Online Appendix to "Information Disclosure and Pricing Policies for Sales of Network Goods"

A. Proofs

This online appendix provides all proofs of propositions, and lemmas in our paper. Since it is a trivial case for our analysis that $s(M) = \mathsf{E}s(M)$ in probability 1, in this appendix we always assume $\mathrm{Var}(s(M)) > 0$.

Proof of Lemma 1. Let $H(\alpha) = \bar{G}(p - s(\alpha m))$ for any $m \geq 0$. We have

$$H'(\alpha) = g(p - s(\alpha m))s'(\alpha m)m. \tag{A.1}$$

By Assumption 1, it is easy to see that $H(\alpha)$ is nondecreasing and (weakly) convex in α if $p > v^0 + s(m)$ and $m \leq d^0$.

Since $p > v^0 + s(m)$, it follows that $H(1) = \bar{G}(p - s(m)) < \bar{G}(v^0) \leq 1$. Also we have $H(0) \geq 0$. Therefore, $H(\alpha)$ can only have one fixed point on $[0, 1)$ in this case. □

Proof of Lemma 2. We prove the properties in part (i) and (ii) only for $\alpha_F(\cdot)$, since the properties of $d_F(\cdot)$ are direct sequences of the properties of $\alpha_F(\cdot)$.

By Equation (1), we have

$$\frac{\partial \alpha_F(m)}{\partial m} = \frac{g\big(p - s(\alpha_F(m)m)\big)s'(\alpha_F(m)m)\alpha_F(m)}{1 - g\big(p - s(\alpha_F(m)m)\big)s'(\alpha_F(m)m)m}. \tag{A.2}$$

Note that $g\big(p - s(\alpha_F(m)m)\big)s'(\alpha_F(m)m)m = H'(\alpha_F(m))$ where $H^(\cdot)$ is defined by Equation (A.1). If $p > v^0 + s(m)$ and $m \leq d^0$, by the convexity of $H(\cdot)$, we have

$$H'(\alpha_F(m)) \leq \frac{H(1) - H(\alpha_F(m))}{1 - \alpha_F(m)} < \frac{1 - \alpha_F(m)}{1 - \alpha_F(m)} = 1;$$

hence, the right-hand side of (A.2) is well defined and nonnegative. Then from (A.2) and Assumption 1, we can see that $\alpha_F(m)$ is (weakly) increasing and (weakly) convex in m as long as $p > v^0 + s(m)$ and $m \leq d^0$; if we also have $\bar{G}(p) > 0$ and $s'(\cdot) > 0$, then $\frac{\alpha_F(m)}{m} > 0$ and strictly increasing, that is, "increasing" and "convex" properties here are in a strict sense.

For part (ii), we have $H(1) = \bar{G}(p - s(m)) = 1$ if $p \leq \underline{v} + s(m)$. Thus the largest fixed point of $H(\alpha) = \alpha$, i.e., $\alpha_F(m)$, is 1. Therefore, the proof is completed. □

Proof of Lemma 3. The proof is analogous to the proof of Lemma 1. □

Before we prove Proposition 1, we give the following lemma:

LEMMA A.1. *Let X be a random variable and $f(\cdot)$ and $g(\cdot)$ be increasing functions. Suppose $f(X)$, $g(X)$ and $f(X)g(X)$ all have finite expectations. Then $\mathsf{E}[f(X)g(X)] \geq \mathsf{E}f(X) \cdot \mathsf{E}g(X)$.*

Proof of Lemma A.1. Define $x_0 = \inf_x\{g(x) \geq \mathsf{E}g(X)\}$. We have

$$\begin{aligned}
\mathsf{E}[f(X)(g(X)-\mathsf{E}g(X))] &= \mathsf{E}[f(X)(g(X)-\mathsf{E}g(X)); X > x_0] + \mathsf{E}[f(X)(g(X)-\mathsf{E}g(X)); X < x_0] \\
&\quad + \mathsf{E}[f(X)(g(X)-\mathsf{E}g(X)); X = x_0] \\
&\geq \mathsf{E}[f(x_0)(g(X)-\mathsf{E}g(X)); X > x_0] + \mathsf{E}[f(x_0)(g(X)-\mathsf{E}g(X)); X < x_0] \\
&\quad + \mathsf{E}[f(x_0)(g(X)-\mathsf{E}g(X)); X = x_0] \\
&= f(x_0) \cdot \mathsf{E}[g(X)-\mathsf{E}g(X)] = 0.
\end{aligned}$$

Thus the lemma is proved. □

Proof of Proposition 1. Part (ii) holds because, if $p \leq \underline{v} + \mathsf{E}s(M)$, $\alpha_F(m) \leq \alpha_N = 1$ for any $m \geq 0$, which yields $R_N \geq R_F$. Next we will prove part (i).

From lemma A.1, we see $\mathsf{E}d_F(M) \geq \mathsf{E}\alpha_F(M)\mathsf{E}M$. Consequently, to prove $\mathsf{E}d_F(M) \geq d_N$, it suffices to prove $\mathsf{E}\alpha_F(M) \geq \alpha_N$.

Given $p \geq v^0 + s(\overline{m})$, we have $G(\cdot)$ is concave on $[v^0, \infty)$. From the definition of $\alpha_F(M)$, i.e., Equation (1), we have

$$\mathsf{E}\alpha_F(M) = \mathsf{E}\bar{G}\Big(p - s\big(\alpha_F(M)M\big)\Big) \geq \bar{G}\Big(p - \mathsf{E}s\big(\alpha_F(M)M\big)\Big).$$

Note that when $p > v^0 + s(\overline{m})$, by Lemma 3, α_N is the unique solution to (2). Thus it suffices to prove

$$\mathsf{E}\alpha_F(M) \geq \bar{G}\Big(p - \mathsf{E}s\big(M\mathsf{E}\alpha_F(M)\big)\Big). \tag{A.3}$$

In fact, if (A.3) holds, then α_N must be between 0 and $\mathsf{E}\alpha_F(M)$. To show (A.3), it suffices to show that

$$\mathsf{E}s\big(\alpha_F(M)M\big) \geq \mathsf{E}s\big(M\mathsf{E}\alpha_F(M)\big). \tag{A.4}$$

Now we prove (A.4). For any M, because of the convexity of $s(\cdot)$, we have

$$s\big(M\alpha_F(M)\big) \geq s\big(M\mathsf{E}\alpha_F(M)\big) + [\alpha_F(M) - \mathsf{E}\alpha_F(M)] \cdot M \cdot s'\big(M\alpha_F(M)\big). \tag{A.5}$$

By Lemma A.1 and taking expectation of both sides of (A.5) with respect to M_1, we have (A.4). This completes the proof. □

Proof of Lemma 4. We will only prove the properties of $\alpha_F(\cdot)$ in part (i) and (ii), because the properties of $d_F(\cdot)$ follow directly.

From Equation (5) we see that $\alpha_F(p_1,p_2,m)=\tilde{\alpha}_F(p_1,p_2,m)$ if $m\geq m_0$. By comparing Equations (1) and (4), we have that $\tilde{\alpha}_F(p_1,p_2,m)$, a function of m, has the same properties as $\alpha_F(m)$, defined as the largest fixed point of (4). Therefore, this lemma follows from Lemma 2. □

Proof of Lemma 5. (i) We have known that

$$\begin{aligned} H_F(x)-H_N(x)=&\delta\big[-G^{-1}(1-x)+p_2-\mathsf{E}s\big(\max\{x,\alpha_N(p_2)\}M\big)\big]^+\\ &-\delta\mathsf{E}\big[-G^{-1}(1-x)+p_2-s\big(\max\{x,\tilde{\alpha}_F(p_2,M)\}M\big)\big]^+. \end{aligned}$$

For $x\leq\alpha_N(p_2)$, $G^{-1}(1-x)-p_2+\mathsf{E}s\big(\max\{x,\alpha_N(p_2)\}M\big)\geq G^{-1}(1-\alpha_N(p_2))-p_2+\mathsf{E}s\big(\alpha_N(p_2)M\big)=0$. Thus in this case we have $H_F(x)-H_N(x)\leq 0$, which yields $\alpha_{1,F}(p_1,p_2)\leq\alpha_{1,N}(p_1,p_2)$.

We now consider the case when $x>\alpha_N(p_2)$. In this case, we have

$$\begin{aligned} &\mathsf{E}\big[-G^{-1}(1-x)+p_2-s\big(\max\{x,\tilde{\alpha}_F(p_2,M)\}M\big)\big]^+\\ &=\mathsf{E}\Big[\big[-G^{-1}(1-x)+p_2-s\big(\max\{x,\tilde{\alpha}_F(p_2,M)\}M\big)\big]\cdot 1_{\{\tilde{\alpha}_F(p_2,M)\leq x\}}\Big], \end{aligned}$$

and

$$\big[-G^{-1}(1-x)+p_2-\mathsf{E}s\big(\max\{x,\alpha_N(p_2,)\}M\big)\big]^+=\mathsf{E}\big[-G^{-1}(1-x)+p_2-s(xM)\big].$$

Since

$$\begin{aligned} &\big[-G^{-1}(1-x)+p_2-s\big(\max\{x,\tilde{\alpha}_F(p_2,M)\}M\big)\big]\cdot 1_{\{\tilde{\alpha}_F(p_2,M)\leq x\}}\\ &\geq -G^{-1}(1-x)+p_2-s(xM),\ \forall M, \end{aligned}$$

it follows that $H_F(x)-H_N(x)\leq 0$. Hence we still have $\alpha_{1,F}(p_1,p_2)\leq\alpha_{1,N}(p_1,p_2)$.

(ii) If $p_1\geq p_2$, we have $\alpha_{1,N}(p_1,p_2)\leq\alpha_N(p_2)$. Thus we have

$$0\leq H_N(\alpha_{1,N})=G^{-1}(1-\alpha_{1,N})-p_1-\delta[G^{-1}(1-\alpha_{1,N})-p_2],$$

which yields $G^{-1}(1-\alpha_{1,N})\geq p_1\geq p_2$. Here, we write $\alpha_{1,N}(p_1,p_2)$ as $\alpha_{1,N}$ for short. Therefore, we conclude that $H_F(\alpha_{1,N})-H_N(\alpha_{1,N})=0$, which yields $\alpha_{1,F}(p_1,p_2)=\alpha_{1,N}(p_1,p_2)$. □

Proof of Proposition 2. (i) As alluded to in the proof of Proposition 1, $\mathsf{E}\tilde{d}_F(p_2,M)\geq d_N(p_2)$ if $p_2\geq v^0+s(\overline{m})$ and $\overline{m}\leq d^0$. Since $d_F(p_1,p_2,M)\geq\tilde{d}_F(p_2,M)$, we have that $\mathsf{E}d_F(p_1,p_2,M)\geq d_N(p_2)$.

Consider $p_1 \geq p_2$. In this case, we have $d_N(p_1,p_2) = d_N(p_2) \leq \mathsf{E}d_F(p_1,p_2,M)$. By Lemma 5, we also have $d_{1,F}(p_1,p_2) = d_{1,N}(p_1,p_2)$. Therefore, $R_F(p_1,p_2) \geq R_N(p_1,p_2)$.

Then consider $p_1 < p_2$. We also consider $\delta = 0$. In this case, we have $H_F(x) = H_N(x)$ for all $x \in [0,1]$, which yields $d_{1,F}(p_1,p_2) = d_{1,N}(p_1,p_2)$. Since we have assumed $\mathrm{Var}(s(M)) > 0$ at the beginning of this appendix, either of the following two results holds: (a) $\mathsf{E}d_{2,F}(p_1,p_2,M) > d_{2,N}(p_1,p_2)$, (b) $d_{2,N}(p_1,p_2) = d_{2,F}(p_1,p_2,\overline{m}) = 0$. Thus we must have either that $R_F(p_1,p_2) > R_N(p_1,p_2)$ for $\delta = 0$ or that $R_F(p_1,p_2) = R_N(p_1,p_2)$ for all δ. Therefore, there must be a threshold $\delta_c > 0$ such that $R_F(p_1,p_2) \geq R_N(p_1,p_2)$ when $\delta \leq \delta_c$.

(ii) If $p_2 \leq \min\{p_1, \underline{v} + \mathsf{E}s(M)\}$, then we have $d_{1,F}(p_1,p_2) = d_{1,N}(p_1,p_2)$ by Lemma 5 and $\mathsf{E}d_F(p_1,p_2,M) \leq d_N(p_1,p_2)$ by Lemma 4(ii). Therefore, $R_F(p_1,p_2) \leq R_N(p_1,p_2)$. □

Proof of Lemma 6. First we prove $\alpha^*_{2,N} > 0$. Suppose $\alpha^*_{2,N} = 0$. Thus $\alpha^*_{1,N} = \alpha^*_N$ must be the largest fixed point of the following equation:

$$\alpha = \bar{G}\big(p^*_{1,N} - \delta \mathsf{E}s(\alpha M)\big),$$

and hence $R_N(p^*_{1,N}, p^*_{2,N}) = p^*_{1,N}\alpha^*_N$. If we take $p_{2,N} = p^*_{1,N}$ and take $p_{1,N}$ such that $p_{1,N} > p^*_{2,N}$ and $\bar{G}(p_{1,N}) > 0$, then $R_N(p_{1,N}, p_{2,N}) > R_N(p^*_{1,N}, p^*_{2,N})$. That is a contradiction. Hence we have $\alpha^*_{2,N} > 0$.

Consider $\alpha_{2,N}(p_{1,N}, p_{2,N}) > 0$. Thus α_N is the largest fixed point of the following equation:

$$\alpha = \bar{G}\big(p_{2,N} - \mathsf{E}s(\alpha M)\big),$$

which implies that α_N depends only on $p_{2,N}$. Hence, $R_N(p_{1,N}, p_{2,N}) = p_{2,N}\alpha_N(p_{2,N}) + (p_{1,N} - p_{2,N})\alpha_{1,N}(p_{1,N}, p_{2,N})$. Since $R_N(p_{2,N}, p_{2,N}) > R_N(p_{1,N}, p_{2,N})$ if $p_{1,N} < p_{2,N}$, it follows that $p^*_{1,N} \geq p^*_{2,N}$. □

Proof of Proposition 3. (i) From Proposition 2(i) and Lemma 6, we see that, to prove $R^*_F \geq R_F(p^*_{1,N}, p^*_{2,N}) \geq R^*_N$, it suffices to prove $p^*_{2,N} \geq v^0 + s(\overline{m})$.

We first show $p^*_{1,N} \geq r$. Suppose $p^*_{1,N} < r$, which yields $p^*_{2,N} < r$. We have $R_N(r_1, r_1) \geq r_1 \cdot \bar{G}(r_1)\mathsf{E}M \geq r\mathsf{E}M > R_N(p^*_{1,N}, p^*_{2,N})$. That is a contradiction. Hence, $p^*_{1,N} \geq r$.

Now we prove $p^*_{2,N} \geq v^0 + s(\overline{m})$. Suppose $p^*_{2,N} < v^0 + s(\overline{m})$. Then we have that $\alpha_{1,N}(p^*_{1,N}, r_2) \geq \alpha_{1,N}(p^*_{1,N}, p^*_{2,N})$, which yields $R_{1,N}(p^*_{1,N}, r_2) \geq R^*_{1,N}$, and that $R_{2,N}(p^*_{1,N}, r_2) \geq r_2 \cdot \mathsf{P}\{r_2 \leq V \leq \frac{r - \delta r_2}{1-\delta}\} \geq (v^0 + s(\overline{m}))\mathsf{E}M > R^*_{2,N}$. That is a contradiction. Therefore, we must have $p^*_{2,N} \geq v^0 + s(\overline{m})$, which completes the proof.

(ii) Now we show that, for any $p_{1,F}$ and $p_{2,F}$, there always exist $p_{1,N}$ and $p_{2,N}$ such that $R_N(p_{1,N}, p_{2,N}) \geq R_F(p_{1,F}, p_{2,F})$. We first consider the case in which $p_{2,F} \leq \underline{v} + \mathsf{E}s(M)$. Take $p_{2,N} = p_{2,F}$ and $p_{1,N} = p_{1,F}$. Thus $d_N(p_{2,F}) = \mathsf{E}M \geq \mathsf{E}d_F(p_{1,F}, p_{2,F}, M)$. By Lemma 5(i) we have $R_N(p_{1,N}, p_{2,N}) \geq R_F(p_{1,F}, p_{2,F})$.

Then we consider $p_{2,F} = r(\underline{v} + \mathsf{E}s(M))$ where $r > 1$. Consider also $p_{1,F} > \underline{v} + \mathsf{E}s(M)$. Since $\mathsf{P}\{V \leq \beta \mathsf{E}M\} = 1$, it follows that 0 is the unique fixed point of (7). Hence, $\alpha_{1,F}(p_{1,F}, p_{2,F}) \leq \alpha_{1,N}(p_{1,F}, p_{2,F}) = 0$. The F setting boils down to the N setting, earning 0 profit. Next, we consider $p_{1,F} \leq \underline{v} + \mathsf{E}s(M)$.

If $s(M) < \min\{p^*_{2,F} - v^0, s(d^0)\}$, then we can see from Lemma 1 that $\tilde{\alpha}_F(p^*_{2,F}, M)$ is the unique fixed point of (4), and futhermore we have $\alpha_F(p^*_{2,F}, M) = 0$ because $\mathsf{P}\{V \leq \beta \mathsf{E}M\} = 1$. Thus we have

$$
\begin{aligned}
R_F(p_{1,F}, p_{2,F}) &= p_{1,F}\alpha_{1,F}\mathsf{E}M + r(\underline{v} + \mathsf{E}s(M))\mathsf{E}[\max\{0, \tilde{\alpha}_F(p_{2,F}, M) - \alpha_{1,F}\}M] \\
&\leq p_{1,F}\alpha_{1,F}\mathsf{E}M + r(\underline{v} + \mathsf{E}s(M))(1-\alpha_{1,F})\mathsf{E}[M \cdot 1_{\{s(M) \geq \min\{p_{2,F} - v^0, s(d^0)\}\}}] \\
&\leq (\underline{v} + \mathsf{E}s(M))\alpha_{1,F}\mathsf{E}M + (\underline{v} + \mathsf{E}s(M))(1-\alpha_{1,F})\mathsf{E}M \\
&= (\underline{v} + \mathsf{E}s(M))\mathsf{E}M \\
&= R_N(\underline{v} + \mathsf{E}s(M), \underline{v} + \mathsf{E}s(M)),
\end{aligned}
$$

where the last inequality uses the fact that $\mathsf{E}[M \cdot 1_{\{s(M) \geq \min\{p_{2,F} - v^0, s(d^0)\}\}}] \leq \frac{\mathsf{E}M}{r}$. Therefore, we conclude that $R^*_N \geq R^*_F$. □

Before proving Corollary 1, we first introduce some symbols and some lemmas (see below). Let $v_{1,N}(p_1, p_2)$ and $v_{2,N}(p_1, p_2)$ be the lowest valuations among consumers buying respectively in periods 1 and 2 in the N setting. If they are interior points of the support of V, then we must have

$$
\begin{aligned}
v_{1,N}(p_1, p_2) &= \frac{p_1 - \delta p_2}{1-\delta} = \frac{p_1 - p_2}{1-\delta} + p_2, \\
v_{2,N}(p_1, p_2) &= p_2 - \mathsf{E}s\big(\alpha_N(p_2)M\big)
\end{aligned}
$$

Under state-independent pricing, the firm's problem in the N setting is equivalent to

$$
\max_{p_1, p_2} R_N(p_1, p_2) = p_1 \bar{G}(v_{1,N})\mathsf{E}M + p_2\Big(G(v_{1,N}) - G(v_{2,N})\Big)\mathsf{E}M. \tag{A.6}
$$

Let $v^*_{1,N}$ and $v^*_{2,N}$ be values of $v_{1,N}(\cdot)$ and $v_{2,N}(\cdot)$ at $p^*_{1,N}$, $p^*_{2,N}$.

Proof of Corollary 1. By the FOCs of (A.6), we have

$$
\bar{G}(v^*_{1,N}) = (p^*_{1,N} - p^*_{2,N})g(v^*_{1,N})\frac{1}{1-\delta}.
$$

Since $g(\cdot)$ is nonincreasing on $(0,+\infty)$, it follows that $(v^*_{1,N} - v^*_{2,N})g(v^*_{1,N}) \le G(v^*_{1,N}) - G(v^*_{2,N})$. Since $v^*_{1,N} - v^*_{2,N} \ge \frac{p^*_{1,N} - p^*_{2,N}}{1-\delta}$, we have

$$\bar{G}(v^*_{1,N}) \le G(v^*_{1,N}) - G(v^*_{2,N}),$$

which yields $\alpha^*_{1,N} \le \alpha^*_{2,N}$ and $\alpha^*_{1,N} \le 1/2$.

To prove this corollary, it suffices to prove $p^*_{2,N} \ge v^0 + s(\overline{m})$. Suppose $p^*_{2,N} < v^0 + s(\overline{m})$. Then we must have $\frac{R^*_N}{\mathsf{E}M} < p^*_{1,N}\bar{G}(v^*_{1,N}) + (v^0 + s(\overline{m}))\cdot G(v^*_{1,N})$. Take $p_{2,N} = r(v^0 + s(\overline{m}))$ where $r > 1$ and take $p_{1,N}$ such that $p_{1,N} - p^*_{1,N} = \delta(r(v^0 + s(\overline{m})) - p^*_{2,N})$, which imples $v_{1,N}(p_{1,N}, p_{2,N}) = v^*_{1,N}$ if $v_{2,N} \le v^*_{1,N}$. Next we prove $R_N(p_{1,N}, p_{2,N}) > R^*_N$ for some $r > 1$.

If $v_{2,N} \le v^*_{1,N}$, then we have

$$\begin{aligned}
\frac{R_N(p_{1,N}, p_{2,N}) - R^*_N}{\mathsf{E}M} &> r(v^0 + s(\overline{m}))[G(v_{1,N}) - G(v_{2,N})] + p_{1,N}\bar{G}(v_{1,N}) - p^*_{2,N}G(v^*_{1,N}) - p^*_{1,N}\bar{G}(v^*_{1,N}) \\
&= r(v^0 + s(\overline{m}))[G(v_{1,N}) - G(v_{2,N})] - p^*_{2,N}G(v^*_{1,N}) + (p_{1,N} - p^*_{1,N})\bar{G}(v^*_{1,N}) \\
&\ge r(v^0 + s(\overline{m}))[G(v_{1,N}) - G(v_{2,N})] - (v^0 + s(\overline{m}))G(v^*_{1,N}) \\
&= (v^0 + s(\overline{m}))\big[(r-1)G(v^*_{1,N}) - rG(v_{2,N})\big] \\
&\ge (v^0 + s(\overline{m}))\big[\frac{(r-1)}{2} - rG(v_{2,N})\big].
\end{aligned}$$

where the last inequality uses the fact that $\alpha^*_{1,N} \le 1/2$. Since $v_{2,N} \le p_{2,N} = r(v^0 + s(\overline{m}))$, we have

$$\begin{aligned}
R_N(p_{1,N}, p_{2,N}) > R^*_N &\Leftarrow G(v_{2,N}) \le \min\{\frac{(r-1)}{2r}, G(v^*_{1,N})\} \\
&\Leftarrow G\big(r(v^0 + s(\overline{m}))\big) \le \frac{(r-1)}{2r} \\
&\Leftarrow \mathsf{P}\{V \ge r(v^0 + s(\overline{m}))\} \ge \frac{r+1}{2r}.
\end{aligned}$$

Therefore, if $\mathsf{P}\{V \ge r(v^0 + s(\overline{m}))\} \ge \frac{r+1}{2r}$ for some $r > 1$, then $R^*_F \ge R^*_N$. □

Proof of Proposition 4. Let $p^*_{1,N}$ and $p^*_{2,N}$ be the optimal prices in the N setting. And let $\alpha^*_{1,N}$ and $\alpha^*_{2,N}$ be the corresponding adoption fractions. Let $\alpha^*_N = \alpha^*_{1,N} + \alpha^*_{2,N}$. We have $p^*_{1,N} \ge p^*_{2,N}$. By the definition of $\alpha_{2,N}(\cdot)$, we have

$$p^*_{2,N} = G^{-1}(1 - \alpha^*_N) + \beta\alpha^*_N\mathsf{E}M.$$

where $G^{-1}(x) = \inf_{y\ge 0}\{y : G(y) \ge x\}$. We will prove that there exist $p_{1,F}$ and $\{p_{2,F}(M)\}$ such that $R_F(p_{1,F}, \boldsymbol{p_{2,F}}) \ge R^*_N$. Note that this proposition will hold if we prove this result.

Take $p_{1,F} = p^*_{1,N}$ and $p_{2,F}(M) = G^{-1}(1 - \alpha^*_N) + s(\alpha^*_N M)$. Thus at least α^*_N fraction of customers will buy the good in the F setting for any $M > 0$, since $p_{2,F}(M)$ and α^*_N satisfy the REE condition

(9), implying $\tilde{\alpha}_F(p_{2,F}(M),M) \geq \alpha_N^*$ and furthermore $\alpha_F(p_{1,F},\boldsymbol{p_{2,F}},M) \geq \alpha_N^*$ for any $M>0$ by Equation (10). Hence, $d_F(p_{1,F},\boldsymbol{p_{2,F}},M) \geq d_N^*$ for any $M>0$.

For any M, we have

$$\begin{aligned}
&G^{-1}(1-\alpha_{1,N}^*)-p_{2,F}(M)+s\Big(\max\{\alpha_{1,N}^*,\tilde{\alpha}(p_{2,F}(M),M)\}M\Big)\\
&=G^{-1}(1-\alpha_{1,N}^*)-p_{2,F}(M)+s(\alpha_N^*M)\\
&\geq G^{-1}(1-\alpha_N^*)-p_{2,F}(M)+s(\alpha_N^*M)\\
&=0.
\end{aligned}$$

Since $\mathsf{E}p_{2,F}(M)=p_{2,N}^*$, we can see from Equation (11) that $\alpha_{1,F}(p_{1,F},\boldsymbol{p_{2,F}})=\alpha_{1,N}^*$, which yields $d_{1,F}(p_{1,F},\boldsymbol{p_{2,F}}) \geq d_{1,N}^*$. We also have

$$\begin{aligned}
R_F(p_{1,F},\boldsymbol{p_{2,F}}) &= (p_{1,F}-\mathsf{E}p_{2,F}(M))d_{1,F}(p_{1,F},\boldsymbol{p_{2,F}})+\mathsf{E}[p_{2,F}(M)d_F(p_{1,F},\boldsymbol{p_{2,F}},M)]\\
&\geq (p_{1,N}^*-p_{2,N}^*)d_{1,N}^*+p_{2,N}^*d_N^*\\
&=R_N^*,
\end{aligned}$$

which completes the proof. □

Proof of Lemma 7. As in the proof of Proposition 4, we take $p_{2,F}(M)=G^{-1}(1-\alpha_N^*)+s(\alpha_N^*M)$ where $\alpha_N^* \geq \alpha_1$. Thus we have

$$R_{2,F}^*(\alpha_1,M)=\max_{p_2}\ p_2\alpha_{2,F}(\alpha_1,p_2,M)M \geq p_{2,F}(M)(\alpha_N^*-\alpha_1)M.$$

It follows that $\mathsf{E}R_{2,F}^*(\alpha_1,M) \geq \mathsf{E}\big[p_{2,F}(M)(\alpha_N^*-\alpha_1)M\big] \geq p_{2,N}^*(\alpha_N^*-\alpha_1)\mathsf{E}M=R_{2,N}^*(\alpha_1)$, where the last inequality holds by Lemma A.1. □

The proof of Proposition 5 makes use of the following lemma.

Lemma A.2. *Suppose* $\sup\left\{\frac{\bar{G}(x)}{g(x)}:g(x)>0\right\}$ *is finite. Then* R_N^* *is continuous in* $\delta\in[0,1)$*;* $\alpha_{1,N}^*$ *and* $R_{1,N}^*$ *are left-continuous in* $\delta\in(0,1)$*; furthermore,* $\alpha_{1,N}^*\to 0$ *and* $R_{1,N}^*\to 0$*, as* $\delta\to 1^-$.

Proof of Lemma A.2. Let $y_2(\alpha_{1,N},p_{2,N})=p_{2,N}[\alpha_N(p_{2,N})-\alpha_{1,N}]$. We note that the second-period problem in the N setting is equivalent to

$$\max\Big\{\max_{p_{2,N}}\ y_2(\alpha_{1,N},p_{2,N}),\mathsf{E}s(M)(1-\alpha_{1,N})\Big\}.$$

We can see that y_2^* is continuous in $\alpha_{1,N}$. By the submodularity of y_2 with respect to α_1 and $p_{2,N}$, we have $p_{2,N}^*(\alpha_{1,N})$ decreasing in α_1. We can also have $p_{2,N}^*$ right-contiguous in $\alpha_{1,N}$. Let

$$\hat{y}_2^*(\alpha_{1,N}) = p_{2,N}^*(\alpha_{1,N})\alpha_N^*(\alpha_{1,N}) = y_2^*(\alpha_{1,N}) + p_{2,N}^*(\alpha_{1,N})\alpha_{1,N}.$$

It can be seen that $\hat{y}_2^*$ is right-continuous in $\alpha_{1,N}$. By the Envelope theorem, we have

$$\frac{\partial \hat{y}_2^*}{\partial \alpha_{1,N}} = \frac{\partial y_2^*(\alpha_{1,N})}{\alpha_{1,N}} + p_{2,N}^*(\alpha_{1,N}) + \frac{\partial p_{2,N}^*(\alpha_{1,N})}{\partial \alpha_{1,N}}\alpha_{1,N} = \frac{\partial p_{2,N}^*(\alpha_{1,N})}{\partial \alpha_{1,N}}\alpha_{1,N} \leq 0.$$

Hence $\hat{y}_2^*$ is decreasing in $\alpha_{1,N}$.

Let $y_1(\delta, v_{1,N}) = \frac{R_N(\delta, v_{1,N})}{\mathsf{E}M}$. Since $v_{1,N} - p_{2,N} = \frac{p_{1,N} - p_{2,N}}{1-\delta}$ for $\delta < 1$, it follows that

$$\begin{aligned} y_1(\delta, v_{1,N}) &= [p_{1,N} - p_{2,N}^*(v_{1,N})]\bar{G}(v_{1,N}) + \hat{y}_2^*(v_{1,N}) \\ &= (1-\delta)(v_{1,N} - p_{2,N}^*(v_{1,N}))\bar{G}(v_{1,N}) + \hat{y}_2^*(v_{1,N}). \end{aligned}$$

We note that $\max_v v\bar{G}(v)$ is bounded if $\sup\left\{\frac{\bar{G}(x)}{g(x)} : g(x) > 0\right\}$ is finite. Thus, $(v_{1,N}^* - p_{2,N}^*(v_{1,N}^*))\bar{G}(v_{1,N}^*)$ and $\hat{y}_2^*(v_{1,N}^*)$ must be bounded for all $\delta < 1$. It follows that $(1-\delta)(v_{1,N}^* - p_{2,N}^*(v_{1,N}^*))\bar{G}(v_{1,N}^*)$ is continuous in $\delta \in (0,1)$. Since

$$\left|\max_v\ y_1(\delta_1, v) - \max_v\ y_1(\delta_2, v)\right| \leq \max_v \left|y_1(\delta_1, v) - y_1(\delta_2, v)\right|,$$

we can deduce that $y_1^*(\delta) = y_1(\delta, v_{1,N}^*)$ is continuous in $\delta \in [0,1)$, i.e., $R_N^*(\delta)$ is continuous in $\delta \in [0,1)$. Hence $\hat{y}_2^*(v_{1,N}^*)$ is continuous in $\delta \in [0,1)$. Since $\hat{y}_2^*(v_{1,N})$ is increasing and left-continuous in $v_{1,N}$, it follows that $v_{1,N}^*$ is left-continuous in $\delta \in (0,1)$, implying $\alpha_{1,N}^*$ is left-continuous in $\delta \in (0,1)$. We have shown $y_2^*(v_{1,N})$ is continuous in $v_{1,N}$. It follows that $y_2^*(v_{1,N}^*)$ is left-continuous in $\delta \in (0,1)$, i.e., $R_{2,N}^*(\delta)$ is left-continuous in $\delta \in (0,1)$, which yields that $R_{1,N}^*(\delta)$ is left-continuous in $\delta \in (0,1)$.

Consider $\delta \to 1^-$. By the monotonicity of $\hat{y}_2^*(v_{1,N})$, we must have $v_{1,N}^* \to \infty$ (or the supremum of the support of $g(\cdot)$), implying $\alpha_{1,N}^* \to 0$. We note that $p_{2,N}^*$ is bounded for all δ because $\max_v v\bar{G}(v)$ is bounded. Hence $y_1(1, v_{1,N}^*) = \hat{y}_2^*(v_{1,N}^*) = p_{2,N}^*(\alpha_{1,N}^* + \alpha_{2,N}^*) \to p_{2,N}^*\alpha_{2,N}^*$, therefore it follows that $R_{1,N}^* \to 0$. □

Proof of Proposition 5. It can be seen from Lemma A.2 that, for any sufficiently small $\epsilon > 0$, there exists $\delta_c < 1$ such that $\alpha_{1,N}^*(\delta) < \epsilon$ and $R_{1,N}^*(\delta) < \epsilon$ if $1 > \delta > \delta_c$. For any $\delta > \delta_c$ ($\delta < 1$), take $p_{1,F}(\delta)$ such that $\alpha_{1,F}^*(\delta) = \alpha_{1,N}^*(\delta)$ (> 0). In this case, the total profits in the F and N setting mainly come from the second-period profits. More precisely, we have

$$R_F(\delta, p_{1,F}(\delta)) - R_N^*(\delta) \geq -\epsilon + \mathsf{E}R_{2,F}^*(\alpha_{1,N}^*, M) - R_{2,N}^*(\alpha_{1,N}^*)$$

As alluded to in the proof of Lemma 7, $R^*_{2,F}(\alpha_1, M)$ is strictly convex in M, which yields that $\mathsf{E}R^*_{2,F}(\alpha_1, M) - R^*_{2,N}(\alpha_1) > 0$ for all α_1 as long as $\mathrm{Var}(M) > 0$. Since $\mathsf{E}R^*_{2,F}(\alpha_1, M) - R^*_{2,N}(\alpha_1) > 0$, it follows that there exists $\epsilon > 0$ such that $\epsilon < \mathsf{E}R^*_{2,F}(\alpha_1, M) - R^*_{2,N}(\alpha_1)$ for all $0 < \alpha_1 < \epsilon$. Therefore, there must be a $\delta_c < 1$ such that $R_F(\delta, p_{1,F}(\delta)) - R^*_N(\delta) \geq 0$ if $1 > \delta > \delta_c$. $\square$