Online Appendix A to

"Efficient Ignorance: Information Heterogeneity in a Queue"

Proof of Theorem 5. First, following the same approach in the proof of Lemma 1, it can be shown that the expected sojourn time $W(q)$ in the heterogenous case strictly increases in q too. As a result, it is easy to further demonstrate that there exists a unique joining equilibrium $q^* \in [0,1]$ for uninformed customers. In the case of $q^* = 0$ or 1, the demonstration of the monotonicity of $\lambda(q^*)$ is parallel to that of the homogeneous reward case, in which $R_{\mathtt{I}} = R_{\mathtt{U}} = R$ and $c_{\mathtt{I}} = c_{\mathtt{U}} = c$. Thus, for the rest of the proof, we only consider the cases in which $q^* \in (0,1)$.

Since $\lambda(q) = \mu(1 - p_0(q))$, the monotonicity of $\lambda(q^*)$ in γ is opposite to that of $p_0(q^*)$. Thus, instead of directly proving that $\lambda(q^*)$ is strictly increasing in γ, we will show that $p_0(q^*)$ is strictly decreasing in γ in two steps: (i) From the expression of $R_{\mathtt{U}} = c_{\mathtt{U}} W(q)$, derive γ as a function of $\rho_{\mathtt{C}}$ and prove $\frac{d\rho_{\mathtt{C}}}{d\gamma} > 0$; (ii) From the expression of $p_0(q)$, prove $\frac{dp_0}{d\rho_{\mathtt{C}}} < 0$. Then, combining these two results, we obtain $\frac{dp_0}{d\gamma} = \frac{dp_0}{d\rho_{\mathtt{C}}}\frac{d\rho_{\mathtt{C}}}{d\gamma} < 0$.

Step (i). Rewrite $R_{\mathtt{U}} = c_{\mathtt{U}} W(q)$ as

$$H(\rho_{\mathtt{C}})(1 - \rho_{\mathtt{C}} + \gamma\rho)^2 + (\nu_{\mathtt{U}} - n_{\mathtt{I}})(1 - \rho_{\mathtt{C}} + \gamma\rho) - 1 = 0, \tag{OA.1}$$

where

$$H(\rho_{\mathtt{C}}) = \frac{(\nu_{\mathtt{U}} - n_{\mathtt{I}})(\rho_{\mathtt{C}} - 1)\rho_{\mathtt{C}}^{n_{\mathtt{I}}} + \nu_{\mathtt{U}} - \nu_{\mathtt{U}}\rho_{\mathtt{C}} + \rho_{\mathtt{C}}^{n_{\mathtt{I}}} - 1}{(\rho_{\mathtt{C}} - 1)^2 \rho_{\mathtt{C}}^{n_{\mathtt{I}}}} = \frac{(\nu_{\mathtt{U}} - n_{\mathtt{I}})\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^i}{\rho_{\mathtt{C}}^{n_{\mathtt{I}}}} + \frac{\sum_{i=1}^{n_{\mathtt{I}}-1}\sum_{j=0}^{i-1}\rho_{\mathtt{C}}^j}{\rho_{\mathtt{C}}^{n_{\mathtt{I}}}}. \tag{OA.2}$$

For further discussion, we derive some properties of $H(\rho_{\mathtt{C}})$.

Lemma 3. *If there exists a $q^* \in (0,1)$ such that $R_{\mathtt{U}} = c_{\mathtt{U}} W(q^*)$, it must be that $H(\rho_{\mathtt{C}}) > 0$. Moreover, $H(\rho_{\mathtt{C}}) > 0$ if and only if $\sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^i / \sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^i < \nu_{\mathtt{U}}$ and $H(\rho_{\mathtt{C}})$ strictly decreases in $\rho_{\mathtt{C}}$ when $H(\rho_{\mathtt{C}}) > 0$.*

Proof of Lemma 3. Consider (OA.1) as a quadratic equation in $(1 - \rho_{\mathtt{C}} + \gamma\rho)$.

- If $(\nu_{\mathtt{U}} - n_{\mathtt{I}})^2 + 4H(\rho_{\mathtt{C}}) < 0$, (OA.1) has no real roots.
- If $(\nu_{\mathtt{U}} - n_{\mathtt{I}})^2 + 4H(\rho_{\mathtt{C}}) \geq 0$ and $H(\rho_{\mathtt{C}}) < 0$, we must have $\nu_{\mathtt{U}} - n_{\mathtt{I}} < 0$ by (OA.2) and both roots of (OA.1) $\frac{-(\nu_{\mathtt{U}} - n_{\mathtt{I}}) \pm \sqrt{(\nu_{\mathtt{U}} - n_{\mathtt{I}})^2 + 4H(\rho_{\mathtt{C}})}}{2H(\rho_{\mathtt{C}})}$ are negative.
- If $(\nu_{\mathtt{U}} - n_{\mathtt{I}})^2 + 4H(\rho_{\mathtt{C}}) \geq 0$ and $H(\rho_{\mathtt{C}}) = 0$, we must have $\nu_{\mathtt{U}} - n_{\mathtt{I}} < 0$ by (OA.2) and (OA.1) only has one negative root $1 - \rho_{\mathtt{C}} + \gamma\rho = \frac{1}{\nu_{\mathtt{U}} - n_{\mathtt{I}}} < 0$, which is invalid because $1 - \rho_{\mathtt{C}} + \gamma\rho > 0$.
- If $H(\rho_{\mathtt{C}}) > 0$, which also implies $(\nu_{\mathtt{U}} - n_{\mathtt{I}})^2 + 4H(\rho_{\mathtt{C}}) \geq 0$, (OA.1) has one positive root $\frac{-(\nu_{\mathtt{U}} - n_{\mathtt{I}}) + \sqrt{(\nu_{\mathtt{U}} - n_{\mathtt{I}})^2 + 4H(\rho_{\mathtt{C}})}}{2H(\rho_{\mathtt{C}})}$.

Therefore, if there exists a $q^* \in (0,1)$ such that $R_{\mathtt{U}} = c_{\mathtt{U}} W(q^*)$, it must be the last case.

We next consider the monotonicity of $H(\rho_{\mathtt{C}})$. Since $\sum_{i=1}^{n_{\mathtt{I}}-1}\sum_{j=0}^{i-1}\rho_{\mathtt{C}}^j = \sum_{i=0}^{n_{\mathtt{I}}-1}(n_{\mathtt{I}} - 1 - i)\rho_{\mathtt{C}}^i$, rewrite $H(\rho_{\mathtt{C}})$ in an alternative form

$$H(\rho_{\mathtt{C}}) = \left(\nu_{\mathtt{U}} - \frac{\sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^i}{\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^i}\right)\frac{\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^i}{\rho_{\mathtt{C}}^{n_{\mathtt{I}}}}. \tag{OA.3}$$

Clearly, $\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}/\rho_{\mathtt{C}}^{n_{\mathtt{I}}}$ is positive and strictly decreasing in $\rho_{\mathtt{C}}$. By (OA.3), $H(\rho_{\mathtt{C}})>0 \Leftrightarrow \sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}/\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}<\nu_{\mathtt{U}}$. Moreover,

$$\begin{aligned}
\frac{\sum_{i=1}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}}{\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}} &= n_{\mathtt{I}}-\frac{\sum_{i=1}^{n_{\mathtt{I}}-1}\left(\rho_{\mathtt{C}}^{i}-1\right)+\rho_{\mathtt{C}}-1}{\rho_{\mathtt{C}}^{n}-1}\\
&= n_{\mathtt{I}}-\frac{\sum_{i=1}^{n_{\mathtt{I}}-1}\sum_{j=0}^{i-1}\rho_{\mathtt{C}}^{j}+1}{\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}}\\
&= n_{\mathtt{I}}-\sum_{i=1}^{n_{\mathtt{I}}-1}\left(1-\frac{\sum_{j=i}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{j}}{\sum_{j=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{j}}\right)-\frac{1}{\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}}\\
&= n_{\mathtt{I}}-\sum_{i=1}^{n_{\mathtt{I}}-1}\left(1-\frac{\sum_{j=0}^{n_{\mathtt{I}}-i-1}\rho_{\mathtt{C}}^{j}}{\sum_{j=0}^{i-1}\rho_{\mathtt{C}}^{j-i}+\sum_{j=0}^{n_{\mathtt{I}}-i-1}\rho_{\mathtt{C}}^{j}}\right)-\frac{1}{\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}}\\
&= n_{\mathtt{I}}-\sum_{i=1}^{n_{\mathtt{I}}-1}\left(1-\frac{1}{\frac{\sum_{j=0}^{i-1}\rho_{\mathtt{C}}^{j-i}}{\sum_{j=0}^{n_{\mathtt{I}}-i-1}\rho_{\mathtt{C}}^{j}}+1}\right)-\frac{1}{\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}},
\end{aligned}\tag{OA.4}$$

which is strictly increasing in $\rho_{\mathtt{C}}$. Consequently, we have that $H(\rho_{\mathtt{C}})$ strictly decreases in $\rho_{\mathtt{C}}$ when $\sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}/\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}<\nu_{\mathtt{U}}$, which is equivalent to $H(\rho_{\mathtt{C}})>0$. $\square$

Solving (OA.1), we obtain γ as a function of $\rho_{\mathtt{C}}$, i.e.,

$$\gamma(\rho_{\mathtt{C}})=\frac{1}{\rho}\left(2\left((\nu_{\mathtt{U}}-n_{\mathtt{I}})+\sqrt{(\nu_{\mathtt{U}}-n_{\mathtt{I}})^{2}+4H(\rho_{\mathtt{C}})}\right)^{-1}+\rho_{\mathtt{C}}-1\right).$$

Since we have shown that $H(\rho_{\mathtt{C}})$ strictly decreases in $\rho_{\mathtt{C}}$ when $H(\rho_{\mathtt{C}})>0$ in Lemma 3, $\gamma(\rho_{\mathtt{C}})$ then strictly increases in $\rho_{\mathtt{C}}$, i.e., $\frac{d\gamma}{d\rho_{\mathtt{C}}}>0$, which implies $\frac{d\rho_{\mathtt{C}}}{d\gamma}>0$.

Step (ii). We now show that $\frac{dp_0}{d\rho_{\mathtt{C}}}<0$. Write $p_0(q^*)$ as a function of $\rho_{\mathtt{C}}$:

$$\begin{aligned}
p_0(q^*) &= \left(\frac{\rho_{\mathtt{C}}^{n_{\mathtt{I}}}-1}{\rho_{\mathtt{C}}-1}+\frac{\rho_{\mathtt{C}}^{n_{\mathtt{I}}}}{1-\rho_{\mathtt{C}}+\gamma\rho}\right)^{-1}\\
&= \left(\frac{\rho_{\mathtt{C}}^{n_{\mathtt{I}}}-1}{\rho_{\mathtt{C}}-1}+\frac{\rho_{\mathtt{C}}^{n_{\mathtt{I}}}}{2}\left((\nu_{\mathtt{U}}-n_{\mathtt{I}})+\sqrt{(\nu_{\mathtt{U}}-n_{\mathtt{I}})^{2}+4H(\rho_{\mathtt{C}})}\right)\right)^{-1}\\
&= \left(\frac{\rho_{\mathtt{C}}^{n_{\mathtt{I}}}-1}{\rho_{\mathtt{C}}-1}+\frac{1}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}}(\nu_{\mathtt{U}}-n_{\mathtt{I}})+\sqrt{\frac{1}{4}\rho_{\mathtt{C}}^{2n_{\mathtt{I}}}(\nu_{\mathtt{U}}-n_{\mathtt{I}})^{2}+\rho_{\mathtt{C}}^{2n_{\mathtt{I}}}H(\rho_{\mathtt{C}})}\right)^{-1}\\
&= \left(\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}+\frac{1}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}}(\nu_{\mathtt{U}}-n_{\mathtt{I}})+\sqrt{\frac{1}{4}\rho_{\mathtt{C}}^{2n_{\mathtt{I}}}(\nu_{\mathtt{U}}-n_{\mathtt{I}})^{2}+\nu_{\mathtt{U}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}}\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}-\rho_{\mathtt{C}}^{n_{\mathtt{I}}}\sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}}\right)^{-1}\\
&= \left(\frac{1}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}}(\nu_{\mathtt{U}}-n_{\mathtt{I}})+\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}+\sqrt{\left(\frac{1}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}}(\nu_{\mathtt{U}}-n_{\mathtt{I}})+\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}\right)^{2}-\sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}}\right)^{-1}.
\end{aligned}\tag{OA.5}$$

We notice that $\frac{1}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}}(\nu_{\mathtt{U}}-n_{\mathtt{I}})+\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}>0$ when $H(\rho_{\mathtt{C}})>0$. To see this, take the derivative in $\rho_{\mathtt{C}}$,

$$\begin{aligned}
\left(\frac{1}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}}(\nu_{\mathtt{U}}-n_{\mathtt{I}})+\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}\right)' &= \frac{1}{2}n_{\mathtt{I}}(\nu_{\mathtt{U}}-n_{\mathtt{I}})\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1}+\frac{n_{\mathtt{I}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}}-\rho_{\mathtt{C}}^{n_{\mathtt{I}}}-n_{\mathtt{I}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1}+1}{(\rho_{\mathtt{C}}-1)^{2}}\\
&> \frac{1}{2}n_{\mathtt{I}}\left(\frac{\sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}}{\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}}-n_{\mathtt{I}}\right)\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1}+\frac{n_{\mathtt{I}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}}-\rho_{\mathtt{C}}^{n_{\mathtt{I}}}-n_{\mathtt{I}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1}+1}{(\rho_{\mathtt{C}}-1)^{2}}\\
&= \frac{1}{2}n_{\mathtt{I}}\frac{n_{\mathtt{I}}\rho_{\mathtt{C}}-n_{\mathtt{I}}-\rho_{\mathtt{C}}^{n_{\mathtt{I}}}+1}{(\rho_{\mathtt{C}}-1)\left(\rho_{\mathtt{C}}^{n_{\mathtt{I}}}-1\right)}\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1}+\frac{n_{\mathtt{I}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}}-\rho_{\mathtt{C}}^{n_{\mathtt{I}}}-n_{\mathtt{I}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1}+1}{(\rho_{\mathtt{C}}-1)^{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\frac{n_{\mathtt{I}}^2}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1} + \frac{n_{\mathtt{I}}}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1}\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i} - \left(\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}\right)^2}{\rho_{\mathtt{C}}^{n_{\mathtt{I}}} - 1} \\
&= \frac{\frac{(n_{\mathtt{I}}-1)n_{\mathtt{I}}}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1} - \sum_{i=0}^{n_{\mathtt{I}}-2}(i+1)\rho_{\mathtt{C}}^{i} + \frac{n_{\mathtt{I}}}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1}\sum_{i=1}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i} - \rho_{\mathtt{C}}^{n_{\mathtt{I}}-1}\sum_{i=1}^{n_{\mathtt{I}}-1}(n_{\mathtt{I}}-i)\rho_{\mathtt{C}}^{i}}{\rho_{\mathtt{C}}^{n_{\mathtt{I}}} - 1} \\
&= \frac{\frac{(n_{\mathtt{I}}-1)n_{\mathtt{I}}}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1} - \sum_{i=0}^{n_{\mathtt{I}}-2}(i+1)\rho_{\mathtt{C}}^{i}}{\rho_{\mathtt{C}}^{n_{\mathtt{I}}} - 1} + \frac{\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1}}{2\left(\rho_{\mathtt{C}}^{n_{\mathtt{I}}} - 1\right)}\sum_{i=1}^{n_{\mathtt{I}}-1}(2i-n_{\mathtt{I}})\rho_{\mathtt{C}}^{i} \\
&= \frac{\frac{(n_{\mathtt{I}}-1)n_{\mathtt{I}}}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1} - \sum_{i=0}^{n_{\mathtt{I}}-2}(i+1)\rho_{\mathtt{C}}^{i}}{\rho_{\mathtt{C}}^{n_{\mathtt{I}}} - 1} + \frac{\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1}}{2\left(\rho_{\mathtt{C}}^{n_{\mathtt{I}}} - 1\right)}\sum_{i=\left\lfloor\frac{n_{\mathtt{I}}+1}{2}\right\rfloor}^{n_{\mathtt{I}}-1}(2i-n_{\mathtt{I}})\left(\rho_{\mathtt{C}}^{2i-n_{\mathtt{I}}} - 1\right)\rho_{\mathtt{C}}^{n_{\mathtt{I}}-i} \\
&> 0,
\end{aligned}
$$

where the first inequality results from the fact that $\sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}/\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i} < \nu_{\mathtt{U}}$ by Lemma 3 and the last inequality stems from $\rho_{\mathtt{C}} \geq 0$, which is implied by the monotonicity of $H(\rho_{\mathtt{C}})$ and $H(\rho_{\mathtt{C}}) > 0$. Since $\frac{1}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}}(\nu_{\mathtt{U}} - n_{\mathtt{I}}) + \sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i} = 1$ at $\rho_{\mathtt{C}} = 0$. By the monotonicity, $\frac{1}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}}(\nu_{\mathtt{U}} - n_{\mathtt{I}}) + \sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i} > 0$ for $\rho_{\mathtt{C}} \geq 0$, i.e., $H(\rho_{\mathtt{C}}) > 0$.

Given the positiveness of $\frac{1}{2}\rho_{\mathtt{C}}^{n_{\mathtt{I}}}(\nu_{\mathtt{U}} - n_{\mathtt{I}}) + \sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}$ when $H(\rho_{\mathtt{C}}) > 0$, for the ease of exposition, let

$$
f = \left(\frac{1}{2}(\nu_{\mathtt{U}} - n_{\mathtt{I}})\rho_{\mathtt{C}}^{n_{\mathtt{I}}} + \sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}\right)^2 = \left(\frac{n_{\mathtt{I}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}+1} - \nu_{\mathtt{U}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}+1} - n_{\mathtt{I}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}} + \nu_{\mathtt{U}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}} - 2\rho_{\mathtt{C}}^{n_{\mathtt{I}}} + 2}{2(\rho_{\mathtt{C}} - 1)}\right)^2 \tag{OA.6}
$$

and

$$
g = \sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i} = \frac{n_{\mathtt{I}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}+1} - (n_{\mathtt{I}}+1)\rho_{\mathtt{C}}^{n_{\mathtt{I}}} + 1}{(\rho_{\mathtt{C}} - 1)^2}. \tag{OA.7}
$$

By (OA.5), we can write $p_0(q^*) = \left(\sqrt{f} + \sqrt{f-g}\right)^{-1}$. To prove $p_0(q^*)$ is strictly decreasing in $\rho_{\mathtt{C}}$, it is sufficient to show that $\sqrt{f} + \sqrt{f-g}$ is a strictly increasing function, i.e.,

$$
\frac{f'}{\sqrt{f}} + \frac{f'-g'}{\sqrt{f-g}} = \frac{f'}{\sqrt{f}} - \frac{g'-f'}{\sqrt{f-g}} > 0.
$$

Apparently, the inequality holds for $f' \geq g'$. We next consider the case $g' > f'$.

Note that f is strictly increasing in $\nu_{\mathtt{U}}$ and g is independent of $\nu_{\mathtt{U}}$. One can also readily show that $\frac{f'}{\sqrt{f}}$ and $-\frac{g'-f'}{\sqrt{f-g}}$ are both strictly increasing in $\nu_{\mathtt{U}}$. Thus, if $\frac{f'}{\sqrt{f}} - \frac{g'-f'}{\sqrt{f-g}} > 0$ for $\nu_{\mathtt{U}} = \sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}/\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}$, it must be true for all $\sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}/\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i} < \nu_{\mathtt{U}}$ by the monotonicity. As a result, $p_0(q^*)$ will be strictly decreasing in $\rho_{\mathtt{C}}$, i.e., $\frac{dp_0}{d\rho_{\mathtt{C}}} < 0$.

By the above argument, we only need to justify that $\frac{f'}{\sqrt{f}} - \frac{g'-f'}{\sqrt{f-g}} > 0$ for $\nu_{\mathtt{U}} = \sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}/\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}$ to complete the proof. Note that

$$
g' = \frac{n_{\mathtt{I}}^2\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1} + n_{\mathtt{I}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}-1} - 2\sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}}{(\rho_{\mathtt{C}} - 1)}.
$$

Moreover, at $\nu_{\mathtt{U}} = \sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}/\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}$,

$$
f = \left(\frac{\rho_{\mathtt{C}}^{2n_{\mathtt{I}}} + n_{\mathtt{I}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}+1} - 3\rho_{\mathtt{C}}^{n_{\mathtt{I}}} - n_{\mathtt{I}}\rho_{\mathtt{C}}^{n_{\mathtt{I}}} + 2}{2(\rho_{\mathtt{C}} - 1)(\rho_{\mathtt{C}}^{n_{\mathtt{I}}} - 1)}\right)^2,
$$

and

$$f' = \frac{\left(n_{\text{I}}\rho_{\text{C}}^{n_{\text{I}}+1} - 3\rho_{\text{C}}^{n_{\text{I}}} + \rho_{\text{C}}^{2n_{\text{I}}} - n_{\text{I}}\rho_{\text{C}}^{n_{\text{I}}} + 2\right)\left(n_{\text{I}}\rho_{\text{C}}^{n_{\text{I}}-1}\sum_{i=0}^{n_{\text{I}}-1}\rho_{\text{C}}^{i} - 2\left(\sum_{i=0}^{n_{\text{I}}-1}\rho_{\text{C}}^{i}\right)^{2} + n_{\text{I}}^{2}\rho_{\text{C}}^{n_{\text{I}}-1}\right)}{2\left(\rho_{\text{C}}-1\right)\left(\rho_{\text{C}}^{n_{\text{I}}}-1\right)^{2}}.$$

Thus, evaluated at $\nu_{\text{U}} = \sum_{i=0}^{n_{\text{I}}-1}(i+1)\rho_{\text{C}}^{i}/\sum_{i=0}^{n_{\text{I}}-1}\rho_{\text{C}}^{i}$,

$$\begin{aligned}
\frac{f'}{\sqrt{f}} - \frac{g'-f'}{\sqrt{f-g}} &= \frac{2}{\left(\rho_{\text{C}}-1\right)^{2}}\frac{\left(\rho_{\text{C}}^{n_{\text{I}}}-1\right)}{\left(\sum_{i=0}^{n_{\text{I}}-1}\rho_{\text{C}}^{i}-n_{\text{I}}\right)}\left(n_{\text{I}}\rho_{\text{C}}^{n_{\text{I}}-1} - \sum_{i=0}^{n_{\text{I}}-1}\rho_{\text{C}}^{i} + n_{\text{I}} - \sum_{i=0}^{n_{\text{I}}-1}\rho_{\text{C}}^{i}\right)\\
&= \frac{2}{\left(\rho_{\text{C}}-1\right)^{2}}\frac{\left(\rho_{\text{C}}^{n_{\text{I}}}-1\right)}{\left(\sum_{i=0}^{n_{\text{I}}-1}\rho_{\text{C}}^{i}-n_{\text{I}}\right)}\left(\rho_{\text{C}}-1\right)\sum_{i=0}^{n_{\text{I}}-2}\left(2i+2-n_{\text{I}}\right)\rho_{\text{C}}^{i}\\
&= \frac{2}{\left(\rho_{\text{C}}-1\right)^{2}}\frac{\left(\rho_{\text{C}}^{n_{\text{I}}}-1\right)}{\left(\sum_{i=0}^{n_{\text{I}}-1}\rho_{\text{C}}^{i}-n_{\text{I}}\right)}\left(\rho_{\text{C}}-1\right)\sum_{i=\left\lfloor\frac{n_{\text{I}}+1}{2}\right\rfloor}^{n_{\text{I}}-2}\left(2i+2-n_{\text{I}}\right)\left(\rho_{\text{C}}^{2i+2-n_{\text{I}}}-1\right)\rho_{\text{C}}^{n_{\text{I}}-i-2}\\
&> 0. \quad \square
\end{aligned}$$

Proof of Theorem 6. The social welfare for each customer segment is

$$S_{\text{I}}\left(q^{*}\right) = \left[\sum_{i=0}^{n_{\text{I}}-1} p_{i}\left(q^{*}\right)\left(R_{\text{I}} - c_{\text{I}}\frac{i+1}{\mu}\right)\right]\cdot\gamma\Lambda \text{ and } S_{\text{U}}\left(q^{*}\right) = \left[q^{*}\sum_{i=0}^{\infty} p_{i}\left(q^{*}\right)\left(R_{\text{U}} - c_{\text{U}}\frac{i+1}{\mu}\right)\right]\cdot(1-\gamma)\Lambda$$

Analogous to Theorem 4, we discuss the following cases in order: $q^{*}=0$, $q^{*}\in(0,1)$, and $q^{*}=1$.

When $q^{*}=0$, $\rho_{\text{C}}=\gamma\rho$. Since uninformed customers do not join, $S_{\text{U}}\left(q^{*}\right)=0$ and total social welfare is identical to informed individuals' contribution

$$\begin{aligned}
S_{\text{I}}\left(q^{*}=0\right) &= \left[\sum_{i=0}^{n_{\text{I}}-1} p_{i}(0)\left(R_{\text{I}} - c_{\text{I}}\frac{i+1}{\mu}\right)\right]\cdot\gamma\Lambda\\
&= \left(\frac{1-(\gamma\rho)^{n_{\text{I}}}}{1-\gamma\rho} + (\gamma\rho)^{n_{\text{I}}}\right)^{-1}\left(R_{\text{I}}\frac{1-(\gamma\rho)^{n_{\text{I}}}}{1-\gamma\rho} - \frac{c_{\text{I}}}{\mu}\frac{1-\left(n_{\text{I}}+1\right)(\gamma\rho)^{n_{\text{I}}} + n(\gamma\rho)^{n_{\text{I}}+1}}{(1-\gamma\rho)^{2}}\right)\cdot\gamma\Lambda.
\end{aligned}$$

Notice that $S_{\text{I}}(q^{*}=0)$ is independent of R_{U}. Thus, we can apply the same discussion in the proof of Theorem 4(i) to show that $S_{\text{I}}\left(q^{*}=0\right)+S_{\text{U}}\left(q^{*}=0\right)$ strictly decreases in γ.

When $q^{*}\in(0,1)$, the social welfare yielded by uninformed customers equals zero as well, i.e., $S_{\text{U}}\left(q^{*}\right)=0$. Thus, we only need to consider $S_{\text{I}}\left(q^{*}\right)$.

$$\begin{aligned}
S_{\text{I}}\left(q^{*}\right) &= \left[\sum_{i=0}^{n_{\text{I}}-1} p_{i}\left(q^{*}\right)\left(R_{\text{I}} - c_{\text{I}}\frac{i+1}{\mu}\right)\right]\cdot\gamma\Lambda\\
&= p_{0}\left(q^{*}\right)\left(R_{\text{I}}\frac{1-\rho_{\text{C}}^{n_{\text{I}}}}{1-\rho_{\text{C}}} - \frac{c_{\text{I}}}{\mu}\frac{1-\left(n_{\text{I}}+1\right)\rho_{\text{C}}^{n_{\text{I}}}+n\rho_{\text{C}}^{n_{\text{I}}+1}}{\left(1-\rho_{\text{C}}\right)^{2}}\right)\cdot\gamma\Lambda\\
&= c_{\text{I}}p_{0}\left(q^{*}\right)\left(\nu_{\text{I}}\frac{1-\rho_{\text{C}}^{n_{\text{I}}}}{1-\rho_{\text{C}}} - \frac{1-\left(n_{\text{I}}+1\right)\rho_{\text{C}}^{n_{\text{I}}}+n_{\text{I}}\rho_{\text{C}}^{n_{\text{I}}+1}}{\left(1-\rho_{\text{C}}\right)^{2}}\right)\cdot\rho\gamma\left(\rho_{\text{C}}\right)\\
&= c_{\text{I}}\rho_{\text{C}}^{n_{\text{I}}}\left(H\left(\rho_{\text{C}}\right) - \frac{\left(\nu_{\text{U}}-\nu_{\text{I}}\right)\sum_{i=0}^{n_{\text{I}}-1}\rho_{\text{C}}^{i}}{\rho_{\text{C}}^{n_{\text{I}}}}\right)\cdot p_{0}\left(q^{*}\right)\cdot\rho\gamma\left(\rho_{\text{C}}\right)\\
&= c_{\text{I}}\rho_{\text{C}}^{n_{\text{I}}}H\left(\rho_{\text{C}}\right)\cdot p_{0}\left(q^{*}\right)\cdot\rho\gamma\left(\rho_{\text{C}}\right) - c_{\text{I}}\left(\nu_{\text{U}}-\nu_{\text{I}}\right)\sum_{i=0}^{n_{\text{I}}-1}\rho_{\text{C}}^{i}\cdot p_{0}\left(q^{*}\right)\cdot\rho\gamma\left(\rho_{\text{C}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= c_{\mathtt{I}}\left(1-(\nu_{\mathtt{U}}-\nu_{\mathtt{I}})\frac{\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}}{\rho_{\mathtt{C}}^{n_{\mathtt{I}}}H(\rho_{\mathtt{C}})}\right)\rho_{\mathtt{C}}^{n_{\mathtt{I}}}H(\rho_{\mathtt{C}})\cdot p_0(q^*)\cdot\rho\gamma(\rho_{\mathtt{C}})\\
&= c_{\mathtt{I}}\left[1-(\nu_{\mathtt{U}}-\nu_{\mathtt{I}})\left(\nu_{\mathtt{U}}-\frac{\sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}}{\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}}\right)^{-1}\right]\rho_{\mathtt{C}}^{n_{\mathtt{I}}}H(\rho_{\mathtt{C}})\cdot p_0(q^*)\cdot\rho\gamma(\rho_{\mathtt{C}}) \qquad \text{(OA.8)}
\end{aligned}
$$

We first observe that $\rho_{\mathtt{C}}^{n_{\mathtt{I}}}H(\rho_{\mathtt{C}})\cdot p_0(q^*)\cdot\rho\gamma(\rho_{\mathtt{C}})$ strictly increases in $\rho_{\mathtt{C}}$. Substitute $p_0(q^*)$ with (OA.5),

$$
\begin{aligned}
\rho_{\mathtt{C}}^{n_{\mathtt{I}}}H(\rho_{\mathtt{C}})p_0(q^*)\rho\gamma(\rho_{\mathtt{C}}) &= \rho_{\mathtt{C}}^{n_{\mathtt{I}}}H(\rho_{\mathtt{C}})\frac{2\left((\nu_{\mathtt{U}}-n_{\mathtt{I}})+\sqrt{(\nu_{\mathtt{U}}-n_{\mathtt{I}})^2+4H(\rho_{\mathtt{C}})}\right)^{-1}+\rho_{\mathtt{C}}-1}{\frac{1-\rho_{\mathtt{C}}^{n_{\mathtt{I}}}}{1-\rho_{\mathtt{C}}}+\frac{\rho_{\mathtt{C}}^{n_{\mathtt{I}}}}{2}\left((\nu_{\mathtt{U}}-n_{\mathtt{I}})+\sqrt{(\nu_{\mathtt{U}}-n_{\mathtt{I}})^2+4H(\rho_{\mathtt{C}})}\right)}\\
&= \left(2\rho_{\mathtt{C}}^{n_{\mathtt{I}}}H(\rho_{\mathtt{C}})+(\nu_{\mathtt{U}}-n_{\mathtt{I}})\rho_{\mathtt{C}}^{n_{\mathtt{I}}}H(\rho_{\mathtt{C}})(\rho_{\mathtt{C}}-1)+\rho_{\mathtt{C}}^{n_{\mathtt{I}}}H(\rho_{\mathtt{C}})(\rho_{\mathtt{C}}-1)\sqrt{(\nu_{\mathtt{U}}-n_{\mathtt{I}})^2+4H(\rho_{\mathtt{C}})}\right)\times\\
&\quad \left((\nu_{\mathtt{U}}-n_{\mathtt{I}})\frac{1-\rho_{\mathtt{C}}^{n_{\mathtt{I}}}}{1-\rho_{\mathtt{C}}}+(\nu_{\mathtt{U}}-n_{\mathtt{I}})^2\rho_{\mathtt{C}}^{n_{\mathtt{I}}}+2\rho_{\mathtt{C}}^{n_{\mathtt{I}}}H(\rho_{\mathtt{C}})+\right.\\
&\quad \left.\left(\frac{1-\rho_{\mathtt{C}}^{n_{\mathtt{I}}}}{1-\rho_{\mathtt{C}}}+(\nu_{\mathtt{U}}-n_{\mathtt{I}})\rho_{\mathtt{C}}^{n_{\mathtt{I}}}\right)\sqrt{(\nu_{\mathtt{U}}-n_{\mathtt{I}})^2+4H(\rho_{\mathtt{C}})}\right)^{-1}\\
&= \left(1-\nu_{\mathtt{U}}\frac{(\nu_{\mathtt{U}}-n_{\mathtt{I}})+\sqrt{(\nu_{\mathtt{U}}-n_{\mathtt{I}})^2+4H(\rho_{\mathtt{C}})}}{2\rho_{\mathtt{C}}^{n_{\mathtt{I}}}H(\rho_{\mathtt{C}})+\left(\frac{1-\rho_{\mathtt{C}}^{n_{\mathtt{I}}}}{1-\rho_{\mathtt{C}}}+(\nu_{\mathtt{U}}-n_{\mathtt{I}})\rho_{\mathtt{C}}^{n_{\mathtt{I}}}\right)\left((\nu_{\mathtt{U}}-n_{\mathtt{I}})+\sqrt{(\nu_{\mathtt{U}}-n_{\mathtt{I}})^2+4H(\rho_{\mathtt{C}})}\right)}\right)\\
&= \left(1-\frac{\nu_{\mathtt{U}}}{\frac{1-\rho_{\mathtt{C}}^{n_{\mathtt{I}}}}{1-\rho_{\mathtt{C}}}+\frac{\rho_{\mathtt{C}}^{n_{\mathtt{I}}}}{2}\left((\nu_{\mathtt{U}}-n_{\mathtt{I}})+\sqrt{(\nu_{\mathtt{U}}-n_{\mathtt{I}})^2+4H(\rho_{\mathtt{C}})}\right)}\right)\\
&= 1-\nu_{\mathtt{U}}p_0(q^*). \qquad \text{(OA.9)}
\end{aligned}
$$

We have already demonstrated that $p_0(q^*)$ strictly decreases in $\rho_{\mathtt{C}}$ in the proof of Theorem 5. Therefore, $\rho_{\mathtt{C}}^{n_{\mathtt{I}}}H(\rho_{\mathtt{C}})\cdot p_0(q^*)\cdot\rho\gamma(\rho_{\mathtt{C}})$ also strictly increases in $\rho_{\mathtt{C}}$.

Next, we consider the monotonicity of the term in the square bracket of (OA.8). Recall that $\sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}\big/\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}$ strictly increases in $\rho_{\mathtt{C}}$ as shown in the proof of Theorem 5. Then,

- If $\nu_{\mathtt{U}}\le\nu_{\mathtt{I}}$, $1-(\nu_{\mathtt{U}}-\nu_{\mathtt{I}})\left(\nu_{\mathtt{U}}-\frac{\sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}}{\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}}\right)^{-1}$ is increasing in $\rho_{\mathtt{C}}$. In this case, $S_{\mathtt{I}}(q^*)$ is increasing in $\rho_{\mathtt{C}}$. Due to the fact that $\frac{d\rho_{\mathtt{C}}}{d\gamma}>0$, we have $S_{\mathtt{I}}(q^*)$ is increasing in γ.
- If $\nu_{\mathtt{U}}>\nu_{\mathtt{I}}$, $1-(\nu_{\mathtt{U}}-\nu_{\mathtt{I}})\left(\nu_{\mathtt{U}}-\frac{\sum_{i=0}^{n_{\mathtt{I}}-1}(i+1)\rho_{\mathtt{C}}^{i}}{\sum_{i=0}^{n_{\mathtt{I}}-1}\rho_{\mathtt{C}}^{i}}\right)^{-1}$ is decreasing in $\rho_{\mathtt{C}}$. Then $S_{\mathtt{I}}(q^*)$ might be unimodal in $\rho_{\mathtt{C}}$, which leads to that $S_{\mathtt{I}}(q^*)$ might be unimodal in γ.

When $q^*=1$, $\rho_{\mathtt{C}}=\rho$. The total social welfare is

$$
\begin{aligned}
S_{\mathtt{I}}(q^*)+S_{\mathtt{U}}(q^*) &= \left[\sum_{i=0}^{n_{\mathtt{I}}-1}p_i(1)\left(R_{\mathtt{I}}-c_{\mathtt{I}}\frac{i+1}{\mu}\right)\right]\cdot\gamma\Lambda+\left[\sum_{i=0}^{\infty}p_i(1)\left(R_{\mathtt{U}}-c_{\mathtt{U}}\frac{i+1}{\mu}\right)\right]\cdot(1-\gamma)\Lambda\\
&\overset{\text{(A.8)}}{=} \frac{\Lambda c_I}{\mu}\rho^{n_{\mathtt{I}}}p_0(1)\gamma L(\rho)+(1-\gamma)\Lambda(R_{\mathtt{U}}-c_{\mathtt{U}}W(1))\\
&= \frac{\Lambda c_{\mathtt{I}}}{\mu}\rho^{n_{\mathtt{I}}}p_0(1)\gamma L(\rho)+\Lambda\frac{c_{\mathtt{I}}}{\mu}\rho^{n_{\mathtt{I}}}p_0(1)(1-\gamma)\left(H(\rho)+\frac{\nu_{\mathtt{U}}-n_{\mathtt{I}}}{1-\rho+\gamma\rho}-\frac{1}{(1-\rho+\gamma\rho)^2}\right)\\
&\overset{\text{(A.1),(OA.2)}}{=} \frac{\Lambda c_{\mathtt{I}}}{\mu}\rho^{n_{\mathtt{I}}}p_0(1)\underbrace{\left[L(\rho)+\frac{(\nu_{\mathtt{U}}-n_{\mathtt{I}})(1-\gamma)}{1-\rho+\gamma\rho}-\frac{1-\gamma}{(1-\rho+\gamma\rho)^2}+\frac{(1-\gamma)(\nu_{\mathtt{U}}-\nu_{\mathtt{I}})(1-\rho^{n_{\mathtt{I}}})}{(1-\rho)\rho^{n_{\mathtt{I}}}}\right]}_{:=\Upsilon(\gamma)}
\end{aligned}
$$

Since $p_0(q^*=1)=\left(\frac{1-\rho^{n_{\mathtt{I}}}}{1-\rho}+\frac{\rho^{n_{\mathtt{I}}}}{1-\rho+\gamma\rho}\right)^{-1}$ strictly increases in γ, we only need to explore the monotonicity of $\Upsilon(\gamma)$. Since $S_{\mathtt{I}}(q^*)+S_{\mathtt{U}}(q^*)>0$, $\Upsilon(\gamma)>0$ as well. Since $L(\rho)$ is independent of γ,

$$\frac{\partial\Upsilon}{\partial\gamma}=\frac{1}{(1-\rho+\gamma\rho)^2}\left(\frac{1+\rho-\gamma\rho}{1-\rho+\gamma\rho}-(\nu_{\mathtt{U}}-n_{\mathtt{I}})-\frac{(\nu_{\mathtt{U}}-\nu_{\mathtt{I}})(1-\rho^{n_{\mathtt{I}}})(1-\rho+\gamma\rho)^2}{(1-\rho)\rho^{n_{\mathtt{I}}}}\right)$$

Note that $(1-\gamma)\rho$ is the workload caused by uninformed customers. Due to the fact that uninformed customers join the queue with probability 1, the server must have enough capacity to handle all of them, i.e., $(1-\gamma)\rho<1 \Leftrightarrow 1-\rho+\gamma\rho\geq 0$. Thus,

- If $(\nu_{\mathtt{U}}-\nu_{\mathtt{I}})\frac{(1-\rho^{n_{\mathtt{I}}})(1-\rho+\gamma\rho)^2}{(1-\rho)\rho^{n_{\mathtt{I}}}}+\nu_{\mathtt{U}}-n_{\mathtt{I}}\leq\frac{1+\rho-\gamma\rho}{1-\rho+\gamma\rho}$, we have $\Upsilon(\gamma)$ is increasing. Then, $S_{\mathtt{I}}(q^*=1)+S_{\mathtt{U}}(q^*=1)$ is increasing in γ. Note that this also covers the homogenous case, since $\nu-n\in[0,1)$.
- If $(\nu_{\mathtt{U}}-\nu_{\mathtt{I}})\frac{(1-\rho^{n_{\mathtt{I}}})(1-\rho+\gamma\rho)^2}{(1-\rho)\rho^{n_{\mathtt{I}}}}+\nu_{\mathtt{U}}-n_{\mathtt{I}}>\frac{1+\rho-\gamma\rho}{1-\rho+\gamma\rho}$, we have $\Upsilon(\gamma)$ is decreasing. In this case, the monotonicity of $S_{\mathtt{I}}(q^*=1)+S_{\mathtt{U}}(q^*=1)$ may change and the social welfare might be unimodal.

It can be readily shown that $(\nu_{\mathtt{U}}-\nu_{\mathtt{I}})\frac{(1-\rho^{n_{\mathtt{I}}})(1-\rho+\gamma\rho)^2}{(1-\rho)\rho^{n_{\mathtt{I}}}}+\nu_{\mathtt{U}}-n_{\mathtt{I}}\leq\frac{1+\rho-\gamma\rho}{1-\rho+\gamma\rho}$ implies $\nu_{\mathtt{I}}\geq\nu_{\mathtt{U}}-\left(\frac{1+\rho-\gamma\rho}{1-\rho+\gamma\rho}-\langle\nu_{\mathtt{I}}\rangle\right)\Big/\left(\frac{(1-\rho^{n_{\mathtt{I}}})(1-\rho+\gamma\rho)^2}{(1-\rho)\rho^{n_{\mathtt{I}}}}+1\right)$. Thus, the proof completes. □

Online Appendix B to
"Efficient Ignorance: Information Heterogeneity in a Queue"

LEMMA B1. *The function $L(\rho,\nu)$ defined in Corollary 2 is strictly decreasing in ρ.*

Proof of Lemma B1. For notation simplicity, we suppress $L(\rho,\nu)$'s dependence on ν and write $L(\rho)$ or simply L. By the definition of $L(\rho)$,

$$\frac{dL}{d\rho} = \frac{\phi(\rho)}{\rho^{n+1}(\rho-1)^3},$$

where

$$\phi(\rho) \equiv \nu(n+1)\rho^2 + (2-\nu-2n\nu+n)\rho + n\nu - n - \langle\nu\rangle\rho^{n+2} + (\langle\nu\rangle - 2)\rho^{n+1}.$$

Taking first and second derivatives of $\phi(\rho)$ with respect to ρ, we have

$$\phi'(\rho) = \frac{d\phi}{d\rho} = 2\nu(n+1)\rho + (2-\nu-2n\nu+n) - (n+2)\langle\nu\rangle\rho^{n+1} + (n+1)(\langle\nu\rangle-2)\rho^n \tag{OA.10}$$

and

$$\begin{aligned}\phi''(\rho) = \frac{d^2\phi}{d\rho^2} &= 2\nu(n+1) - (n+1)(n+2)\langle\nu\rangle\rho^n + n(n+1)(\langle\nu\rangle-2)\rho^{n-1} \\ &= (n+1)\left[2\nu - 2n\rho^{n-1} - \langle\nu\rangle\left(2\rho^n + n\rho^n - n\rho^{n-1}\right)\right] \\ &\overset{\nu=n+\langle\nu\rangle}{=} (n+1)\left[2n(1-\rho^{n-1}) + \langle\nu\rangle\left(2(1-\rho^n) + n\rho^{n-1}(1-\rho)\right)\right] \\ &= (n+1)(1-\rho)\left[2n\sum_{i=0}^{n-2}\rho^i + \langle\nu\rangle\left(2\sum_{i=0}^{n-1}\rho^i + n\rho^{n-1}\right)\right],\end{aligned} \tag{OA.11}$$

where $\sum_{i=0}^{n-2}\rho^i$ is understood as 0 for $n=1$. Moreover, note that $\phi(1) = \phi'(1) = 0$. Hence, by Eq.(OA.10) and (OA.11),

$$\begin{cases}\phi''(\rho) > 0, \text{ if } 0<\rho<1 \\ \phi''(\rho) < 0, \text{ if } \rho>1\end{cases} \Longrightarrow \begin{cases}\phi'(\rho) < \phi'(1) = 0, \text{ if } 0<\rho<1 \\ \phi'(\rho) < \phi'(1) = 0, \text{ if } \rho>1\end{cases} \Longrightarrow \begin{cases}\phi(\rho) > \phi(1) = 0, \text{ if } 0<\rho<1 \\ \phi(\rho) < \phi(1) = 0, \text{ if } \rho>1\end{cases}.$$

Therefore, $\frac{dL}{d\rho} = \frac{\phi(\rho)}{\rho^{n+1}(\rho-1)^3} < 0$ for $0<\rho<1$ and $\rho>1$. Finally, by L'Hôpital's rule, $\lim_{\rho\to 1}\frac{dL}{d\rho} = n(n+1)(n+2-3\nu)/6$, which is negative for all $\nu>1$ and is zero for $\nu=1$. We thus conclude that $\frac{dL}{d\rho}<0$ for $\rho>0$ (almost surely except for the point $\rho=1$ when $\nu=1$), i.e., $L(\rho)$ is strictly decreasing in ρ (note that the derivative being equal to 0 at one point does not affect the strict monotonicity of a function). □

LEMMA B2. *In the neighborhood where full participation is not adopted by uninformed customers in equilibrium, i.e., $q^*\in[0,1)$, for any information level γ', there exists $k<n$ such that*

$$\left.\frac{dp_i(q^*(\gamma))}{d\gamma}\right|_{\gamma=\gamma'} < 0 \text{ for } 0\le i\le k \text{ and } \left.\frac{dp_i(q^*(\gamma))}{d\gamma}\right|_{\gamma=\gamma'} \ge 0 \text{ for } k<i<n.$$

Proof of Lemma B2. We have shown, in Lemma 2, that $p_0(q^*(\gamma))$ strictly decreases in γ for $0 \le q^* < 1$. At $\gamma = \gamma'$, if for any $i = 1, \dots, n-1$, $dp_i(q^*(\gamma'))/d\gamma < 0$. Then, $k = n-1$.

If there exists $k < n-1$, such that $dp_k(q^*(\gamma'))/d\gamma \ge 0$ at γ', then the statement holds as long as for any $i = k, k+1, \dots, n-1$, $dp_i(q^*(\gamma'))d\gamma \ge 0$. Let $\rho_{\mathsf{c}}(\gamma) = \gamma\rho + q^*(\gamma)(1-\gamma)\rho$, where $0 \le q^*(\gamma) < 1$. By Eq. (3), $p_i(q^*(\gamma)) = p_k(q^*(\gamma))\rho_{\mathsf{c}}^{i-k}(\gamma) = p_0(q^*(\gamma))\rho_{\mathsf{c}}^{k}(\gamma)\rho_{\mathsf{c}}^{i-k}(\gamma)$. Thereby, for $i = k, k+1, \dots, n-1$,

$$\frac{dp_i(q^*(\gamma))}{d\gamma} = \frac{dp_k(q^*(\gamma))}{d\gamma} \underbrace{\rho_{\mathsf{c}}^{i-k}(\gamma)}_{\ge 0} + \underbrace{p_k(q^*(\gamma))(i-k)\rho_{\mathsf{c}}^{i-k-1}(\gamma)}_{\ge 0} \frac{d\rho_{\mathsf{c}}(\gamma)}{d\gamma}.$$

At γ', $dp_k(q^*(\gamma'))/d\gamma \ge 0$ by assumption. Hence, if $d\rho_{\mathsf{c}}(\gamma')/d\gamma \ge 0$, $dp_i(q^*(\gamma'))/d\gamma \ge 0$. Note that

$$\frac{dp_k(q^*(\gamma))}{d\gamma} = \frac{dp_0(q^*(\gamma))}{d\gamma}\rho_{\mathsf{c}}^{k}(\gamma) + p_0(q^*(\gamma))k\rho_{\mathsf{c}}^{k-1}(\gamma)\frac{d\rho_{\mathsf{c}}(\gamma)}{d\gamma} \ge 0.$$

The first term is negative since $p_0(q^*(\gamma))$ strictly decreases in γ. Hence, $dp_k(q^*(\gamma'))/d\gamma \ge 0$ implies $d\rho_{\mathsf{c}}(\gamma')/d\gamma \ge 0$, which further leads to $dp_i(q^*(\gamma'))/d\gamma \ge 0$ for $i = k, k+1, \dots, n-1$. □

PROPOSITION B1 (COMPARATIVE STATICS OF ACCESSIBILITY FOR INFORMED CUSTOMERS).

(i) If $0 \le q^* < 1$, the probability $\sum_{i=0}^{n-1} p_i(q^*)$ that an informed customer joins the queue is strictly decreasing in γ.

(ii) If $q^* = 1$, the probability $\sum_{i=0}^{n-1} p_i(q^*)$ that an informed customer joins the queue is strictly increasing in γ.

Proof of Proposition B1. (i) When $q^* = 0$, $\sum\limits_{i=0}^{n-1} p_i(q^* = 0) = p_0(q^* = 0)\sum\limits_{i=0}^{n-1}(\gamma\rho)^i = \frac{1-(\gamma\rho)^n}{1-(\gamma\rho)^{n+1}}$. It is straightforward to verify that $\frac{1-(\gamma\rho)^n}{1-(\gamma\rho)^{n+1}}, n \ge 1$ is strictly decreasing in γ.

Consider the case where $0 < q^* < 1$. Again, let $\rho_{\mathsf{c}} = \rho(\gamma + q^*(1-\gamma))$. Recall that we have shown $d\rho_{\mathsf{c}}/d\gamma > 0$ in the proof of Theorem 2. If $\sum_{i=0}^{n-1} p_i(q^*)$ is strictly decreasing in ρ_{c}, by the chain rule, it must be strictly decreasing in γ. Hence, it is sufficient to prove that $\sum_{i=0}^{n-1} p_i(q^*)$ is strictly decreasing in ρ_{c}. We rewrite

$$\sum_{i=0}^{n-1} p_i(q^*) = \sum_{i=0}^{n-1} p_0(q^*)\rho_{\mathsf{c}}^i = p_0(q^*)\frac{1-\rho_{\mathsf{c}}^n}{1-\rho_{\mathsf{c}}} \overset{(6)}{=} \frac{1-\rho_{\mathsf{c}}^n}{1-\rho_{\mathsf{c}}} \Big/ \left(\frac{1-\rho_{\mathsf{c}}^n}{1-\rho_{\mathsf{c}}} + \frac{\rho_{\mathsf{c}}^n}{1-\rho_{\mathsf{c}}+\gamma\rho}\right) = \left(1 + \frac{\rho_{\mathsf{c}}^n}{1-\rho_{\mathsf{c}}+\gamma\rho} \cdot \frac{1-\rho_{\mathsf{c}}}{1-\rho_{\mathsf{c}}^n}\right)^{-1}.$$

By Eq.(A.6),

$$\begin{aligned}
\frac{\rho_{\mathsf{c}}^n}{1-\rho_{\mathsf{c}}+\gamma\rho} \cdot \frac{1-\rho_{\mathsf{c}}}{1-\rho_{\mathsf{c}}^n} &= \frac{\rho_{\mathsf{c}}^n(1-\rho_{\mathsf{c}})}{1-\rho_{\mathsf{c}}^n} \cdot \frac{1}{2}\left(\langle\nu\rangle + \sqrt{\langle\nu\rangle^2 + 4L(\rho_{\mathsf{c}})}\right) \\
&= \frac{1}{2}\langle\nu\rangle\frac{\rho_{\mathsf{c}}^n(1-\rho_{\mathsf{c}})}{1-\rho_{\mathsf{c}}^n} + \sqrt{\left(\frac{1}{2}\langle\nu\rangle\frac{\rho_{\mathsf{c}}^n(1-\rho_{\mathsf{c}})}{1-\rho_{\mathsf{c}}^n}\right)^2 + \frac{\rho_{\mathsf{c}}^{2n}(1-\rho_{\mathsf{c}})^2}{(1-\rho_{\mathsf{c}}^n)^2}L(\rho_{\mathsf{c}})} \\
&\overset{(A.5)}{=} \frac{1}{2}\langle\nu\rangle\frac{\rho_{\mathsf{c}}^n(1-\rho_{\mathsf{c}})}{1-\rho_{\mathsf{c}}^n} + \sqrt{\left(\frac{1}{2}\langle\nu\rangle\frac{\rho_{\mathsf{c}}^n(1-\rho_{\mathsf{c}})}{1-\rho_{\mathsf{c}}^n}\right)^2 + \frac{\rho_{\mathsf{c}}^{2n}(1-\rho_{\mathsf{c}})^2}{(1-\rho_{\mathsf{c}}^n)^2} \cdot \frac{\langle\nu\rangle(\rho_{\mathsf{c}}-1)\rho_{\mathsf{c}}^n + \nu - \nu\rho_{\mathsf{c}} + \rho_{\mathsf{c}}^n - 1}{(1-\rho_{\mathsf{c}})^2\rho_{\mathsf{c}}^n}} \\
&= \frac{1}{2}\langle\nu\rangle\frac{\rho_{\mathsf{c}}^n(1-\rho_{\mathsf{c}})}{1-\rho_{\mathsf{c}}^n} + \sqrt{\left(\frac{1}{2}\langle\nu\rangle\frac{\rho_{\mathsf{c}}^n(1-\rho_{\mathsf{c}})}{1-\rho_{\mathsf{c}}^n}\right)^2 + \langle\nu\rangle\frac{\rho_{\mathsf{c}}^n(1-\rho_{\mathsf{c}})}{1-\rho_{\mathsf{c}}^n} + \rho_{\mathsf{c}}^n\frac{n - n\rho_{\mathsf{c}} + \rho_{\mathsf{c}}^n - 1}{(1-\rho_{\mathsf{c}}^n)^2}}.
\end{aligned}$$

It is apparent that $\dfrac{\rho_{\mathsf{c}}^n(1-\rho_{\mathsf{c}})}{1-\rho_{\mathsf{c}}^n} = \left(\sum_{i=1}^n \rho_{\mathsf{c}}^{-i}\right)^{-1}$ is strictly increasing in ρ_{c}. Therefore, to show $\sum_{i=0}^{n-1} p_i(q^*)$ is strictly decreasing in ρ_{c}, it suffices to justify $\rho_{\mathsf{c}}^n\dfrac{n - n\rho_{\mathsf{c}} + \rho_{\mathsf{c}}^n - 1}{(1-\rho_{\mathsf{c}}^n)^2}$ increases in ρ_{c}.

$$\left(\rho_{\mathsf{c}}^n\frac{n - n\rho_{\mathsf{c}} + \rho_{\mathsf{c}}^n - 1}{(1-\rho_{\mathsf{c}}^n)^2}\right)' = \frac{n\rho_{\mathsf{c}}^{n-1}}{(1-\rho_{\mathsf{c}}^n)^3}\left((n+1)\rho_{\mathsf{c}}^n - (n-1)\rho_{\mathsf{c}}^{n+1} - (n+1)\rho_{\mathsf{c}} + n - 1\right)$$

Let $\chi(\rho_{\mathsf{c}}) = (n+1)\rho_{\mathsf{c}}^n - (n-1)\rho_{\mathsf{c}}^{n+1} - (n+1)\rho_{\mathsf{c}} + n - 1$. Then, $\chi'(\rho_{\mathsf{c}}) = (n+1)\left(n\rho_{\mathsf{c}}^{n-1} + (1-n)\rho_{\mathsf{c}}^n - 1\right)$ and $\chi''(\rho_{\mathsf{c}}) = n(n^2-1)(1-\rho_{\mathsf{c}})\rho_{\mathsf{c}}^{n-2}$. Hence,

$$\begin{cases} \chi''(\rho_{\mathsf{c}}) > 0, \text{ if } 0 < \rho_{\mathsf{c}} < 1 \\ \chi''(\rho_{\mathsf{c}}) < 0, \text{ if } \rho_{\mathsf{c}} > 1 \end{cases} \Longrightarrow \begin{cases} \chi'(\rho_{\mathsf{c}}) < \chi'(1) = 0, \text{ if } 0 < \rho_{\mathsf{c}} < 1 \\ \chi'(\rho_{\mathsf{c}}) < \chi'(1) = 0, \text{ if } \rho_{\mathsf{c}} > 1 \end{cases} \Longrightarrow \begin{cases} \chi(\rho_{\mathsf{c}}) > \chi(1) = 0, \text{ if } 0 < \rho_{\mathsf{c}} < 1 \\ \chi(\rho_{\mathsf{c}}) < \chi(1) = 0, \text{ if } \rho_{\mathsf{c}} > 1 \end{cases}.$$

Thus, $\left(\rho_{\mathsf{c}}^n \dfrac{n - n\rho_{\mathsf{c}} + \rho_{\mathsf{c}}^n - 1}{(1-\rho_{\mathsf{c}}^n)^2}\right)' > 0$ for $\rho_{\mathsf{c}} > 0$ but $\rho_{\mathsf{c}} \neq 1$. Moreover, by L'Hôpital's rule, $\lim\limits_{\rho_{\mathsf{c}} \to 1} \left(\rho_{\mathsf{c}}^n \dfrac{n - n\rho_{\mathsf{c}} + \rho_{\mathsf{c}}^n - 1}{(1-\rho_{\mathsf{c}}^n)^2}\right)' = (n^2-1)/(6n) \geq 0$ with equality only if $n = 1$ and $\rho_{\mathsf{c}} = 1$. Consequently, $\rho_{\mathsf{c}}^n \dfrac{n - n\rho_{\mathsf{c}} + \rho_{\mathsf{c}}^n - 1}{(1-\rho_{\mathsf{c}}^n)^2}$ is strictly increasing in ρ_{c}, which implies $\sum_{i=0}^{n-1} p_i(q^*)$ is strictly decreasing in γ.

(ii) When $q^* = 1$, $\sum\limits_{i=0}^{n-1} p_i(q^* = 1) = p_0(q^* = 1) \sum\limits_{i=0}^{n-1} \rho^i = \left(\frac{1-\rho^n}{1-\rho} + \frac{\rho^n}{1-\rho+\gamma\rho}\right)^{-1} \frac{1-\rho^n}{1-\rho}$, which clearly is strictly increasing in γ. □

LEMMA B3. *For $\nu \geq 2$ and $\bar{\nu} = \nu + i$ for any $i \in N$, we have $y^*(\bar{\nu}) > y^*(\nu) \geq 1$.*

Proof of Lemma B3. It suffices to prove that (i) $y^*(\nu) \geq 1$ for $\nu \in [2,3)$; (ii) $y^*(\nu+1) > y^*(\nu)$ for $\nu \geq 2$. Note from the proof of Corollary 1 that $f(y,\nu) = n + 1 + \frac{1}{1-y} - \frac{n+1}{1-y^{n+1}}$ strictly increases in y, and $\lim_{y \to 1^+} f(y,\nu) = \frac{1}{2}n + 1$.

When $\nu \in [2,3)$, we have $\lim_{y \to 1^+} f(y,\nu) = 2$, i.e., $y^*(\nu = 2) = 1$. Then, due to $f(y,\nu)$'s monotonicity, we have that $y^*(\nu)$ is an increasing function of ν. Thus, $y^*(\nu) \geq 1$, for $\nu \in [2,3)$.

When $\nu \in [n, n+1)$, we have $f(y^*(\nu), \nu) - \nu = 0$, and

$$\begin{aligned} f(y^*(\nu), \nu+1) - (\nu+1) &= n + 1 - \nu + \frac{1}{1-y^*(\nu)} - \frac{n+2}{1-(y^*(\nu))^{n+2}} \\ &< n + 1 - \nu + \frac{1}{1-y^*(\nu)} - \frac{n+1}{1-(y^*(\nu))^{n+1}} = f(y^*(\nu), \nu) - \nu = 0, \end{aligned}$$

where the inequality is from

$$\frac{n+2}{1-y^{n+2}} - \frac{n+1}{1-y^{n+1}} = \frac{\sum_{i=0}^{n} y^i \sum_{j=0}^{n-i} y^j}{\sum_{i=0}^{n} y^i \sum_{i=0}^{n+1} y^i} > 0.$$

Then, from $f(y,\nu)$'s monotonicity, we have $y^*(\nu+1) > y^*(\nu)$. □