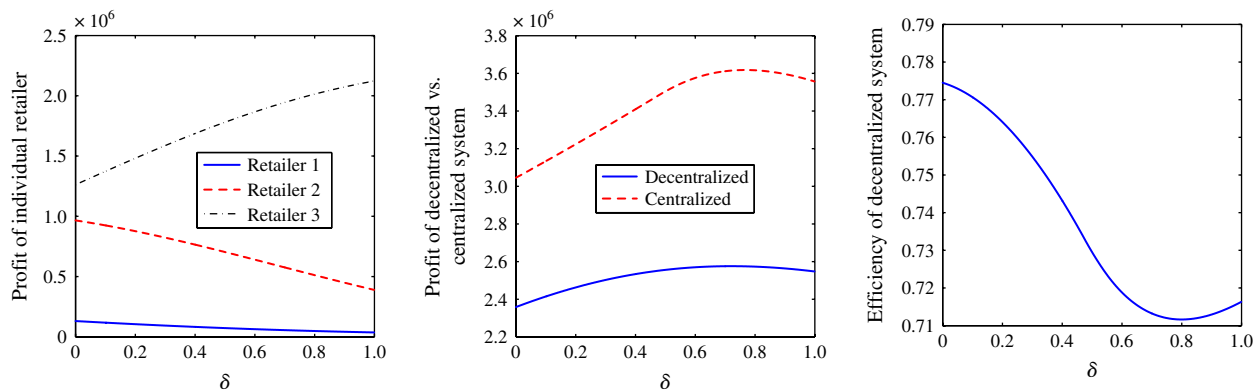
Figure 4.   (Color online) 2 firms, 4 firms, 10 firms.
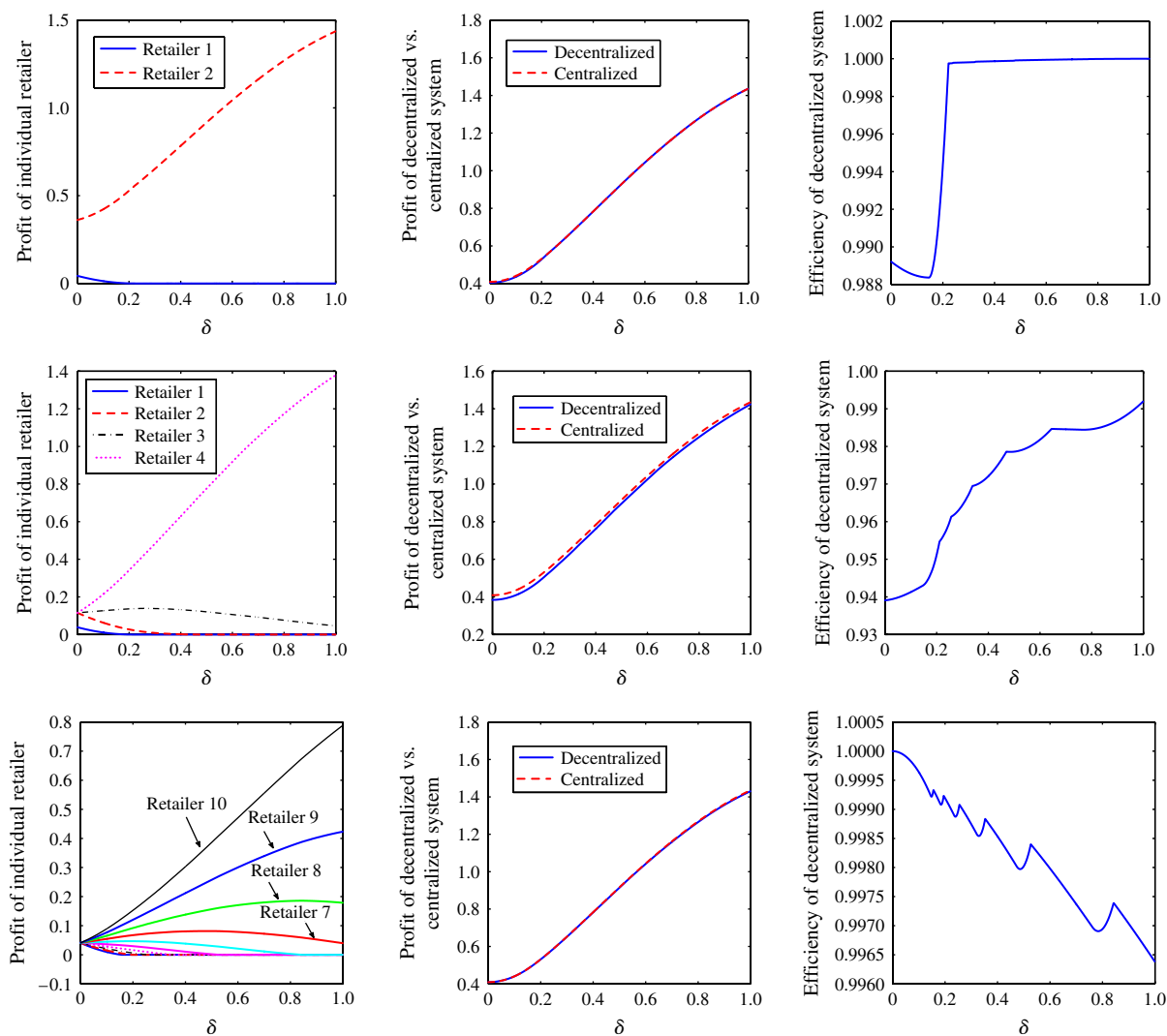
4. Conclusions and Extensions

We have characterized the equilibrium behavior in a retailer price competition model in which each retailer selects a product assortment from a set of potential products, along with associated retail prices. The demand functions are the unique regular extension of a set of functions that are affine on the "effective price polyhedron" P, a given polyhedron of the price space: The demand volumes and product assortments associated with a general price vector p, are obtained by applying this affine function to the projection of p onto the polyhedron P.

We have provided a complete characterization of the equilibrium behavior under an arbitrary cost rate vector w, and minimal conditions for the model primitives: A positive definite matrix R that has nonpositive off-diagonal elements, since the different products represent substitutes. (See, however Footnote 1, for generalizations allowing for certain types of complementarities.) We have shown that there exists exactly one pure Nash equilibrium in P.

Depending on the cost rate vector w, there may be multiple, in fact sometimes infinitely many, pure equilibria. However, there is always a component-wise smallest price equilibrium, and all price equilibria are equivalent in the sense of generating identical equilibrium sales volumes, product assortments and identical profit levels for the retailers. This smallest price equilibrium may be obtained by applying an affine function, either directly to the vector of cost rates, or to its projection onto a specific effective wholesale price polyhedron.

The results in this paper provide the foundation for several interesting analyses. For example, under the same general consumer demand model, they may be used to characterize the equilibrium behavior in multi-echelon multi-product supply chain networks where any number of firms compete in each of the echelons, selecting product assortments and prices, see Federgruen and Hu (2013). The crisp characterizations of the equilibria also allow for clear quantitative and qualitative insights about the system-wide

impacts of changes in various model primitives, other than the exogenous cost rates pursued in §3. As a final example, the above multi-echelon results may be used to characterize the impacts of various types of vertical integration.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi .org/10.1287/opre.2015.1380.

Acknowledgments

The authors thank the area editor Chung Piaw Teo, the associate editor, and three anonymous reviewers for their guidance and constructive comments.

Endnotes

1. Assumption (Z) can be relaxed to allow for certain types of complementarities, see Federgruen and Hu (2013).
2. Farahat and Perakis (2011) consider the additional special case wherein $w \in \partial P$. They establish that $p^*(w)$ is a Nash equilibrium in this case as well. The paper does not address whether this equilibrium is unique or what kind of assortment it is associated with. Instead, almost invariably when $w \in \partial P$, $w \in W^o$ so that the UF characterization prevails, see the first row of Table 1. However, in very rare cases, $w \in \partial P \cap \partial W$, in which case there exist multiple equilibria that are associated with incomplete product assortments, see the middle row of Table 1 and see point C in Example 1.
3. We confine ourselves to the case wherein $w \in W^o$. The proposition may be generalized for arbitrary wholesale prices w, i.e., for $w \in \partial W$ or $w \notin W$. However, this generalization requires the construction of price sensitivity matrices associated with the prevailing equilibrium product assortment $\mathscr{A} \subseteq \mathscr{N}$. Moreover, for vectors w where a marginal wholesale price increase alters the product assortment, the marginal cost pass-through rate may be different when a wholesale price increases as opposed to decreases.

References

Allon G, Federgruen A (2007) Competition in service industries. *Oper. Res.* 55(1):37–55.

Besanko D, Dubé J-P, Gupta S (2005) Own-brand and cross-brand retail pass-through. *Marketing Sci.* 24(1):123–137.

Besbes O, Sauré D (2014) Product assortment and price competition under multinomial logit demand. *Production Oper. Management.* Forthcoming.

Cottle RW, Pang JS, Stone RE (1992) *The Linear Complementarity Problem* (Academic Press, Boston).

Dubé, J-P, Gupta S (2008) Cross-brand pass-through in supermarket pricing. *Marketing Sci.* 27(3):324–333.

Dubé J-P, Manchanda P (2005) Differences in dynamic brand competition across markets: An empirical analysis. *Marketing Sci.* 24(1):81–95.

Farahat A, Perakis G (2009) Profit loss in differentiated oligopolies. *Oper. Res. Lett.* 37(1):43–46.

Farahat A, Perakis G (2010) A nonnegative extension of the affine demand function and equilibrium analysis for multiproduct price competition. *Oper. Res. Lett.* 38(4):280–286.

Farahat A, Perakis G (2011) A comparison of bertrand and cournot profits in oligopolies with differentiated products. *Oper. Res.* 59(2):507–513.

Farrell J, Shapiro C (2010a) Antitrust evaluation of horizontal mergers: An economic alternative to market definition. *BE J. Theor. Econ.* 10(1):1–41.

Farrell J, Shapiro C (2010b) Recapture, pass-through, and market definition. *Antitrust Law J.* 76(3):585–604.

Federgruen A, Hu M (2013) Price competition in sequential multi-product oligopolies. Unabridged memo.

Li J, Rusmevichientong P, Simester D, Tsitsiklis JN, Zoumpoulis SI (2015) The value of field experiments. *Management Sci.*, ePub ahead of print February 19, http://dx.doi.org/10.1287/mnsc.2014.2066.

Mangasarian OL (1976) Linear complementarity problems solvable by a single linear program. *Math. Programming* 10(1):263–270.

McGuire TW, Staelin R (2008) An industry equilibrium analysis of downstream vertical integration. *Marketing Sci.* 27(1):115–130. Reprint of *Marketing Sci.* 2(2):161–191.

Moorthy S (2005) A general theory of pass-through in channels with category management and retail competition. *Marketing Sci.* 24(1): 110–122.

Rusmevichientong P, Shen Z-JM, Shmoys DB (2010) Dynamic assortment optimization with a multinomial logit choice model and capacity constraint. *Oper. Res.* 58(6):1666–1680.

Rusmevichientong P, Shmoys DB, Tong C, Topaloglu H (2014) Assortment optimization under the multinomial logit model with random choice parameters. *Production Oper. Management* 23(11): 2023–2039.

Shubik M, Levitan R (1980) *Market Structure and Behavior* (Harvard University Press, Cambridge, MA).

Soon W, Zhao G, Zhang J (2009) Complementarity demand functions and pricing models for multi-product markets. *Eur. J. Appl. Math.* 20(5):399–430.

Vilcassim NJ, Kadiyali V, Chintagunta PK (1999) Investigating dynamic multifirm market interactions in price and advertising. *Management Sci.* 45(4):499–518.

Awi Federgruen is the Charles E. Exley professor of management at the Graduate School of Business at Columbia University.

Ming Hu is an assistant professor of operations management at Rotman School of Management, University of Toronto. His research interests include revenue management, supply chain management, and service operations management. Most recently, he studies operations management in the context of social buying, crowdfunding, crowdsourcing, and two-sided markets.