Online Appendix to "Sequential Multi-Product Price Competition in Supply Chain Networks"

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A. Preliminaries.

A matrix is a P-matrix, if all of its principal minors are positive. It is well known that a positive definite matrix is a P-matrix. A matrix which is both a Z-matrix and a P-matrix is referred to as a ZP-matrix. We use the following properties of ZP-matrices.

Lemma A.1 (Properties of ZP-matrices). *Let X be a ZP-matrix and Y be a Z-matrix such that $X \leq Y$, i.e., $Y - X \geq 0$. Then*

(a) X^{-1} exists and $X^{-1} \geq 0$;

(b) Y is a ZP-matrix and $Y^{-1} \leq X^{-1}$;

(c) XY^{-1} and $Y^{-1}X$ are ZP-matrices; and

(d) If D is a positive diagonal matrix, then DX, XD and $X + D$ are ZP-matrices.

Proof of Lemma A.1. (a)-(d). By Horn and Johnson (1991, Theorem 2.5.3), a ZP-matrix is a nonsingular, so-called, M-matrix. Properties (a)-(d) of ZP-matrices can be found in Horn and Johnson (1991, Section 2.5) as properties of M-matrices. $\square$

We need the following properties of the projection operator.

Lemma A.2 (Projection). *(a) $\Omega(p) \in P$; if $p \in P$, $\Omega(p) = p$.*

(b) If $p \notin P$, $\Omega(p)$ is on the boundary of P.

(c) $\Omega(p)$ may be computed by minimizing *any linear objective $\phi^T t$ with $\phi > 0$ over the polyhedron, described by (3).*

(d) The projection operator $\Omega(\cdot)$ is monotonically increasing, and each component of $\Omega(\cdot)$ is a jointly concave function.

Proof of Lemma A.2. (a) See Lemma 2 in Federgruen and Hu (2013).

(b) Since $p \notin P$, the correction vector $t \neq 0$; thus, there exists a product l with $t_l > 0$ and by (3), $[a - R(p-t)]_l = 0$, implying that $\Omega(p)$ is on the boundary of P.

(c) Follows from Theorem 2 in Mangasarian (1976), since R is a Z-matrix.

(d) Let $p^1 \leq p^2$. Fix a product l. To show $\Omega(p^1)_l \leq \Omega(p^2)_l$, choose $\phi \in \mathbb{R}^N$ as follows: let $\phi_l = 1$ and $\phi_{l'} = \epsilon$ for all $l' \neq l$ and $\epsilon > 0$ arbitrarily small. Note that with the change of variables $u \equiv p - t$, the Linear Program described in part (c) is equivalent to

$$z_\epsilon(p) \equiv \max\ \phi^T u$$

$$\text{s.t.} \quad u \le p,$$
$$a - Ru \ge 0.$$

Clearly, $z_\epsilon(p^1) \le z_\epsilon(p^2)$, since the feasible region under $p = p^2$ contains that under $p = p^1$. Thus, $\Omega(p^1)_l = \lim_{\epsilon \downarrow 0} z_\epsilon(p^1) \le \lim_{\epsilon \downarrow 0} z_\epsilon(p^2) = \Omega(p^2)_l$. Finally, by a standard argument, $z_\epsilon(p)$ is a jointly concave function, for any $\epsilon > 0$, and the same applies to $\Omega(p)_l = \lim_{\epsilon \downarrow 0} z_\epsilon(p)$. $\square$

B. Proofs.

Proof of Proposition 1. Parts (a) and (b) follow from Theorems 2 and 3 in Federgruen and Hu (2013). The same pair of theorems also show part (c), i.e., $\Omega(\tilde{p}) = p^*$ for any equilibrium $\tilde{p}$. This implies that all equilibria share the same retailer sales volumes $d(p^*)$; moreover, since $p^* = \tilde{p} - t$ with $t \ge 0$ and $d(\tilde{p} - t) = q(p^*)$; we have

$$\begin{aligned}
\pi_i(\tilde{p}) &= \sum_{(j,k)\in\mathcal{N}(i)} (p^*_{ijk} + t_{ijk} - w_{ijk})[q(\tilde{p} - t)]_{ijk} \\
&= \sum_{(j,k)\in\mathcal{N}(i)} (p^*_{ijk} - w_{ijk})[q(p^*)]_{ijk} + \sum_{(j,k)\in\mathcal{N}(i)} t_{ijk}[q(\tilde{p} - t)]_{ijk} \\
&= \sum_{(j,k)\in\mathcal{N}(i)} (p^*_{ijk} - w_{ijk})[q(p^*)]_{ijk} = \pi_i(p^*),
\end{aligned}$$

where the one next to last identity holds because of (3).

The expression of the component-wise smallest equilibrium for $w \in W$ and $w \notin W$ follows from Proposition 4(a) and Theorem 3 in Federgruen and Hu (2013), respectively. $\square$

Proof of Theorem 1. (a) We need to show that the regular extension $D(w)$ of the affine functions $Q(w) = b - Sw$, is obtained by applying the affine functions to the projection $\Theta(w)$. The latter result was shown in Soon et al. (2009), when the matrix S is a positive definite Z-matrix. By Lemma 1 below, S is a Z-matrix. Moreover,

$$S \equiv \Psi(R)R = T(R)[R + T(R)]^{-1}R = T(R)[\mathtt{I} + R^{-1}T(R)]^{-1} = [T(R)^{-1} + R^{-1}]^{-1} \tag{B.1}$$

is positive definite, since R, and hence $T(R)$, are positive definite and the inverse of a positive definite matrix is positive definite.

If R is symmetric, so is S: By (B.1), $S = [R^{-1} + T(R)^{-1}]^{-1}$. Since R is symmetric, so is $T(R)$ and so are their inverses R^{-1} and $T(R)^{-1}$; the same applies to $[R^{-1} + T(R)^{-1}]$ and its inverse.

Finally, $b = \Psi(R)a \ge 0$ follows from $\Psi(R) \ge 0$, a result shown in Federgruen and Hu (2013, Proposition 4(e)).

Parts (b) and (c). The proof is analogous to that of Theorem 1 in Federgruen and Hu (2013). The proof of part (b) only requires that S is positive definite, a property just verified. The proof of part (c) merely requires that $b \geq 0$ and S is a positive definite Z-matrix, properties verified in part (a). $\square$

Proof of Theorem 2. (a) Clearly $Q(c^0) = b - S(S^{-1}b) = 0$. Moreover, since by Theorem 1(a), S is a positive definite Z-matrix, $S^{-1} \geq 0$, see, e.g., Horn and Johnson (1991, Theorem 2.5.3). By Theorem 1(a), $b \geq 0$, so that $0 \leq c^0 = S^{-1}b = R^{-1}\Psi(R)^{-1}b = R^{-1}a$. (To verify the second equality, note that both R and $\Psi(R)$ are invertible: R is invertible because it is positive definite, by assumption (P); $\Psi(R) = T(R)[T(R)+R]^{-1}$ is invertible as the product of two invertible matrices, with $T(R)$ invertible because it is positive definite, as well.)

(b) Analogous to the proof of Theorem 2 in Federgruen and Hu (2013).

(c) Analogous to the proof of Theorem 3 in Federgruen and Hu (2013), after establishing that

$$\Psi(S)b \geq 0 \ \text{ and } \ \Psi(S)S \text{ is a positive definite } Z\text{-matrix} \tag{B.2}$$

to ensure that the projection onto the polyhedron C in the space of cost rate vectors, is well defined, in the sense that any vector $c \in \mathbb{R}^N_+$ is projected onto a *non-negative* vector c'. By Theorem 1(a), S is a symmetric positive definite Z-matrix and $b \geq 0$. Thus, the induced demand functions $D(w)$ are the unique regular extension of the system of affine functions $Q(w) = b - Sw$ with (b, S) sharing the same properties as (a, R). Applying the above arguments to the functions $Q(\cdot)$, (B.2) follows. $\square$

Proof of Proposition 2. $P \subseteq W$: Since $W = \{w \geq 0 : \Psi(R)q(w) \geq 0\}$, it suffices to show that $\Psi(R) \geq 0$: if $0 \leq x \in P$, $q(x) \geq 0$ and $\Psi(R)q(x) \geq 0$, i.e., $x \in W$. But, $\Psi(R) \geq 0$ follows from Proposition 4(e) in Federgruen and Hu (2013). The proof of $W \subseteq C$ is analogous. $\square$

Proof of Proposition 3. (a) We show that $\{P^{(e)}, e = 1, 2, \ldots, m+1\}$ is nested. Analogously to (8), for any e,

$$a^{(e+1)} - R^{(e+1)}p = \Psi^{(e+1)}(R^{(e)})(a^{(e)} - R^{(e)}p),$$

see (12). Since $R = R^{(1)}$ is symmetric, $\Psi(R^{(1)}) \geq 0$, a result shown in Federgruen and Hu (2013, Proposition 4(e)). Recursively, for any e, $R^{(e)}$ is symmetric and hence, $\Psi^{(e+1)}(R^{(e)}) \geq 0$. Therefore, for any e,

$$P^{(e)} = \{p \geq 0 \mid a^{(e)} - R^{(e)}p \geq 0\} \subseteq \{p \geq 0 \mid \Psi^{(e+1)}(R^{(e)})(a^{(e)} - R^{(e)}p) \geq 0\} = P^{(e+1)}.$$

(b) We show that $\{P^{(e)}, e = 1, 2, \ldots, m+1\}$ is contained in hypercube H. For any $p \in P^{(e)}$, $p \geq 0$ and $a^{(e)} - R^{(e)}p \geq 0$. Since $R^{(e)}$ is a ZP-matrix, ${R^{(e)}}^{-1} \geq 0$, see Lemma A.1(a). Then for any $p \in P^{(e)}$, $p \geq 0$ and $p \leq {R^{(e)}}^{-1}a^{(e)} = \cdots = R^{-1}a$, i.e., $p \in H$.

(c) An alternative characterization of polyhedron $P^{(e)}$ is by its extreme points. Note that $P^{(e)}$ is an N-dimensional polyhedron with $2N$ linear constraints: $p \geq 0$ and $a^{(e)} - R^{(e)}p \geq 0$. An extreme point is the intersection of N hyperplanes corresponding with N constraints chosen from the total of these $2N$ constraints. The set of constraints may be referred to by a pair of index sets $(\mathcal{A}^1, \mathcal{A}^2)$, where $\mathcal{A}^1 \subseteq \mathcal{N}$ is the index set for the set of constraints $p \geq 0$ and $\mathcal{A}^2 \subseteq \mathcal{N}$ is the index set for the set of constraints $a^{(e)} - R^{(e)}p \geq 0$, which are binding at the extreme point:

$$p_{\mathcal{A}^1} = 0 \;\text{ and }\; [q^{(e)}(p)]_{\mathcal{A}^2} = [a^{(e)} - R^{(e)}p]_{\mathcal{A}^2} = 0.$$

Note that $\mathcal{A}^1$ and $\mathcal{A}^2$ must be mutually exclusive, if $a > 0$: When a product l has its price equal to 0, since $a_l > 0$, its demand cannot be equal to zero; Thus, since $|\mathcal{A}^1 \cup \mathcal{A}^2| = N$, $\mathcal{A}^2 = \mathcal{N} \setminus \mathcal{A}^1$. If for some product l, $a_l = 0$, the extreme points may be degenerate, the set of products that have zero prices may be strictly larger than $\mathcal{A}^1$. Nevertheless, it is still sufficient to use one index set $\mathcal{A} \subseteq \mathcal{N}$ to characterize an extreme point. That is, an extreme point, denoted by $z^{(e)}(\mathcal{A})$, is the unique solution of the system of linear equations:

$$p_{\bar{\mathcal{A}}} = 0 \;\text{ and }\; [q^{(e)}(p)]_{\mathcal{A}} = [a^{(e)} - R^{(e)}p]_{\mathcal{A}} = 0.$$

(Note that for degenerate extreme points, there exists an index set $\mathcal{S} \supset \bar{\mathcal{A}}$ such that $p_{\mathcal{S}} = 0$.) Since $p_{\bar{\mathcal{A}}} = 0$,

$$[q^{(e)}(p)]_{\mathcal{A}} = [a^{(e)} - R^{(e)}p]_{\mathcal{A}} = a^{(e)}_{\mathcal{A}} - R^{(e)}_{\mathcal{A},\mathcal{A}} p_{\mathcal{A}} = 0.$$

Hence, for any e and $\mathcal{A}$,

$$[z^{(e)}(\mathcal{A})]_{\mathcal{A}} = [R^{(e)}_{\mathcal{A},\mathcal{A}}]^{-1} a^{(e)}_{\mathcal{A}} \geq 0, \tag{B.3}$$

where, because of Lemma A.1(a), the inequality is due to the fact that $R^{(e)}$ is a ZP-matrix and $a^{(e)} \geq 0$, as shown in part (b) (since $R^{(e)}$ is a ZP-matrix, so is $R^{(e)}_{\mathcal{A},\mathcal{A}}$). This also verifies that the extreme points are indeed non-negative.

The extreme point, $z^{(e+1)}(\mathcal{A})$, for polyhedron $P^{(e+1)}$ satisfies: $[z^{(e+1)}(\mathcal{A})]_{\bar{\mathcal{A}}} = [z^{(e)}(\mathcal{A})]_{\bar{\mathcal{A}}} = 0$ and

$$[q^{(e+1)}(p)]_{\mathcal{A}} = [\Psi^{(e+1)}(R^{(e)})(a^{(e)} - R^{(e)} z^{(e+1)}(\mathcal{A}))]_{\mathcal{A}} = 0.$$

For notational simplicity, we temporarily denote $\Psi^{(e+1)}(R^{(e)})$ by Ψ. Moreover, since $[z^{(e+1)}(\mathcal{A})]_{\bar{\mathcal{A}}} = 0$, $z^{(e+1)}(\mathcal{A})$ satisfies:

$$0 = [\Psi^{(e+1)}(R^{(e)})(a^{(e)} - R^{(e)}p)]_{\mathcal{A}} = \Psi_{\mathcal{A},\mathcal{A}}(a^{(e)}_{\mathcal{A}} - R^{(e)}_{\mathcal{A},\mathcal{A}} p_{\mathcal{A}}) + \Psi_{\mathcal{A},\bar{\mathcal{A}}}(a^{(e)}_{\bar{\mathcal{A}}} - R^{(e)}_{\bar{\mathcal{A}},\mathcal{A}} p_{\mathcal{A}}),$$

i.e.,

$$\Psi_{\mathcal{A},\mathcal{A}} a^{(e)}_{\mathcal{A}} + \Psi_{\mathcal{A},\bar{\mathcal{A}}} a^{(e)}_{\bar{\mathcal{A}}} = [\Psi_{\mathcal{A},\mathcal{A}} R^{(e)}_{\mathcal{A},\mathcal{A}} + \Psi_{\mathcal{A},\bar{\mathcal{A}}} R^{(e)}_{\bar{\mathcal{A}},\mathcal{A}}][z^{(e+1)}(\mathcal{A})]_{\mathcal{A}}.$$

We write:

$$
\begin{aligned}
\Psi_{\mathcal{A},\mathcal{A}} a^{(e)}_{\mathcal{A}} &\le \Psi_{\mathcal{A},\mathcal{A}} a^{(e)}_{\mathcal{A}} + \Psi_{\mathcal{A},\bar{\mathcal{A}}} a^{(e)}_{\bar{\mathcal{A}}} \\
&= [\Psi_{\mathcal{A},\mathcal{A}} R^{(e)}_{\mathcal{A},\mathcal{A}} + \Psi_{\mathcal{A},\bar{\mathcal{A}}} R^{(e)}_{\bar{\mathcal{A}},\mathcal{A}}][z^{(e+1)}(\mathcal{A})]_{\mathcal{A}} \\
&\le \Psi_{\mathcal{A},\mathcal{A}} R^{(e)}_{\mathcal{A},\mathcal{A}}[z^{(e+1)}(\mathcal{A})]_{\mathcal{A}}, \qquad \text{(B.4)}
\end{aligned}
$$

where the first inequality is due to $\Psi \ge 0$, hence $\Psi_{\mathcal{A},\bar{\mathcal{A}}} \ge 0$ and $a^{(e)} \ge 0$, and the second inequality is due to the fact that $R^{(e)}$ is a Z-matrix, hence $R^{(e)}_{\bar{\mathcal{A}},\mathcal{A}} \le 0$, while $\Psi_{\mathcal{A},\bar{\mathcal{A}}} \ge 0$ and $[z^{(e+1)}(\mathcal{A})]_{\mathcal{A}} \ge 0$ (see (B.3)).

Analogous to the fact that $\Psi^{(e+1)}(R^{(e)})R^{(e)}$ is a ZP-matrix, we can show that $\Psi_{\mathcal{A},\mathcal{A}} R^{(e)}_{\mathcal{A},\mathcal{A}}$ is a ZP-matrix since $R^{(e)}_{\mathcal{A},\mathcal{A}}$ is a symmetric ZP-matrix. Hence, $[\Psi_{\mathcal{A},\mathcal{A}} R^{(e)}_{\mathcal{A},\mathcal{A}}]^{-1} \ge 0$. By (B.3) and (B.4),

$$
[z^{(e+1)}(\mathcal{A})]_{\mathcal{A}} \ge [R^{(e)}_{\mathcal{A},\mathcal{A}}]^{-1} a^{(e)}_{\mathcal{A}} = [z^{(e)}(\mathcal{A})]_{\mathcal{A}}.
$$

Hence, the series $\{z^{(e)}(\mathcal{A})\}$ is monotone.

By part (b), this series of extreme points $\{z^{(e)}(\mathcal{A})\}$ is bounded above by the corresponding extreme point, $h(\mathcal{A})$, of the hypercube H, characterized by $[h(\mathcal{A})]_{\mathcal{A}} = [R^{-1}a]_{\mathcal{A}}$ and $[h(\mathcal{A})]_{\bar{\mathcal{A}}} = 0$. By the Monotone Convergence Theorem, this series of extreme points $\{z^{(e)}(\mathcal{A})\}$ converges. This argument holds for an *arbitrary* index set $\mathcal{A}$, and the corresponding extreme point. Since a polyhedron can be characterized by its extreme points, the sequence of polyhedra $\{P^{(e)}\}$ converges. $\square$

Proof of Theorem 3. (a) Let $\gamma \equiv \Psi(S)b$ and $U \equiv \Psi(S)S$. When the suppliers choose the vector of cost rates $[c + \delta e_l]$, the resulting equilibrium sales volumes are obtained as the unique regular extension $D^S(c + \delta e_l)$ of the affine functions $Q^S(c + \delta e_l) = \gamma - U[c + \delta e_l]$. It follows from (B.2) that $\gamma \ge 0$ while U is a positive definite Z-matrix, i.e., U has positive diagonal elements and non-positive off-diagonal elements. Thus, as c_l increases by δ, $Q^S_l(c + \delta e_l)$ decreases *linearly*. Let Δ^+ denote the root of the equation $Q^S_l(c + \delta e_l) = 0$. Thus $D^S(c + \Delta^+ e_l) = Q^S(c + \Delta^+ e_l) = 0$. By the definition of regularity, any increase of δ beyond Δ^+ has *no* impact on any of the demand volumes, see Definition 1. When $\delta \le \Delta^+$, the remaining monotonicity properties follow from Proposition 2 in Federgruen and Hu (2013), applied to the system of demand functions $D^S(\cdot)$.

(b) It follows from part (a) that when $\delta > \Delta^+$, an increase in δ has *no* impact on any of the products' demand volumes, and, a fortiori, on the equilibrium assortment. When $\delta \le \Delta^+$, it follows from part (a), that the demand volume of all other products $l' \ne l$ increases (weakly), while that of product l remains positive, by the definition of the root Δ^+. This implies that, for $\delta \le \Delta^+$, the product assortment remains the same or *expands*.

Finally, let $\Delta \equiv \max\{0 \le \delta \le \Delta^+ : \text{assortment } \mathcal{A}^0 \text{ is the assortment under the cost rate vector } c = c^0 + \delta e_l\}$. Thus, when $\delta \in [0, \Delta]$, the product assortment remains given by the set $\mathcal{A}^0$. This implies that the demand volume of any product $l \notin \mathcal{A}^0$ remains equal to 0, while that of the products in $\mathcal{A}^0$ varies *linearly* with δ, see Proposition 2 in Federgruen and Hu (2013) applied to the affine functions $Q^S(\cdot)$ and the set $\mathcal{A}^0$.

(c) It follows from Theorem 2(b) that w^*, the component-wise smallest equilibrium in the supplier competition game, has $w^* \in W$. Recall from Proposition 1 and (10) that

$$p^* = [R + T(R)]^{-1} a + [R + T(R)]^{-1} T(R) w^*, \tag{B.5}$$

$$w^* = [S + T(S)]^{-1} b + [S + T(S)]^{-1} T(S) \Gamma(c). \tag{B.6}$$

By Lemma A.2(d) applied to the projection operator, $\Gamma(\cdot)$ is a monotonically increasing operator. The fact that both p^* and w^* are increasing vector-functions of c, thus follows by using the fact that $[R + T(R)]^{-1} T(R) \ge 0$ and $[S + T(S)]^{-1} T(S) \ge 0$, see Proposition 1 parts (d) and (f).

Finally, we have shown that every product l's equilibrium retail and wholesale price p_l^* and w_l^* is an increasing affine function of $\Gamma(c)$, while Lemma A.2(d) shows that $\Gamma(\cdot)$ is a vector of jointly concave functions. This establishes that p_l^* and w_l^* are jointly concave functions of c. $\square$

Proof of Corollary 1. (a) If $w \in W^o$, there exists a ball around the vector w which is contained within W. The result is immediate from (B.5) and Theorem 3(c).

(b) If $c \in C^o$, there exists a ball around the vector c which is contained within C so that, in this ball, $\Gamma(c) = c$. The result then follows by substituting (B.6) into (B.5), while the sign of the elements of the pass-through matrix follows from Theorem 3(c).

(c) When $c \notin C^o$, the marginal pass-through rates of cost changes are no longer immediate from (B.5) and (B.6). To address this case, we first need the following lemma.

LEMMA B.1. *Fix $c \in \mathbb{R}_+^N$. Let $\mathcal{A}$ denote the (unique) assortment associated with the equilibria, under c. Let $P_{\mathcal{A}}^o$ denote the subspace of the retail price space $\mathbb{R}_+^N$ on which the same assortment $\mathcal{A}$ arises.*

(a) $P_{\mathcal{A}}^o = \{p \in \mathbb{R}_+^N \mid d_{\mathcal{A}}(p) = a^{\mathcal{A}} - R^{\mathcal{A}} p_{\mathcal{A}} > 0 \text{ and } p_{\bar{\mathcal{A}}} \ge R_{\bar{\mathcal{A}},\bar{\mathcal{A}}}^{-1}[a_{\bar{\mathcal{A}}} - R_{\bar{\mathcal{A}},\mathcal{A}} p_{\mathcal{A}}]\}$, *where* $a^{\mathcal{A}} \equiv a_{\mathcal{A}} - R_{\mathcal{A},\bar{\mathcal{A}}} R_{\bar{\mathcal{A}},\bar{\mathcal{A}}}^{-1} a_{\bar{\mathcal{A}}} \ge 0$, *and* $R^{\mathcal{A}} \equiv R_{\mathcal{A},\mathcal{A}} - R_{\mathcal{A},\bar{\mathcal{A}}} R_{\bar{\mathcal{A}},\bar{\mathcal{A}}}^{-1} R_{\bar{\mathcal{A}},\mathcal{A}}$ *is a positive definite Z-matrix.*

(b)

$$p_{\mathcal{A}}^* = [R^{\mathcal{A}} + T(R^{\mathcal{A}})]^{-1} [a^{\mathcal{A}} + T(R^{\mathcal{A}}) w_{\mathcal{A}}^*], \tag{B.7}$$

$$w_{\mathcal{A}}^* = [S^{\mathcal{A}} + T(S^{\mathcal{A}})]^{-1} [b^{\mathcal{A}} + T(S^{\mathcal{A}}) c_{\mathcal{A}}], \tag{B.8}$$

where $b^{\mathcal{A}} = \Psi(R^{\mathcal{A}}) a^{\mathcal{A}}$ *and* $S^{\mathcal{A}} = \Psi(R^{\mathcal{A}}) R^{\mathcal{A}}$.

Proof of Lemma B.1. (a) Fix $p \in P^o_{\mathcal{A}}$. Let t denote the unique price correction vector such that $d(p) = q(p-t)$. It follows from (3) that $t_{\mathcal{A}} = 0$, since $d_{\mathcal{A}}(p) > 0$. It follows from $0 = d_{\bar{\mathcal{A}}}(p) = a_{\bar{\mathcal{A}}} - R_{\bar{\mathcal{A}},\mathcal{A}} p_{\mathcal{A}} - R_{\bar{\mathcal{A}},\bar{\mathcal{A}}}(p_{\bar{\mathcal{A}}} - t_{\bar{\mathcal{A}}})$ and $t_{\bar{\mathcal{A}}} \geq 0$, that

$$p_{\bar{\mathcal{A}}} - t_{\bar{\mathcal{A}}} = R^{-1}_{\bar{\mathcal{A}},\bar{\mathcal{A}}}[a_{\bar{\mathcal{A}}} - R_{\bar{\mathcal{A}},\mathcal{A}} p_{\mathcal{A}}], \tag{B.9}$$

and $p_{\bar{\mathcal{A}}} \geq R^{-1}_{\bar{\mathcal{A}},\bar{\mathcal{A}}}[a_{\bar{\mathcal{A}}} - R_{\bar{\mathcal{A}},\mathcal{A}} p_{\mathcal{A}}]$. (Since R is positive definite, so is $R_{\bar{\mathcal{A}},\bar{\mathcal{A}}}$, so that $R_{\bar{\mathcal{A}},\bar{\mathcal{A}}}$ is invertible.) This verifies the second set of inequalities in the characterization of $P^o_{\mathcal{A}}$. Substituting (B.9) into $d_{\mathcal{A}}(p) = a_{\mathcal{A}} - R_{\mathcal{A},\mathcal{A}} p_{\mathcal{A}} - R_{\mathcal{A},\bar{\mathcal{A}}}(p_{\bar{\mathcal{A}}} - t_{\bar{\mathcal{A}}})$, we get

$$0 < d_{\mathcal{A}}(p) = (a_{\mathcal{A}} - R_{\mathcal{A},\bar{\mathcal{A}}} R^{-1}_{\bar{\mathcal{A}},\bar{\mathcal{A}}} a_{\bar{\mathcal{A}}}) - (R_{\mathcal{A},\mathcal{A}} - R_{\mathcal{A},\bar{\mathcal{A}}} R^{-1}_{\bar{\mathcal{A}},\bar{\mathcal{A}}} R_{\bar{\mathcal{A}},\mathcal{A}}) p_{\mathcal{A}} = a^{\mathcal{A}} - R^{\mathcal{A}} p_{\mathcal{A}}, \tag{B.10}$$

thus verifying the first set of inequalities in the description of $P^o_{\mathcal{A}}$. By Assumptions (Z) and (P), $R_{\mathcal{A},\bar{\mathcal{A}}} \leq 0$, $R_{\bar{\mathcal{A}},\mathcal{A}} \leq 0$ and $R_{\bar{\mathcal{A}},\bar{\mathcal{A}}}$ is a positive definite Z-matrix. By Lemma A.1(a), $R^{-1}_{\bar{\mathcal{A}},\bar{\mathcal{A}}} \geq 0$ and hence $a^{\mathcal{A}} = a_{\mathcal{A}} - R_{\mathcal{A},\bar{\mathcal{A}}} R^{-1}_{\bar{\mathcal{A}},\bar{\mathcal{A}}} a_{\bar{\mathcal{A}}} \geq a_{\mathcal{A}} \geq 0$. Since $R^{\mathcal{A}}$ is the Schur complement of a principal submatrix $R_{\mathcal{A},\mathcal{A}}$ of matrix R, $R^{\mathcal{A}}$ is positive definite by Lemma A.2(a) in Federgruen and Hu (2013). Moreover, since $R_{\mathcal{A},\bar{\mathcal{A}}} R^{-1}_{\bar{\mathcal{A}},\bar{\mathcal{A}}} R_{\bar{\mathcal{A}},\mathcal{A}} \geq 0$, $R^{\mathcal{A}} \leq R_{\mathcal{A},\mathcal{A}}$. Therefore, $R^{\mathcal{A}}$ is a Z-matrix.

Conversely, fix $p \in \mathbb{R}^N_+$. Assume

$$a^{\mathcal{A}} - R^{\mathcal{A}} p_{\mathcal{A}} > 0, \tag{B.11}$$

$$p_{\bar{\mathcal{A}}} \geq R^{-1}_{\bar{\mathcal{A}},\bar{\mathcal{A}}}[a_{\bar{\mathcal{A}}} - R_{\bar{\mathcal{A}},\mathcal{A}} p_{\mathcal{A}}]. \tag{B.12}$$

Let $t_{\bar{\mathcal{A}}}$ denote the surplus variables in (B.12) and $t_{\mathcal{A}} = 0$. Thus, $t = (t_{\mathcal{A}}, t_{\bar{\mathcal{A}}}) \geq 0$. It suffices to show that $[a - R(p-t)]_{\mathcal{A}} > 0$ and $[a - R(p-t)]_{\bar{\mathcal{A}}} = 0$. The latter follows immediately from (B.9), while $[a - R(p-t)]_{\mathcal{A}} = a_{\mathcal{A}} - R_{\mathcal{A},\mathcal{A}} p_{\mathcal{A}} - R_{\mathcal{A},\bar{\mathcal{A}}}(p_{\bar{\mathcal{A}}} - t_{\bar{\mathcal{A}}}) = a_{\mathcal{A}} - R_{\mathcal{A},\mathcal{A}} p_{\mathcal{A}} - R_{\mathcal{A},\bar{\mathcal{A}}} R^{-1}_{\bar{\mathcal{A}},\bar{\mathcal{A}}}[a_{\bar{\mathcal{A}}} - R_{\bar{\mathcal{A}},\mathcal{A}} p_{\mathcal{A}}] = a^{\mathcal{A}} - R^{\mathcal{A}} p_{\mathcal{A}} > 0$, by (B.11).

(b) The vector equations (B.7) and (B.8) follow immediately by proving the following result: Consider the retailer competition model under a given wholesale price vector w. Let p^* denote the component-wise smallest equilibrium in this game, and let $\mathcal{A}$ denote the associated assortment of products. In other words, $p^*_{\mathcal{A}} \in P^o_{\mathcal{A}}$. Then

$$p^*_{\mathcal{A}} = [R^{\mathcal{A}} + T(R^{\mathcal{A}})]^{-1}[a^{\mathcal{A}} + T(R^{\mathcal{A}}) w_{\mathcal{A}}]. \tag{B.13}$$

Applying (B.13) to $w = w^*$, we get (B.7). The proof of (B.8) is analogous.

To prove (B.13), for $p \in P^o_{\mathcal{A}}$, $d_{\mathcal{A}}(p) = a^{\mathcal{A}} - R^{\mathcal{A}} p_{\mathcal{A}} > 0$ and $d_{\bar{\mathcal{A}}}(p) = 0$, where $a^{\mathcal{A}} \geq 0$ and $R^{\mathcal{A}}$ is a positive definite Z-matrix. For $p \in P^o_{\mathcal{A}}$,

$$\pi_i(p) = (p_{\mathcal{N}(i)\cap\mathcal{A}} - w_{\mathcal{N}(i)\cap\mathcal{A}})[a^{\mathcal{A}} - R^{\mathcal{A}} p_{\mathcal{A}}]_{\mathcal{N}(i)\cap\mathcal{A}}.$$

p^* *must* satisfy the First Order Conditions (FOC): for $(i,j,k)\in\mathcal{A}$,

$$0=\frac{\partial \pi_i(p)}{\partial p_{ijk}}=[a^{\mathcal{A}}-R^{\mathcal{A}}p_{\mathcal{A}}]_{ijk}-R^{\mathcal{A}}_{ijk,ijk}(p_{ijk}-w_{ijk})-\sum_{(i,j',k')\in\mathcal{A},(j',k')\neq(j,k)}R^{\mathcal{A}}_{ij'k',ijk}(p_{ij'k'}-w_{ij'k'}).$$

Rearranging terms, we get

$$\begin{aligned}2R^{\mathcal{A}}_{ijk,ijk}p_{ijk}+\sum_{(i,j',k')\in\mathcal{A},(j',k')\neq(j,k)}(R^{\mathcal{A}}_{ij'k',ijk}+R^{\mathcal{A}}_{ijk,ij'k'})p_{ij'k'}+\sum_{(i',j',k')\in\mathcal{A},i'\neq i}R^{\mathcal{A}}_{ijk,i'j'k'}p_{i'j'k'}\\ =a^{\mathcal{A}}_{ijk}+R^{\mathcal{A}}_{ijk,ijk}w_{ijk}+\sum_{(i,j',k')\in\mathcal{A},(j',k')\neq(j,k)}R^{\mathcal{A}}_{ij'k',ijk}w_{ij'k'}.\end{aligned}$$

It is convenient to write this system of $|\mathcal{A}|$ linear equations in $|\mathcal{A}|$ unknowns in matrix form as: $[R^{\mathcal{A}}+T(R^{\mathcal{A}})]p_{\mathcal{A}}=a^{\mathcal{A}}+T(R^{\mathcal{A}})w_{\mathcal{A}}$, or equivalently,

$$[R^{\mathcal{A}}+T(R^{\mathcal{A}})](p_{\mathcal{A}}-w_{\mathcal{A}})=a^{\mathcal{A}}-R^{\mathcal{A}}w_{\mathcal{A}}.$$

Hence, we can write

$$\begin{aligned}p^*_{\mathcal{A}}&=w_{\mathcal{A}}+[R^{\mathcal{A}}+T(R^{\mathcal{A}})]^{-1}(a^{\mathcal{A}}-R^{\mathcal{A}}w_{\mathcal{A}})\\&=[R^{\mathcal{A}}+T(R^{\mathcal{A}})]^{-1}a^{\mathcal{A}}+[\mathtt{I}-(R^{\mathcal{A}}+T(R^{\mathcal{A}}))^{-1}R^{\mathcal{A}}]w_{\mathcal{A}}\\&=[R^{\mathcal{A}}+T(R^{\mathcal{A}})]^{-1}a^{\mathcal{A}}+[(R^{\mathcal{A}}+T(R^{\mathcal{A}}))^{-1}(R^{\mathcal{A}}+T(R^{\mathcal{A}}))-(R^{\mathcal{A}}+T(R^{\mathcal{A}}))^{-1}R^{\mathcal{A}}]w_{\mathcal{A}}\\&=[R^{\mathcal{A}}+T(R^{\mathcal{A}})]^{-1}a^{\mathcal{A}}+[R^{\mathcal{A}}+T(R^{\mathcal{A}})]^{-1}T(R^{\mathcal{A}})w_{\mathcal{A}}\\&=[R^{\mathcal{A}}+T(R^{\mathcal{A}})]^{-1}[a^{\mathcal{A}}+T(R^{\mathcal{A}})w_{\mathcal{A}}].\qquad\square\end{aligned}$$

Theorem 3(c) shows that for every product l, the (component-wise smallest) retail and wholesale price equilibrium, p^*_l and w^*_l are concave functions of the cost-rate vector c. This implies that they are differentiable, almost everywhere. Moreover, their left- and right-hand (partial) derivatives with respect to any of the cost rates exist, everywhere. These can be obtained, straightforwardly, from the vector equations (B.7) and (B.8) in Lemma B.1.

Fix a product $l=(i,j,k)$ and consider the marginal impact of a decrease of its cost rate c_l to $c_l-\delta$. As in the proof of Theorem 3(b), one verifies that a *positive* threshold $\Delta>0$ exists, such that the assortment $\mathcal{A}$ remains unchanged, as long as $\delta\le\Delta$. (Unlike for cost increases, the threshold for cost reductions must be positive.) Then, equations (B.7) and (B.8) apply for all $c_l\in(c^0_l-\Delta,c^0_l]$. The left-hand derivative expressions follow immediately. $\square$

Proof of Proposition 4. (a) We write

$$[R^{\mathcal{A}}+T(R^{\mathcal{A}})]^{-1}T(R^{\mathcal{A}})=[[T(R^{\mathcal{A}})]^{-1}R^{\mathcal{A}}+\mathtt{I}]^{-1}.$$

Since R is symmetric, $T(R^{\mathcal{A}})$ is symmetric and then $T(R^{\mathcal{A}}) \geq R^{\mathcal{A}}$. By Lemma A.1(c), since $T(R^{\mathcal{A}})$ is a ZP-matrix and $R^{\mathcal{A}}$ is a Z-matrix, $[T(R^{\mathcal{A}})]^{-1}R^{\mathcal{A}}$ is a ZP-matrix. By Lemma A.1(d), $[T(R^{\mathcal{A}})]^{-1}R^{\mathcal{A}}+\mathtt{I}$ is a ZP-matrix, as well. By Lemma A.1(a),

$$[[T(R^{\mathcal{A}})]^{-1}R^{\mathcal{A}}+\mathtt{I}]^{-1} \geq 0. \tag{B.14}$$

Since $T(R^{\mathcal{A}})$ is a ZP matrix, hence $[T(R^{\mathcal{A}})]^{-1} \geq 0$ by Lemma A.1(a). Then because $R^{\mathcal{A}} \leq T(R^{\mathcal{A}})$, $[T(R^{\mathcal{A}})]^{-1}R^{\mathcal{A}} \leq \mathtt{I}$ and hence $[T(R^{\mathcal{A}})]^{-1}R^{\mathcal{A}}+\mathtt{I} \leq 2\mathtt{I}$. Since $[T(R^{\mathcal{A}})]^{-1}R^{\mathcal{A}}+\mathtt{I}$ is a ZP-matrix, we have by Lemma A.1(b), $[R^{\mathcal{A}}+T(R^{\mathcal{A}})]^{-1}T(R^{\mathcal{A}}) = [[T(R^{\mathcal{A}})]^{-1}R^{\mathcal{A}}+\mathtt{I}]^{-1} \geq \frac{\mathtt{I}}{2}$.

To prove the upper bound in (16), let $\Delta \equiv T(R^{\mathcal{A}})-R^{\mathcal{A}}$. Then $R^{\mathcal{A}} = T(R^{\mathcal{A}})-\Delta$. By the symmetry of $T(R^{\mathcal{A}})$, $\Delta \geq 0$. Then

$$\begin{aligned} R^{\mathcal{A}}[R^{\mathcal{A}}+T(R^{\mathcal{A}})]^{-1}T(R^{\mathcal{A}}) &= [T(R^{\mathcal{A}})-\Delta][R^{\mathcal{A}}+T(R^{\mathcal{A}})]^{-1}T(R^{\mathcal{A}}) \\ &\leq \frac{1}{2}[2T(R^{\mathcal{A}})-\Delta][R^{\mathcal{A}}+T(R^{\mathcal{A}})]^{-1}T(R^{\mathcal{A}}) \\ &= \frac{1}{2}[T(R^{\mathcal{A}})+R^{\mathcal{A}}][R^{\mathcal{A}}+T(R^{\mathcal{A}})]^{-1}T(R^{\mathcal{A}}) = \frac{1}{2}T(R^{\mathcal{A}}), \end{aligned}$$

where the inequality is due to $\Delta \geq 0$ and $[R^{\mathcal{A}}+T(R^{\mathcal{A}})]^{-1}T(R^{\mathcal{A}}) \geq 0$, by (B.14). Since $(R^{\mathcal{A}})^{-1} \geq 0$, we have the desired upper bound.

(b) Analogously, since S is symmetric (see Theorem 1(a)), $T(S^{\mathcal{A}})$ is symmetric. Hence,

$$\frac{\mathtt{I}}{2} \leq [S^{\mathcal{A}}+T(S^{\mathcal{A}})]^{-1}T(S^{\mathcal{A}}) \leq \frac{(S^{\mathcal{A}})^{-1}T(S^{\mathcal{A}})}{2}.$$

The bounds on the supplier cost pass through matrix follow immediately because all the bounds are non-negative matrices. $\square$

LEMMA B.2. *Fix $a^1 \leq a^2$. (a) $P(a^1) \subseteq P(a^2)$. (b) $W(a^1) \subseteq W(a^2)$. (c) $C(a^1) \subseteq C(a^2)$. (d) $\Gamma(c,a^1) \leq \Gamma(c,a^2)$. (e) $p^*(w,a^1) \leq p^*(w,a^2)$. (f) $w^*(c,a^1) \leq w^*(c,a^2)$.*

Proof of Lemma B.2. Rather than assuming that the matrix R is symmetric, we prove the lemma under the far weaker assumptions that $T(R)$ and $T(S)$ are symmetric. (Recall that under R is symmetric, S is symmetric as well, and so are $T(R)$ and $T(S)$.) Part (a) does not require any symmetry assumption. Parts (b) and (e) of the lemma only require that $T(R)$ be symmetric.

(a) Recall that

$$P(a) \equiv \{p \geq 0 \mid a - Rp \geq 0\}.$$

Suppose $p \in P(a^1)$, i.e., $p \geq 0$ and $a^1 - Rp \geq 0$. Since $a^1 \leq a^2$, we have $p \geq 0$ and $a^2 - Rp \geq a^1 - Rp \geq 0$, i.e., $p \in P(a^2)$. Hence, $P(a^1) \subseteq P(a^2)$.

(b) Recall that

$$W(a) \equiv \{w \geq 0 \mid \Psi(R)a - \Psi(R)Rw \geq 0\}.$$

Since $T(R)$ is symmetric, by Proposition 4(e) in Federgruen and Hu (2013), $\Psi(R) \geq 0$. If $a^1 \leq a^2$, then $\Psi(R)a^1 \leq \Psi(R)a^2$. Hence if $w \in W(a^1)$, $w \in W(a^2)$, i.e., $W(a^1) \subseteq W(a^2)$.

(c) Recall that

$$C(a) \equiv \left\{c \geq 0 \mid \Psi(S)\Psi(R)a - \Psi(S)\Psi(R)Rc \geq 0\right\}.$$

Since $T(R)$ and $T(S)$ are symmetric, $\Psi(R) \geq 0$ and $\Psi(S) \geq 0$, so that $\Psi(S)\Psi(R) \geq 0$. The result follows, as in part (b).

(d) The proof is analogous to that of Lemma A.2(d): one easily verifies that the Linear Program to be solved to compute the projection, under $a = a^1$, has a feasible region that is contained within the feasible region of the LP associated with $a = a^2$.

(e) By Proposition 1 parts (d) and (c),

$$p^*(w;a) = [R + T(R)]^{-1}a + [R + T(R)]^{-1}T(R)\Theta(w).$$

The monotonicity in a follows from $[R+T(R)]^{-1} \geq 0$, an immediate consequence of $R+T(R)$ being a ZP-matrix, see Lemma A.1(a).

(f) In view of Theorem 2 and (10), it suffices to show that $\Psi(R) \geq 0$ and $[S+T(S)]^{-1} \geq 0$; the former follows from $T(R)$ being symmetric, and the latter from the symmetry of $T(S)$. □

Proof of Theorem 4. (a)

$$w^*(\Gamma(c,a^1),a^1) \leq w^*(\Gamma(c,a^1),a^2) \leq w^*(\Gamma(c,a^2),a^2),$$

where the first inequality is due to Lemma B.2(f), and the second inequality is due to Lemma B.2(d) and Theorem 3(c), i.e., $w^*(c)$ is increasing in c.

(b)

$$p^*(w^*(\Gamma(c,a^1),a^1),a^1) \leq p^*(w^*(\Gamma(c,a^1),a^1),a^2) \leq p^*(w^*(\Gamma(c,a^2),a^2),a^2),$$

where the first inequality is due to Lemma B.2(e) and the second inequality is due to part (a) and Theorem 1(f), i.e., $p^*(w)$ is increasing in w.

(c) Fix an arbitrary product l. We consider two cases.

Case 1. Suppose $d_l(p^*(w^*(\Gamma(c,a^1),a^1),a^1)) = 0$. Then the claim trivially holds.

Case 2. Suppose $d_l(p^*(w^*(\Gamma(c,a^1),a^1),a^1)) = D_l^S(\Gamma(c,a^1),a^1) > 0$, where $D_l^S(\cdot)$ is defined in the proof of Theorem 3(a). Note that $c_l \geq [\Gamma(c,a^2)]_l \geq [\Gamma(c,a^1)]_l = c_l$, where the first inequality follows from (3), the second one from Lemma B.2(d), and the equality from $D^S_l(\Gamma(c,a^1),a^1) > 0$ and the complementary slackness conditions (3), applied to the demand system $D^S(\cdot)$. Thus,

$$\begin{aligned} 0 < D^S_l(\Gamma(c,a^1),a^1) &= [\Psi(S)b^1]_l - [\Psi(S)S]_{l,\mathcal{N}}\Gamma(c,a^1) \\ &= [\Psi(S)\Psi(R)a^1]_l - [\Psi(S)S]_{l,\mathcal{N}}\Gamma(c,a^1) \\ &\leq [\Psi(S)\Psi(R)a^2]_l - [\Psi(S)S]_{l,\mathcal{N}}\Gamma(c,a^2) = D^S_l(\Gamma(c,a^2),a^2), \end{aligned}$$

where the inequality is due to (i) $\Psi(S) \geq 0$ since $T(S)$ is symmetric and $\Psi(R) \geq 0$ since $T(R)$ is symmetric, see Proposition 4(e) in Federgruen and Hu (2013), which leads to $\Psi(S)\Psi(R)a^1 \leq \Psi(S)\Psi(R)a^2$; (ii) $\Psi(S)S$ is a Z-matrix since $T(S)$ is symmetric by Proposition 5(b) in Federgruen and Hu (2013), and (iii) $[\Gamma(c,a^1)]_{l'} \leq [\Gamma(c,a^2)]_{l'}$ for all $l' \neq l$, and $[\Gamma(c,a^1)]_l = [\Gamma(c,a^2)]_l = c_l$.

(d) Immediate from part (c).

(e) For any $w \in W$, $p^*(w) - w = [R + T(R)]^{-1}q(w)$, see Proposition 1(d). By Proposition 4(a) in Federgruen and Hu (2013), the demand volumes under $p^*(w)$ satisfy $q(p^*(w)) = \Psi(R)q(w)$, with $\Psi(R) = T(R)[R + T(R)]^{-1}$ invertible. Thus,

$$p^*(w) - w = [R + T(R)]^{-1}[\Psi(R)]^{-1}q(p^*(w)) = [T(R)]^{-1}q(p^*(w)) = [T(R)]^{-1}d(p^*(w)),$$

since $p^*(w) \in P$. Moreover, $[T(R)]^{-1} \geq 0$, since $T(R)$ is a ZP-matrix, see Lemma A.1(a). In other words, the vector of the retailers' profit margins for all N products is an increasing function of the vector of equilibrium sales volumes and the latter increases in a by part (c).

(f) Let $c^1 \equiv \Gamma(c,a^1) \in C(a^1) \subseteq C(a^2)$ and $c^2 \equiv \Gamma(c,a^2) \in C(a^2)$. It follows from Lemma B.2(f) that

$$\begin{aligned} w^*(c^1,a^1) - c^1 &\leq w^*(c^1,a^2) - c^1 = w^*(\Gamma(c,a^1),a^2) - \Gamma(c,a^1) \\ &= [S + T(S)]^{-1}\Psi(R)a^2 + [S + T(S)]^{-1}T(S)\Gamma(c,a^1) \\ &\leq [S + T(S)]^{-1}\Psi(R)a^2 + [S + T(S)]^{-1}T(S)\Gamma(c,a^2) = w^*(\Gamma(c,a^2),a^2) - \Gamma(c,a^2), \end{aligned}$$

where the second and last equality follow from Proposition 1(d) applied to the supplier competition game under $a = a^2$, respectively with $c = c^1 \in C(a^1) \subseteq C(a^2)$ and $c = c^2 \in C(a^2)$. The second inequality follows from $[S + T(S)]^{-1}T(S) \geq 0$, as shown in the proof of Proposition 1(f), and $\Gamma(c,a^1) \leq \Gamma(c,a^2)$ by Lemma B.2(d).

(g) Immediate from parts (c), (e) and (f). $\square$

Proof of Theorem 5. Theorem 1(a): The part that $D(w)$ arises as the unique regular extension of the affine functions $Q(w) = b - Sw$ follows by Soon et al. (2009), if the matrix S is a positive definite Z-matrix and $b \geq 0$ (see Soon et al. 2009, Theorem 4 and Lemma 6). By Lemma 1, $b \geq 0$ and S is a Z-matrix, since $T(R)$ is symmetric. Moreover, by the proof of Theorem 1(a), S is positive definite. Hence, all results in Theorem 1(a), except for the symmetry of S, continue to apply.

Theorem 1 parts (b) and (c): The proof is analogous to that of Theorem 1 in Federgruen and Hu (2013). The proof of part (b) only requires that S is positive definite. The proof of part (c) merely requires that $b \geq 0$ and S is a positive definite Z-matrix, properties verified in part (a).

Theorem 2(a): The proof of Theorem 2(a) only requires that $b \geq 0$ and S is a positive definite Z-matrix, properties verified in Theorem 1(a).

Theorem 2(b): The proof of Theorem 2(b) is analogous to the proof of Theorem 2 in Federgruen and Hu (2013), requiring only that S is a positive definite Z-matrix, a property verified in Theorem 1(a).

Theorem 2(c): Since $b \geq 0$ and S is a positive definite Z-matrix, properties verified in Theorem 1(a), the proof of Theorem 2(c) is analogous to the proof of Theorem 3 in Federgruen and Hu (2013), after establishing that

$$\Psi(S)b \geq 0 \text{ and } \Psi(S)S \text{ is a positive definite } Z\text{-matrix,}$$

which can be shown analogously to Lemma 1 under the symmetry assumption of matrix $T(S)$. Since S is positive definite, the positive definiteness of matrix $\Psi(S)S$ can be shown analogously to (B.1).

Proposition 2: The proof of $P \subseteq W$ requires $\Psi(R) \geq 0$, which follows from Proposition 4(e) in Federgruen and Hu (2013) under the symmetry assumption of matrix $T(R)$. Analogously, the proof of $W \subseteq C$ requires $\Psi(S) \geq 0$, which is guaranteed by the symmetry assumption of matrix $T(S)$. $\square$

Proof of Theorem 6. Theorem 3: The proof requires that $\gamma = \Psi(S)b \geq 0$ and $U = \Psi(S)S$ is a positive definite Z-matrix, which can be shown analogously to Lemma 1 under the symmetry assumption of matrix $T(S)$. In addition, for the proof of Theorem 3(c), one needs to show that $[R + T(R)]^{-1}T(R) \geq 0$ and $[S + T(S)]^{-1}T(S) \geq 0$, which can be guaranteed by the symmetry assumptions of matrices $T(R)$ and $T(S)$, see the proof of Proposition 1(f).

Corollary 1 parts (a) and (b): The proof resorts to Theorems 2 and 3, which have been shown to hold under the symmetry assumptions of matrices $T(R)$ and $T(S)$.

Lemma B.1: The proof requires that $a \geq 0$ and R is a positive definite Z-matrix, and that $b \geq 0$ and S is a positive definite Z-matrix. Since $T(R)$ is symmetric, the latter set of properties follow from Lemma 1.

Corollary 1 part (c): The proof resorts to Theorem 3, which has been shown to hold under the symmetry assumptions of matrices $T(R)$ and $T(S)$.

Proposition 4: The proposition was proved under the far weaker assumptions that $T(R)$ and $T(S)$ are symmetric, rather than assuming that the matrix R is symmetric. (Note that if $T(R)$ and $T(S)$ are symmetric, so are $T(R^{\mathcal{A}})$ and $T(S^{\mathcal{A}})$ for any assortment set $\mathcal{A}$.)

Lemma B.2: The lemma was proved under the far weaker assumptions that $T(R)$ and $T(S)$ are symmetric, rather than assuming that the matrix R is symmetric. Part (a) does not require any symmetry assumption. Parts (b) and (e) of the lemma only require that $T(R)$ be symmetric.

Theorem 4: The theorem was proved under the far weaker assumptions that $T(R)$ and $T(S)$ are symmetric, rather than assuming that the matrix R is symmetric. $\square$

C. The Extended Affine Model versus the MNL Model

As mentioned, our demand model is parsimonious, in that it is fully specified by a single $N \times N$-matrix R of price sensitivity coefficients, and a single N-dimensional intercept vector a. Cicchetti et al. (1976), Vilcassim et al. (1999) and Dubé and Manchanda (2005) have used the affine demand model to study, respectively, a ski-resorts, a personal care products and a packaged frozen entrées industry, all with $N \leq 6$ products. Sudhir (2001), Villas-Boas and Zhao (2005) and Pancras and Sudhir (2007) have estimated demand functions in food items industries, each with $N \leq 4$ products, employing a (mixed) MNL-model in each. Since $N \leq 6$, in each of these 6 studies, the effort to estimate an extended affine model involves estimating at most 42 parameters. For large values of N, the task of estimating the N^2 entries of the R-matrix may still be formidable. However, there are various approaches to reduce the effort. For example, the category partitioning approach is to partition the products into $m \ll N$ categories and have $R_{ll'}$ specified as a function of the category identities of products (l, l').

Beyond standard econometric techniques, the parameters may be estimated with field experiments, see Li et al. (2015). Without imposing *any* structure on the R-matrix, the approach requires $O(N \log N)$ experiments. However, the authors identify several approaches to reduce the number of required experiments to $O(\log N)$. Beyond the category partitioning approach, mentioned above, the authors address three approaches that impose a sparsity structure on the R-matrix. Using these field experiments, the authors obtained estimates of the R-matrix in studies involving up to $N = 80$ products. These studies addressed grocery, health and beauty product lines.

It is interesting to compare the above with the complexity of fitting an MNL model to a given industry. The number of parameters in an MNL model is at least equal to K, the number of attribute terms in the specified utility measures. (An attribute term is a single attribute value or a function of the various attribute values considered in the model.) However many MNL specifications assume that at least some of the coefficients of the attribute terms are product specific. For example, several specifications employ a product-dependent price sensitivity coefficient in the utility measures, see, e.g., Gallego and Wang (2014) and Aksoy-Pierson et al. (2013). In such specifications the number of parameters to be estimated is $O(NK)$ or $O(N+K)$.

The MNL and the affine model are similar in that both models allow the user control over the parameter complexity, by adjustments in the specification: in the MNL model, this is done by adjusting the number of attribute terms or by *restricting* oneself to, say, a uniform price sensitivity coefficient (for the price term in the utility measure). In the affine model, this is accomplished by imposing restrictions on the structure of the R-matrix.

It should be noted that the MNL model, intrinsically, comes with many restrictions. For example, it is well known, see, e.g., Berry et al. (1995, p.847) that in this model, the impact of a price change by a given product j, on the demand volume of *any other product* i is one and the same. More specifically, let $\epsilon_{i,j} = [(\partial d_i(p))/(\partial p_j)]/[(d_i(p))/p_j]$ denote the cross price elasticity of the demand for product i due to a price change for product j. Then $\epsilon_{i,j} = b_j d_j p_j/M$, *independent of the identity of product* i; here, b_j denotes the coefficient of the price term in the utility measure for product j and M the total market size. This is very restrictive: it implies that in the automobile industry, for example, the cross price elasticity of a price change of the Honda Accord on the sales of other sedans (e.g. Toyota Camry) is identical to that of coupes or hatchbacks or hybrids, let alone SUVs or minivans. (See also the discussion on page 847 in Berry et al. 1995). Conversely, if one is willing to make such restrictive assumptions, the analogous assumption in the extended affine model, is to specify that all off-diagonal elements in any given column of the R-matrix be identical, thus immediately reducing the number of parameters to $2N$.

Another well-known restriction of the MNL model is the "*Independence of Irrelevant Alternatives*" assumption: the relative market share of a pair of products i and j is not affected by the emergence of close substitutes for product i, say. As a final example, pointed out in the paper, the MNL model assumes that any given product is sold in the market, irrespective of how large its absolute and relative price (compared to those of substitutes). The extended affine model avoids this restriction entirely, as explained in the paper.

D. Example.

Consider the distribution structure analyzed in McGuire and Staelin (2008), as depicted in Figure 1 where supplier i, $i = 1, 2$, sells product i exclusively through retailer i.

Assume that

$$a = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & -\gamma_1 \\ -\gamma_2 & 1 \end{pmatrix}, \quad \text{with } 0 \le \gamma_1, \gamma_2 \le 1.$$

It is easily verified that R is a positive definite Z-matrix. When $\gamma_1 \neq \gamma_2$, R fails to be symmetric but all of the results in Propositions 1 and 2, and Theorems 1 and 2 continue to hold, see §6. Clearly, $T(R) = \mathtt{I}$ and

$$\Psi(R) = [\mathtt{I} + R]^{-1} = \frac{1}{4 - \gamma_1\gamma_2} \begin{pmatrix} 2 & \gamma_1 \\ \gamma_2 & 2 \end{pmatrix}.$$

Then we have

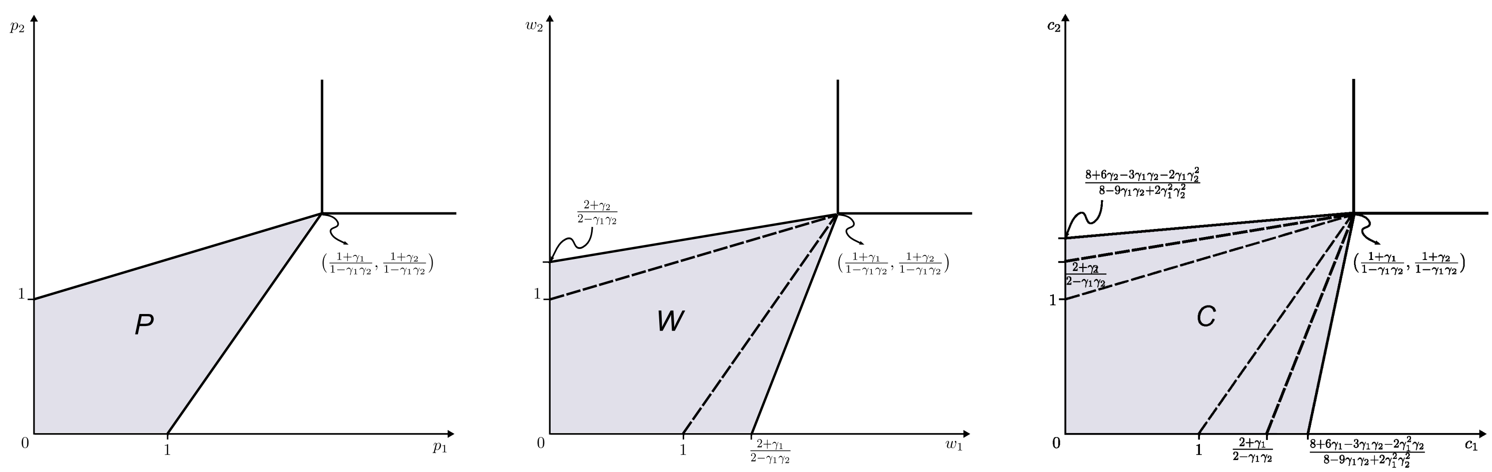
$$S = \Psi(R)R = \frac{1}{4 - \gamma_1\gamma_2} \begin{pmatrix} 2 - \gamma_1\gamma_2 & -\gamma_1 \\ -\gamma_2 & 2 - \gamma_1\gamma_2 \end{pmatrix},$$

$$\Psi(S) = T(S)[S + T(S)]^{-1} = \frac{2 - \gamma_1\gamma_2}{4(2 - \gamma_1\gamma_2)^2 - \gamma_1\gamma_2} \begin{pmatrix} 2(2 - \gamma_1\gamma_2) & \gamma_1 \\ \gamma_2 & 2(2 - \gamma_1\gamma_2) \end{pmatrix}.$$

Hence,

$$C = \left\{ c \ge 0 \;\middle|\; \begin{matrix} (8 + 6\gamma_1 - 3\gamma_1\gamma_2 - 2\gamma_1^2\gamma_2) - (8 - 9\gamma_1\gamma_2 + 2\gamma_1^2\gamma_2^2)c_1 + \gamma_1(2 - \gamma_1\gamma_2)c_2 \ge 0 \\ (8 + 6\gamma_2 - 3\gamma_1\gamma_2 - 2\gamma_1\gamma_2^2) + \gamma_2(2 - \gamma_1\gamma_2)c_1 - (8 - 9\gamma_1\gamma_2 + 2\gamma_1^2\gamma_2^2)c_2 \ge 0 \end{matrix} \right\}.$$

Figure 6 Effective Price Polyhedra



(a) Effective retail price polyhedron P (b) Effective wholesale price polyhedron W (c) Effective supply cost polyhedron C

In Figure 6, we exhibit the effective retail price polyhedron P, the effective wholesale price polyhedron W and the effective marginal cost polyhedron C. As stated in Proposition 2, $P \subseteq W \subseteq C$.

We also provide an example where $c \in C$ and $w^*(c) \in (W \setminus P)$. Let $\gamma_1 = 0.7, \gamma_2 = 0.3$. Then, with

$$a = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & -0.7 \\ -0.3 & 1 \end{pmatrix},$$

it is easily verified that

$$b = \Psi(R)a = \begin{pmatrix} 0.7124 \\ 0.6069 \end{pmatrix} \quad \text{and} \quad S = \Psi(R)R = \begin{pmatrix} 0.4723 & -0.1847 \\ -0.0792 & 0.4723 \end{pmatrix},$$

and moreover,

$$\Psi(S) = T(S)[S+T(S)]^{-1} = \begin{pmatrix} 0.5083 \; 0.0994 \\ 0.0426 \; 0.5083 \end{pmatrix}.$$

Consider $c = (1, 1.5)^T$. It is easily verified that

$$\Psi(S)Q(c) = \Psi(S)(b - Sc) = \begin{pmatrix} 0.2607 \\ 0.0106 \end{pmatrix} > 0,$$

i.e., $c \in C^o$. By Theorem 2,

$$w^*(c) = c + [S+T(S)]^{-1}Q(c) = \begin{pmatrix} 1.5519 \\ 1.5225 \end{pmatrix} \in W^o.$$

By Proposition 1(d),

$$p^*(w^*(c)) = w^*(c) + [R+T(R)]^{-1}q(w^*(c)) = \begin{pmatrix} 1.8125 \\ 1.5331 \end{pmatrix} \in P^o$$

and

$$d(p^*(w^*(c))) = a - Rp^*(w^*(c)) = \begin{pmatrix} 0.2607 \\ 0.0106 \end{pmatrix} > 0.$$

However, note that

$$a - Rw^*(c) = \begin{pmatrix} 0.5139 \\ -0.0569 \end{pmatrix},$$

i.e., $w^*(c) \notin P$.

We now calculate the matrix of cost pass-through rates. Let $\mathcal{A}$ denote the equilibrium assortment. We distinguish between two cases.

Case 1: $\mathcal{A} = \mathcal{N}$. It follows from Corollary 1 and Proposition 4 that

$$\frac{\mathtt{I}}{2} \le \left(\frac{\partial p^*}{\partial w}\right) = [R+T(R)]^{-1}T(R) = [\mathtt{I}+R]^{-1} = \frac{1}{4-\gamma_1\gamma_2}\begin{pmatrix} 2 & \gamma_1 \\ \gamma_2 & 2 \end{pmatrix} \le \frac{1}{2(1-\gamma_1\gamma_2)}\begin{pmatrix} 1 & \gamma_1 \\ \gamma_2 & 1 \end{pmatrix} = \frac{R^{-1}T(R)}{2},$$

$$\begin{aligned}\frac{\mathtt{I}}{2} \le \left(\frac{\partial w^*}{\partial c}\right) = [S+T(S)]^{-1}T(S) &= \frac{2-\gamma_1\gamma_2}{4(2-\gamma_1\gamma_2)^2-\gamma_1\gamma_2}\begin{pmatrix} 2(2-\gamma_1\gamma_2) & \gamma_1 \\ \gamma_2 & 2(2-\gamma_1\gamma_2) \end{pmatrix} \\ &\le \frac{2-\gamma_1\gamma_2}{2[(2-\gamma_1\gamma_2)^2-\gamma_1\gamma_2]}\begin{pmatrix} 2-\gamma_1\gamma_2 & \gamma_1 \\ \gamma_2 & 2-\gamma_1\gamma_2 \end{pmatrix} = \frac{S^{-1}T(S)}{2},\end{aligned}$$

$$\begin{aligned}\frac{\mathtt{I}}{4} \le \left(\frac{\partial p^*}{\partial c}\right) = \left(\frac{\partial p^*}{\partial w}\right)\left(\frac{\partial w^*}{\partial c}\right) &= \frac{2-\gamma_1\gamma_2}{[4(2-\gamma_1\gamma_2)^2-\gamma_1\gamma_2](4-\gamma_1\gamma_2)}\begin{pmatrix} 8-3\gamma_1\gamma_2 & 2\gamma_1(3-\gamma_1\gamma_2) \\ 2\gamma_2(3-\gamma_1\gamma_2) & 8-3\gamma_1\gamma_2 \end{pmatrix} \\ &\le \frac{2-\gamma_1\gamma_2}{4(1-\gamma_1\gamma_2)[(2-\gamma_1\gamma_2)^2-\gamma_1\gamma_2]}\begin{pmatrix} 2 & \gamma_1(3-\gamma_1\gamma_2) \\ \gamma_2(3-\gamma_1\gamma_2) & 2 \end{pmatrix} = \frac{R^{-1}T(R)S^{-1}T(S)}{4}.\end{aligned}$$

Thus, the *own-brand* pass-through rate for the retailers (, in response to an increase of a wholesale price) grows as either γ_1 or γ_2 increases from 0 to 1, from a minimum value of 50% to a maximum value of $\frac{2}{4-1} = 66\frac{2}{3}\%$. The *cross-brand* pass-through rates grow from 0% to $33\frac{1}{3}\%$ as γ_1 and γ_2 increases from 0 to 1 (, their maximum value).

Similarly, the own-brand pass-through rates for the suppliers in response to an increase of their input costs, is given by $\frac{2}{4-\frac{\gamma_1\gamma_2}{(2-\gamma_1\gamma_2)^2}}$, an increasing function of $\gamma_1\gamma_2$, which again increases from 50% to $66\frac{2}{3}\%$. Note that the *cross-brand* pass-through rate of product i due to a cost increase of product j is given by $\gamma_i\big[\frac{(2-\gamma_1\gamma_2)^{-1}}{4-\frac{\gamma_1\gamma_2}{(2-\gamma_1\gamma_2)^2}}\big]$; both the numerator and the denominator of the expression within square brackets are increasing in $(\gamma_1\gamma_2)$. Thus for a given value of γ_i, the cross-brand pass-through rate increases from $\frac{\gamma_i}{8}$ to $\gamma_i\frac{(2-\gamma_i)}{4(2-\gamma_i)^2-\gamma_i}$, as γ_j increases from 0 to 1. Once again, the cross-brand pass-through rate varies between 0 and $33\frac{1}{3}\%$. Finally, the marginal change rate in a product's retail price, due to an increase of its supplier's cost rate, increases from a minimum of 25% to a maximal value of 55.6%, as $\gamma_1\gamma_2$ increases from 0 to 1.

The γ-parameters are a measure for the competitive intensity. The above results show that all cost pass-through rates increase as competition becomes more intense.

Case 2: $\mathcal{A} = \mathcal{N}(1) = \{1\}$, i.e., only one of the products, without loss of generality, product 1, is sold in the market. In this case, $a^{\mathcal{A}} = a_{\mathcal{A}} - R_{\mathcal{A},\bar{\mathcal{A}}}R^{-1}_{\bar{\mathcal{A}},\bar{\mathcal{A}}}a_{\bar{\mathcal{A}}} = 1+\gamma_1$ and $R^{\mathcal{A}} = R_{\mathcal{A},\mathcal{A}} - R_{\mathcal{A},\bar{\mathcal{A}}}R^{-1}_{\bar{\mathcal{A}},\bar{\mathcal{A}}}R_{\bar{\mathcal{A}},\mathcal{A}} = 1-\gamma_1\gamma_2$. Thus, $\left(\frac{\partial w^*_{\mathcal{A}}}{\partial c_{\mathcal{A}}}\right)^- = [S^{\mathcal{A}} + T(S^{\mathcal{A}})]^{-1}T(S^{\mathcal{A}}) = \frac{1}{2}$ and $\left(\frac{\partial p^*_{\mathcal{A}}}{\partial c_{\mathcal{A}}}\right)^- = \left(\frac{\partial p^*_{\mathcal{A}}}{\partial w_{\mathcal{A}}}\right)^- \left(\frac{\partial w^*_{\mathcal{A}}}{\partial c_{\mathcal{A}}}\right)^- = \frac{1}{4}$. In other words, in this monopoly case, the cost pass-through rates are at their minimum levels of 50% and 25%, see Proposition 4.

E. Restrictions on Retailer Prices.

In the US, such restrictions arise potentially because of the Robinson-Patman Act, a Federal law enacted at the start of the 20th century. To appreciate the importance and prevalence of such price restrictions, it is important to note that, for example, in the European Community, there is no direct legislative equivalent to the US Robinson-Patman Act, see, e.g., Spinks (2000) and Whelan and Marsden (2006).

Even in the US, Kirkland and Ellis (2005) write, when reviewing the "realities of the Robinson-Patman Act" that "everyone price discriminates. [...] Manufacturers of all kinds, selling to national accounts and local distributors, do it." The same authors point out that over the past several decades, many economists and federal judges, as well as the antitrust enforcement agencies of the Department of Justice and the Federal Trade Commission, have come "to view the Robinson-Patman Act as itself – potentially – 'anticompetitive,' leading to higher rather than lower prices, hurting rather than benefiting consumers." Based on the same consideration, the Antitrust Modernization Commission (AMC), established by the 2002 Antitrust Modernization Commission Act, recommended in its final report AMC (2007) that Congress finally repeal the Robinson-Patman Act.[6] As a result, there has been *no* government challenge, in the ten years preceding the Kirkland and Ellis (2005) report, to *any* company's price discrimination under the Act, with just one exception, described as being "anomalous". Moreover, "even the number of those [privately originated] challenges has diminished in recent years, reflecting the poor record of success that Robinson-Patman Act claims have experienced in recent years."

Indeed, there are many defenses a "price differentiating" firm may invoke, see, e.g., Kirkland and Ellis (2005), as well as the discussion in Moorthy (2005). In addition, in many industries, there has been a steady increase in the use of retailer specific "private" labels and brand variants, with manufacturers offering different variants of the same product for different retail chains; the differentiation in packaging/labeling/after-sales support is sufficient to consider the products essentially different and protected from the implications of the Robinson-Patman Act. Bergen et al. (1996) discuss this advantage as one of many benefits associated with "branded variants".

In this appendix, we outline how the equilibrium behavior in the base model of Section 3 needs to be adapted in case all suppliers are required to charge uniform prices across all retailers, for each of the products they offer to the market. See Section 3 for a discussion of the limited settings where such price restrictions may prevail.

The above price restrictions require that for each $j \in \mathcal{J}$ and $k \in \mathcal{K}(i,j)$ for some i:

$$w_{ijk} = \bar{w}_{jk} \quad \text{for all retailers } i = 1, \ldots, I \text{ such that } (i,j,k) \in \mathcal{N}. \tag{E.1}$$

The restricted choice of wholesale price vectors has, of course, no bearing, whatsoever, on the equilibrium behavior of the second stage retailer competition game. This implies that Proposition 1 continues to apply. More specifically, any vector of wholesale prices w induces a *unique* set of equilibrium demand volumes. If $w \in W$,

$$D(w) = Q(w) = \Psi(R)a - [\Psi(R)R]w, \tag{E.2}$$

[6] The Commission wrote: "The Commission recommends that Congress finally repeal the Robinson-Patman Act (RPA). This law, enacted in 1936, appears antithetical to core antitrust principles. Its repeal or substantial overhaul has been recommended in three prior reports, in 1955, 1969, and 1977. That is because the RPA protects competitors over competition and punishes the very price discounting and innovation in distribution methods that the antitrust laws otherwise encourage. At the same time, it is not clear that the RPA actually effectively protects the small business constituents that it was meant to benefit. Continued existence of the RPA also makes it difficult for the United States to advocate against the adoption and use of similar laws against U.S. companies operating in other jurisdictions. Small business is adequately protected from truly anticompetitive behavior by application of the Sherman Act."

see (6), where the matrix $S \equiv \Psi(R)R$ is positive definite, as shown in Theorem 1(a). Similarly, if $w \notin W$,

$$D(w) = Q(\Theta(w)). \tag{E.3}$$

Turning, next, to the first stage competition game among the upstream suppliers, it should be noted that the induced demand functions are given in closed form, by (E.2) and (E.3). This, in itself, allows for the numerical exploration of equilibria, for example by the use of a tatônnement scheme, see Topkis (1998) and Vives (1999). To proceed with the equilibrium analysis, recall that it is advantageous to re-sequence the products so that they are lexicographically ranked according to their supplier index (j), product index (k) and, lastly, retailer index (i). Let $n \equiv \big|\{(j,k)|$ product $(i,j,k) \in \mathcal{N}$ for at least one retailer $i\}\big|$ denote the number of distinct supplier/product combinations. Any restricted wholesale price vector $\bar{w}$ can be expanded onto the full price space $\mathbb{R}_+^N$ from the supplier/product space $\mathbb{R}_+^n$, via the transformation $w = A^T\bar{w}$, where the $n \times N$ matrix A is defined as follows:

$$A_{jk,i'j'k'} = \begin{cases} 1 & \text{if } j = j', k = k' \text{ and } (i',j',k') \in \mathcal{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\bar{W} = \{\bar{w} \geq 0 \mid A^T\bar{w} \in W\}$. It is easily verified that on the polyhedron $\bar{W}$, the demand functions $\bar{D}(\cdot)$ are again *affine*, with

$$\bar{D}(\bar{w}) = \bar{Q}(\bar{w}) \equiv A[\Psi(R)a] - (ASA^T)\bar{w}.$$

($\bar{D}(\bar{w})_{jk}$ denotes the aggregate induced demand for product k sold by supplier j, across all of the retailers.) Moreover, since $S = \Psi(R)R$ is positive definite, it is easily verified that the matrix $\bar{S} \equiv ASA^T \in \mathbb{R}^{n\times n}$ is positive definite as well. (Verification is immediate from the definition of positive definiteness; for any $\bar{z} \in \mathbb{R}^n$ with $\bar{z} \neq 0$, $\bar{z}^T\bar{S}\bar{z} = (\bar{z}^TA)S(A^T\bar{z}) = z^TSz$, with $z = A^T\bar{z} \neq 0$. Thus $\bar{z}^T\bar{S}\bar{z} > 0$.)

Unfortunately, while the vector of demand volumes $\bar{D}(\bar{w})$ can be obtained in closed form for all wholesale price vectors, including vectors $\bar{w} \notin \bar{W}$, it is no longer true that the demand volumes $\bar{D}(\bar{w}) = \bar{Q}(\bar{w}')$, with $\bar{w}'$ the projection of $\bar{w}$ onto $\bar{W}$. As a consequence, the characterization of the equilibrium behavior in Theorem 2, no longer applies. However, the following partial characterization of the equilibrium behavior can be obtained if the competition among the suppliers is restricted to the price space $\bar{W}$ on which the demand functions are affine. Recall, the interior of $\bar{W}$ is the set of all wholesale price vectors under which all supplier/product combinations maintain a positive market share: We assume, without loss of generality, that the suppliers' marginal cost rates satisfy the same type of restrictions as (E.1), i.e.,

$$c_{ijk} = \bar{c}_{jk} \quad \text{for all retailers } i = 1, \ldots, I \text{ such that } (i,j,k) \in \mathcal{N}. \tag{E.4}$$

(If (E.4) is violated, this, itself, provides a legal rationale, even within the context of the Robinson-Patman Act, for example, to use differentiated wholesale prices, as in our base model.) Under this cost rate vector $\bar{c}$, the First Order Conditions of the game with affine demand functions (E.2) have the unique solution

$$\bar{w}^*(\bar{c}) = \bar{c} + [\bar{S} + T(\bar{S})]^{-1}\bar{Q}(\bar{c}),$$

see (10). Assume $\bar{c}$ is such that $\bar{w}^*(\bar{c}) \in \bar{W}$. In other words, assume $\bar{c} \in \bar{C} = \{\bar{c} \geq 0 \mid \Psi(\bar{S})\bar{Q}(\bar{c}) \geq 0\}$. Then, $\bar{w}^*(\bar{c})$ is an equilibrium in the *restricted* competition game, and if $\bar{c}$ is chosen in the *interior* of $\bar{C}$, $\bar{w}^*(\bar{c})$ is the *unique* such equilibrium.

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