Online Appendices to "Dynamic Type Matching"

A. Sketches of Proofs

In this online appendix, we present the sketches of proofs of the theorems, propositions, lemmas and corollaries, and leave the full-length proofs to the online supplements (available in an unabridged memo at https://ssrn.com/abstract=2592622).

Sketch of proof of Theorems 1 and 2. For any given state $(\mathbf{x}, \mathbf{y})$ in period t, we will construct an optimal matching decision that satisfies the desired properties for *all* pairs simultaneously comparable by the modified Monge condition. To that end, Let $\mathbf{Q}^{(0)}$ be a feasible matching decision in period t under the state $(\mathbf{x}, \mathbf{y})$. We will modify the matching decision successively, such that after each modification, (i) The matching decision remains feasible; (ii) The corresponding expected total discounted matching reward (until the end of the horizon) improves weakly after each modification.

We now describe the modification procedure. Suppose that we obtain a feasible matching decision $\mathbf{Q}^{(k)}$ after k modifications, with the corresponding post-matching levels given by $(\mathbf{u}^{(k)}, \mathbf{v}^{(k)})$. We consider the following two kinds of transfers of matching quantities.

The first kind of transfers move matching quantities from a weakly preceded pair (i', j) to the corresponding preceding pair (i, j), if $u_i^{(k)} > 0$ and $q_{i'j}^{(k)} > 0$. We construct the feasible matching decision $\mathbf{Q}^{(k+1)} := \mathbf{Q}^{(k)} + \delta^{(k)}\mathbf{e}_{ij} - \delta^{(k)}\mathbf{e}_{i'j}$, where $\delta^{(k)} := \min\left\{u_i^{(k)}, q_{i'j}^{(k)}\right\}$. In Lemma E.23 of Online Supplement E, we show that $\mathbf{Q}^{(k+1)}$ weakly outperforms $\mathbf{Q}^{(k)}$ with respect to the expected discounted reward. Likewise, if $v_j^{(k)} > 0$ and there exists a pair (i, j') weakly preceded by (i, j) such that $q_{ij'}^{(k)} > 0$, we construct $\mathbf{Q}^{(k+1)} := \mathbf{Q}^{(k)} + \delta^{(k)}\mathbf{e}_{ij} - \delta^{(k)}\mathbf{e}_{ij'}$, where $\delta^{(k)} := \min\left\{v_j^{(k)}, q_{ij'}^{(k)}\right\}$. And $\mathbf{Q}^{(k+1)}$ weakly outperforms $\mathbf{Q}^{(k)}$.

The second kind of transfers occur when there exists two pairs (i', j) and (i, j') weakly preceded by (i, j), such that $r_{ij}^t + r_{i'j'}^t \geq r_{i'j}^t + r_{ij'}^t$, $q_{ij'}^{(k)} > 0$ and $q_{i'j}^{(k)} > 0$. We construct the new feasible matching decision $\mathbf{Q}^{(k+1)} := \mathbf{Q}^{(k)} + \delta^{(k)}\mathbf{e}_{ij}^{m\times n} + \delta^{(k)}\mathbf{e}_{i'j'}^{m\times n} - \delta^{(k)}\mathbf{e}_{i'j}^{m\times n} - \delta^{(k)}\mathbf{e}_{ij'}^{m\times n}$, where $\delta^{(k)} := \min\left\{q_{i'j}^{(k)}, q_{ij'}^{(k)}\right\}$. It is easy to see that $\mathbf{Q}^{(k+1)}$ weakly outperforms $\mathbf{Q}^{(k)}$, since it leads to a weakly higher matching reward in period t (because $r_{ij}^t + r_{i'j'}^t \geq r_{i'j}^t + r_{ij'}^t$) than but the same post-matching levels as $\mathbf{Q}^{(k)}$.

It is easy to see that for both kinds of transfers the new matching decision $\mathbf{Q}^{(k+1)}$ is feasible. We let the modification procedure repeatedly apply the first kind of transfers as long as possible for the proof of Theorem 1, and let it repeatedly apply any of the two kinds of transfers as long as possible for the proof of Theorem 2. We observe that for either kind of transfers in the procedure, only the preceded pairs lose matching quantity. In other words, the transfer is "unidirectional". (For a rigorous argument, please refer to the proof of Theorems 1 and 2 in Online Supplement E.) Based on this observation, we can show that the procedure either stops in finite steps or yields a limit point. In either case, we reach a matching decision (i.e., the decision after the last modification in the former case, or the limit point in the latter case) denoted by $\mathbf{Q}^{(\infty)}$, which is feasible (since all $\mathbf{Q}^{(k)}$s are feasible). Moreover, $\mathbf{Q}^{(\infty)}$ satisfies the properties in Theorems 1 and/or 2. This is because in the former case (i.e., the procedure stops in finite steps), it is impossible to further transfer quantities; in the latter case, it is almost impossible to do so (the quantity that can be transferred is increasingly small) for sufficiently large k. (See the detailed argument in Online Supplement E.) Thus $\mathbf{Q}^{(\infty)}$ is weakly better than the original decision $\mathbf{Q}^{(0)}$ and has the desired properties. $\square$

Sketch of Proof of Theorem 3. We focus on the optimal policy that satisfies the properties in Theorems 1 and 2, and prove the theorem by induction. The conclusion clearly holds for the last period. Suppose it is true for period $t+1$. For period t, let us suppose to the contrary that the optimal policy does not greedily match a perfect pair (i,j). By the properties in Theorem 2, the post-matching levels u_i and v_j (for i and j, respectively) are both positive. If we increase the matching quantity in period t between i and j by the amount $\varepsilon := \min\{u_i, v_j\}$, the matching reward in period t will increase by $r_{ij}^t\varepsilon$ and also reduce the available type i demand (resp., type j supply) by $\alpha\varepsilon$ (resp., $\beta\varepsilon$) in the beginning of period $t+1$. Suppose without loss of generality that $\alpha \le \beta$. Based on induction hypothesis (i.e., type i demand and type j supply are matched greedily in period $t+1$), we can show that the reduction in expected reward from period $t+1$ to period T is at most $\beta\varepsilon r_{ij}^{t+1}$ (for a rigorous argument, please see the proof of Theorem 3 in Online Supplement E). Therefore, by increasing the matching quantity in period t between i and j by ε, the increase in expected total discounted reward from period t to period T is at least $r_{ij}^t\varepsilon - \gamma\beta r_{ij}^{t+1}\varepsilon \ge 0$. Thus, it does not hurt the optimality if we match i and j greedily in period t (nor does it change the properties in Theorems 1 and 2). The induction is completed. $\square$

Sketch of Proof of Proposition 1. We focus on matching between the imperfect pair. We also focus on the model with continuous-valued states and decisions, and leave the discrete-valued case to the proof of Proposition 2. In Online Supplement A, we define the transformed state $\mathbf{z} = (z_1, z_2) := (x_1 - y_1, y_2 - x_2)$. The matching quantity between an imperfect pair can be positive only when $z_1 z_1 > 0$. We focus on the case with $z_1 > 0$ and $z_2 > 0$ (the case with $z_1, z_2 < 0$ is symmetric). We use the post-matching level $p_s := z_2 - q$ of type 2 supply (instead of the matching quantity q between type 1 demand and type 2 supply) as the decision variable in period t. Then, the expected total discounted reward from period t to T is $r_{12}^t(z_2 - p_s) + \gamma E V_{t+1}(\alpha(p_s + z_1 - z_2) + D_1^{t+1}, D_2^{t+1}, S_1^{t+1}, \beta p_s + S_2^{t+1})$. Noting that V_{t+1} is convex and that p_s must be at least $(z_2 - z_1)^+ = IB^-$, the match-down-to threshold on type 2 supply is given by $p_{s_2}^t(IB) \in \arg\max_{p_s \ge IB^-} -r_{12}^t p_s + \gamma E V_{t+1}(\alpha(p_s + IB) + D_1^{t+1}, D_2^{t+1}, S_1^{t+1}, \beta p_s + S_2^{t+1})$. $\square$

Sketch of Proof of Proposition 2. In Online Supplement A, we write the expected discounted reward for matching the imperfect pair by q units under the transformed state $\mathbf{z}$ in period t (and using the optimal policy thereafter) as a function $J_t(q,\mathbf{z})$ of q and $\mathbf{z}$ (see its expression in (A.3)).

When $\alpha=\beta$, we further define $\tilde{J}_t(q,\mathbf{z}) := -r_{11}^t z_1^+ - r_{22}^t z_2^+ + J_t(q,\mathbf{z})$. In Lemma A.4, we show that $\tilde{J}_t(q,\mathbf{z})$ is $L^\natural$-concave in (q,z), which implies that the optimal matching quantity between the imperfect pair is increasing in $\mathbf{z}$ at a rate less than or equal to 1. One can also use the post-matching level p_s for the supply type (instead of q) to describe the matching problem in period t. For example, when $z_1,z_2>0$ (the case with $z_1,z_2<0$ is symmetric), we can write $\tilde{J}_t(q,\mathbf{z}) = \check{J}_t(p_s,IB) - (r_{22}^t - r_{12}^t + \gamma\alpha r_{11}^{t+1})z_2 - (r_{11}^t - \gamma\alpha r_{11}^{t+1})z_1$. Define $p_{s_2}^t(IB) := \arg\max_{p_s\ge IB^-} \check{J}_t(p_s,IB)$ and we can show that the optimal matching quantity $q_{12}^* = [z_2 - p_{s_2}^t(IB)]^+$ (this proves Proposition 1 for the discrete-valued model with $\alpha=\beta$; see Online Supplement A.1 for more details). Since $IB = z_1 - z_2$ and q_{12}^* increases in $\mathbf{z}$ (at a rate at most 1), we can show that $p_t^{s_2}(IB)$ decreases in IB at a rate no faster than 1 (see the proof of Proposition 2 in Online Supplement E for details).

When $\alpha=0$, for $z_1,z_2>0$ (the case with $z_1,z_2<0$ is again symmetric), the expected total reward for matching type 1 demand and type 2 supply until the latter reduces to p_s is $r_{12}^t(z_2 - p_s) + \gamma EV_{t+1}(D_1^{t+1}, D_2^{t+1}, S_1^{t+1}, \beta p_s + S_2^{t+1})$. We can show that $EV_{t+1}(D_1^{t+1}, D_2^{t+1}, S_1^{t+1}, \beta p_s + S_2^{t+1})$ is concave in p_s, and thus it is optimal to match down to the state-independent threshold level $\bar{p}_{s_2}^t := \arg\max_{p_s\ge 0} -r_{12}^t p_s + \gamma EV_{t+1}(D_1^{t+1}, D_2^{t+1}, S_1^{t+1}, \beta p_s + S_2^{t+1})$ or as much as possible (see the details in the proof of Lemma A.5 in Online Supplement A.1). This also proves Proposition 1 for the discrete-valued model with $\alpha=0$. □

Sketch of Proof of Lemma 1. To show that (i,j) weakly precedes procedes (i',j) if and only if j is closer to i than to i', along the direction of $\mathcal{L}$, we need to show that the inequality condition in Definition 2 is satisfied by (i,j) and (i',j) if and only if j is closer to i than to i', along the direction of $\mathcal{L}$. To show that the strong modified Monge condition is satisfied, we need to verify the condition in Assumption 1. Noting that $\text{dist}_{l_3\leftarrow l_1} = \text{dist}_{l_3\leftarrow l_2} + \text{dist}_{l_2\leftarrow l_1}$ for any three subsequent locations l_1, l_2 and l_3 along the direction of $\mathcal{L}$, it is relatively straightforward to verify the aforementioned conditions. We leave the details to the proof of Lemma 1 in Online Supplement E. □

Sketch of Proof of Proposition 3. The proof follows directly from Special Case 1. □

Sketch of Proof of Corollary 1. The proof follows directly from Theorem 3. □

Sketch of Proof of Proposition 4. The proof follows directly from Special Case 2. □

Sketch of Proof of Lemma 2. A total matching quantity $\bar{Q}$ in period t implies that the quantity of demand fulfilled and that of supply used are both equal to $\bar{Q}$. Under top-down matching, demand and supply types with smaller indices are matched first. Therefore, after fulfilling demand types $1,\ldots,i-1$, the remaining quantity to be matched is $(\bar{Q}-\sum_{i'=1}^{i-1}x_{i'})^+$. Out of this remaining quantity, the amount $\min\left\{(\bar{Q}-\sum_{i'=1}^{i-1}x_{i'})^+,x_i\right\}$ of type i demand will be matched, which yields the reward $r_{id}^t\min\left\{(\bar{Q}-\sum_{i'=1}^{i-1}x_{i'})^+,x_i\right\}$. Similarly, the reward from type j supply is $r_{js}^t\min\left\{(\bar{Q}-\sum_{j'=1}^{j-1}y_{j'})^+,y_j\right\}$. Noting that the post-matching level for type i demand (resp., type j supply) is $u_i=[x_i-(Q-\sum_{i'=1}^{i-1}x_{i'})^+]^+$ (reps. $v_j=[y_j-(Q-\sum_{j'=1}^{j-1}y_{j'})^+]^+$), we can conclude that the optimal expected total discounted reward is given by (3).

To show that $G_t(\bar{Q},\mathbf{x},\mathbf{y})$ is concave in $\bar{Q}$, we note that $EV_{t+1}(\alpha\mathbf{u}+\mathbf{D}^{t+1},\beta\mathbf{v}+\mathbf{S}^{t+1})$ is concave in $(\mathbf{u},\mathbf{v})$ (it is obvious for the continuous-valued model; for the discrete-valued model, it follows from the $L^\natural$-concavity of V_{t+1} proved in Lemma B.7 of Online Supplement B). Thus, $G_t(Q,\mathbf{x},\mathbf{y})$ is concave in Q within the interior of the ranges $\tilde{x}_{i-1}\le Q<\tilde{x}_i$ and $\tilde{y}_{j-1}\le Q<\tilde{y}_j$. It remains to show that $G_t(Q,\mathbf{x},\mathbf{y})$ is concave near any breakpoints. For example, to show that G_t is concave in the neighborhood of a breakpoint $a=\tilde{x}_i$, we prove $G_t(a+\epsilon,\mathbf{x},\mathbf{y})-G_t(a,\mathbf{x},\mathbf{y})\le G_t(a,\mathbf{x},\mathbf{y})-G_t(a-\epsilon,\mathbf{x},\mathbf{y})$ for sufficiently small $\varepsilon>0$. We leave the details of this inequality to the full proof. □

Sketch of Proof of Proposition 5. In Online Supplement B, we define $\tilde{x}_i=\sum_{k=1}^{i}x_k$ $(i=1,\ldots,m)$ and $\tilde{y}_j=\sum_{k=1}^{j}y_k$ $(j=1,\ldots,n)$ as the transformed state. In vector form, we have $\mathbf{x}\mathbf{U}_m=\tilde{\mathbf{x}}$ and $\mathbf{y}\mathbf{U}_n=\tilde{\mathbf{y}}$, where $\mathbf{U}_k$ is the $k\times k$ upper triangular matrix with all the entries on or above the main diagonal equal to 1. We also define $\tilde{G}_t(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}}):=-\tilde{\mathbf{x}}\mathbf{U}_m^{-1}(\mathbf{r}_d^t)^\intercal-\tilde{\mathbf{y}}\mathbf{U}_n^{-1}(\mathbf{r}_s^t)^\intercal+G_t(Q,\tilde{\mathbf{x}}\mathbf{U}_m^{-1},\tilde{\mathbf{y}}\mathbf{U}_n^{-1})$, with G_t defined as in the lemma. To maximize $G_t(Q,\mathbf{x},\mathbf{y})$ with respect to Q, it is equivalent to maximize $\tilde{G}_t(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. In Lemma B.7 of Online Supplement B, we show that $\tilde{G}_t(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is $L^\natural$-concave in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ for the two cases (i.e., $\alpha=\beta$ and $\alpha=0$), which implies that the optimal total matching quantity $\bar{Q}_t^*$ is increasing in $\tilde{x}_i$ and $\tilde{y}_j$ (for $i=1,\ldots,m$, $j=1,\ldots,n$), with the rate of increase smaller than or equal to 1. Since $\tilde{x}_i$ and $\tilde{y}_j$ are increasing in x_i and y_j, respectively, with rate 1, $\bar{Q}_t^*$ satisfies the properties in part (i). To prove the properties in part (ii), we note that increasing x_i by $\varepsilon>0$ will increase $\tilde{x}_i,\ldots,\tilde{x}_m$ by ε (but will not change the first $i-1$ entries $\tilde{\mathbf{x}}_{[1,i-1]}:=(\tilde{x}_1,\ldots,\tilde{x}_{i-1})$), and increasing x_{i+1} by ε will increase $\tilde{x}_{i+1},\ldots,\tilde{x}_m$ by ε (but will not change $\tilde{\mathbf{x}}_{[1,i]}$). Since $\bar{Q}_t^*$ increases in $\tilde{\mathbf{x}}$, we see that increasing x_i leads to a weakly higher increase in $\bar{Q}_t^*$ than increasing x_{i+1} by the same amount. Likewise, increasing y_j leads to a weakly higher increase in $\bar{Q}_t^*$ than increasing y_{j+1} by the same amount. □

Sketch of Proof of Proposition 6. We first show that the 1-step-lookahead heuristic follows the top-down matching procedure. To that end, we need to show that the policy does not match any demand type i' (resp., supply type j') unless all type i demand such that $i<i'$ (resp., all type j supply such that $j'<j$) are fully matched in a period t. Suppose otherwise. Then, without loss of generality, we assume that there exists $i<i'$ such that the post-matching level $u_i>0$

and the total matched quantity of type i' demand (denoted by $q_{i'\cdot}$) in period t is positive. We will modify the matching decision in period t to comply with top-down matching structure while weakly improving the expected discounted matching reward (assuming greedy matching from the next period). To that end, we can redirect a total matched quantity of $\varepsilon := \min\{u_i, q^t_{i'\cdot}\}$ of type i' demand (with possibly multiple supply types) to type i demand. This increases the reward in period t by $r^t_{id}\varepsilon - r^t_{i'd}\varepsilon$, and changes the expected future discounted reward by $\gamma EV^g_{t+1}(\alpha\mathbf{u} + \mathbf{D}^{t+1} - \alpha\varepsilon\mathbf{e}_i + \alpha\varepsilon\mathbf{e}_{i'}, \beta\mathbf{v} + \mathbf{S}^{t+1}) - \gamma EV^g_{t+1}(\alpha\mathbf{u} + \mathbf{D}^{t+1}, \beta\mathbf{v} + \mathbf{S}^{t+1})$. With the help of Lemma B.9 in Online Supplement B.1, we can show that this difference is at least $-\gamma\alpha(r^{t+1}_{id} - r^{t+1}_{i'd})\varepsilon$. Thus, the modification weakly improves expected discounted reward. By repeatedly performing similar modifications, we will arrive at a policy that complies with the top-down structure.

Next, we show that the 1-step-lookahead heuristic performs weakly better than the greedy matching policy. Let $P^{\mathrm{OSA}[1,t],\mathrm{Greedy}[t+1,T]}$ be the policy that applies the 1-step-lookahead policy up to period t, and uses greedy matching from period $t+1$ to period T. The two policies, $P^{\mathrm{OSA}[1,t],\mathrm{Greedy}[t+1,T]}$ and $P^{\mathrm{OSA}[1,t-1],\mathrm{Greedy}[t,T]}$, coincide with each other in periods $1,\dots,t-1$, and therefore have the same rewards in those periods. The former achieves a higher expected discounted reward from period t to T than the latter. This is because the latter always performs greedy matching while the former improves upon greedy matching in period t (to maximize the expected discounted reward from period t to T). Noting that the 1-step-lookahead policy coincides with $P^{\mathrm{OSA}[1,T-1],\mathrm{Greedy}[T,T]}$ and the greedy matching policy coincides with $P^{\mathrm{OSA}[1,0],\mathrm{Greedy}[1,T]}$, we can obtain the desired result. □

Sketch of Proof of Proposition 7. In Online Supplement B.1, given the transformed state $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ in period t, we define $G^g_t(Q, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ as the expected total discounted reward from period t to period T, if we follow the top-down matching up to the total quantity Q in period t and enforce greedy matching thereafter (see (B.17) in Online Supplement B.1). We also define $\tilde{G}^g_t(Q, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) := -\tilde{\mathbf{x}}\mathbf{U}_m^{-1}(\mathbf{r}^t_d)^\intercal - \tilde{\mathbf{y}}\mathbf{U}_n^{-1}(\mathbf{r}^t_s)^\intercal + G^g_t(Q, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ in Online Supplement B.1. To determine the total matching quantity in period t for the 1-step-lookahead heuristic, we solve $\max_{0\le Q\le \min\{\tilde{x}_m, \tilde{y}_n\}} \tilde{G}^g_t(Q, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$. Since the 1-step-lookahead policy follows the top-down structure, within the range $\max\{\tilde{x}_{i-1}, \tilde{y}_{j-1}\} \le Q \le \min\{\tilde{x}_i, \tilde{y}_j\}$ it matches type i demand with type j. In the proof of Proposition 7 in Online Supplement E, we show that within this range, there exists a function $\tilde{G}^{g,a}_{ij,t}(Q, \tilde{x}_m, \tilde{y}_n)$ (which depends on $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ only through $\tilde{x}_m$ and $\tilde{y}_n$, respectively) such that $\frac{\partial \tilde{G}^{g,a}_{ij,t}(Q,\tilde{x}_m,\tilde{y}_n)}{\partial Q} = \frac{\partial \tilde{G}^g_t(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})}{\partial Q}$, and therefore it is equivalent to maximize $\tilde{G}^{g,a}_{ij,t}(Q, \tilde{x}_m, \tilde{y}_n)$ within the same range. We then further substitute $p := \tilde{y}_n - Q$, which represent the total supply level after matching type i demand with type j supply in period t. We can then rewrite $\tilde{G}^{g,a}_{ij,t}(Q, \tilde{x}_m, \tilde{y}_n)$ as a function $\tilde{G}^{g,b}_{ij,t}(p, IB)$ of p and the total imbalance $IB := \tilde{x}_m - \tilde{y}_n$ in period t. Noting that p cannot go below IB^-, subject to availability of total demand, we define $p^t_{s_{ij}}(IB) := \inf \arg\max_{p \ge IB^-} \tilde{G}^{g,b}_{ij,t}(p, IB)$. Then, assuming greedy matching from period $t+1$, it is optimal to match type i demand and type j to reduce total supply to the threshold $p^t_{s_{ij}}(IB)$ or as much as possible in the range $\tilde{v}^{ij}_{n,L} \le p \le \tilde{v}^{ij}_{n,U}$ (where $\tilde{v}^{ij}_{n,U}$ and $\tilde{v}^{ij}_{n,L}$ represent the available supply when we start to match (i,j) and when we were to match (i,j) to the maximum possible extent, respectively). $\square$

B. The top-down matching procedure with protection levels for the vertical model

We now describe the top-down matching procedure with respect to the protection levels $p^t_{s_{ij}}(IB)$ (on the total supply, for all i, j and t), which is optimal (Proposition 7) when $\alpha = \beta$.

Algorithm B.3 The top-down matching procedure with the protection levels $p^t_{s_{ij}}(IB)$ (for $i = 1, \ldots, m$, $j = 1, \ldots, n$, $t = 1, \ldots, T$) in period t, given the state $(\mathbf{x}, \mathbf{y})$

$i \leftarrow 1$, $j \leftarrow 1$, total imbalance $IB \leftarrow \sum_{k=1}^{m} x_k - \sum_{k=1}^{n} y_k$, total matched quantity $Q \leftarrow 0$
for $i = 1, \ldots, m$, $j = 1, \ldots, n$ **do**
    Remaining type i demand, $u_i \leftarrow x_i$, remaining type j supply, $v_i \leftarrow y_j$
end for
Current available supply (of all types), $\tilde{v}_{n,U} \leftarrow \sum_{k=1}^{n} y_k$
while $i \le m$ and $j \le n$ **do**
    $\tilde{v}_{n,L} \leftarrow \tilde{v}_{n,U} - \min\{u_i, v_j\}$, where $\tilde{v}_{n,L}$ would be the remaining available supply (of all types) if (i,j) is matched greedily
    $q_{ij} \leftarrow \min\left\{\left[\tilde{v}_{n,U} - p^t_{s_{ij}}(IB)\right]^+, \tilde{v}_{n,U} - \tilde{v}_{n,L}\right\}$, match i with j for the quantity q_{ij}
    $Q \leftarrow Q + q_{ij}$, $u_i \leftarrow u_i - q_{ij}$, $v_j \leftarrow v_j - q_{ij}$, $\tilde{v}_{n,U} \leftarrow \tilde{v}_{n,U} - q_{ij}$
    if type i demand runs out **then**
        $i \leftarrow i + 1$
    else if type j supply runs out **then**
        $j \leftarrow j + 1$
    else
        Break
    end if
end while

C. List of notation

In Table C.1, we summarize the notation used in the paper.

Table C.1: List of notation

$\mathbf{1}^k$ (or $\mathbf{1}$)	:=	The k-dimensional row vector of ones (the superscript k may be omitted if the size of the vector can be inferred from the context)
α, β	:=	Fractions of unmatched demand and unmatched supply in a period that will carry-over to the next period, respectively
γ	:=	The discount factor
$\text{dist}_{i\leftarrow j}$	:=	The unidirectional distance from the location of supply type j to the location of type i demand in the horizontal model
D_i^t	:=	The random quantity of type i demand to arrive in period t
$\mathbf{D}^t$	:=	$(D_1^t, \ldots, D_m^t)$
$\mathbf{e}_\ell^k$ (or $\mathbf{e}_\ell$)	:=	The k-dimensional row vector with the ℓth entry equal to 1 and all other entries equal to 0 (the superscript k may be omitted if the size of the vector ca be inferred from the context)
$\mathbf{e}_{ij}^{m\times n}$ (or $\mathbf{e}_{ij}$)	:=	The $m \times n$ matrix with the (i, j)th entry equal to 1 and all other entries equal to 0 (the superscript $m \times n$ may be omitted if it can be inferred from the context)
$G_t(\bar{Q}, \mathbf{x}, \mathbf{y})$	:=	The maximum expected total discounted reward from period t to period T, by matching up to the total quantity $\bar{Q}$ following the top-down matching procedure, given the state $(\mathbf{x}, \mathbf{y})$ in the beginning of period t
$H_t(\mathbf{Q}, \mathbf{x}, \mathbf{y})$	:=	The maximum expected total discounted reward for applying the matching decision $\mathbf{Q}$ in period t, under the state $(\mathbf{x}, \mathbf{y})$ in the beginning of period t
i	:=	Demand type index
(i, j)	:=	The demand-supply pair consist of demand type i and supply type j
IB	:=	Total imbalance between demand and supply (i.e., total demand quantity less total supply quantity) in a period
IB_{ij}	:=	$\breve{x}_i - \breve{y}_j$, the imbalance between type i demand and type j supply immediately before we match the two types; used in the match-down-to heuristic for the horizontal model
j	:=	Supply type index
$\mathcal{L}$	:=	The line segment on which demand and supply types are located, in the horizontal model
m	:=	Number of demand types
n	:=	Number of supply types
$p_{s_j}^t(IB)$	:=	Match-down-to threshold level define on type j $(j = 1, 2)$ supply in period t, in the 2×2 horizontal model
$p_{s_{ij}^t(IB)}$	:=	Protection level used by the 1-step-lookahead heuristic for matching i with j in period t
q_{ij}	:=	Matching quantity between type i demand and type j supply in a period
$\mathbf{Q}$	:=	The matrix of matching quantities $(q_{ij})_{i=1,\ldots,m,j=1,\ldots,n}$
q_{ij}^{t*}	:=	The optimal matching quantity between type i demand and type j supply in period t

$\mathbf{Q}^{t*}$	:=	$(q_{ij}^{t*})_{i=1,\ldots,m,j=1,\ldots,n}$, the matrix of optimal matching quantities in period t
$\bar{Q}^{t*}(\mathbf{x},\mathbf{y})$	:=	The optimal total matching quantity in period t, given the state $(\mathbf{x},\mathbf{y})$
r_{ij}^t	:=	Unit matching reward for matching type i demand with type j supply in period t, and $r_{ij}^{T+1} \equiv 0$ for all i,j
r_{id}^t, r_{js}^t	:=	The rewards for matching one unit of type i demand and one unit of type j supply, respectively, in the vertical model
$\mathbf{R}^t$	:=	$\mathbf{R}^t$, the matrix of unit matching rewards in period t
R_i^t	:=	The baseline unit reward for type i demand in the horizontal model
S_j^t	:=	The random quantity of type j supply to arrive in period t
$\mathbf{S}^t$	:=	$(S_1^t,\ldots,S_m^t)$
T	:=	Total number of periods
u_i	:=	Post-matching level (available quantity after matching) of type i demand at the end of a period
$\mathbf{U}_k$	:=	the $k\times k$ upper triangular matrix with all the entries on or above the main diagonal equal to one
$\mathbf{u}$	:=	$(u_1,\ldots,u_m)$
v_j	:=	Post-matching level (available quantity after matching) of type j supply at the end of a period
$\mathbf{v}$	:=	$(v_1,\ldots,v_n)$
$\tilde{v}_{m,U}^{ij}$	:=	The available supply quantity of all types immediately before we match the pair (i,j) under the top-down matching procedure
$\tilde{v}_{m,L}^{ij}$	:=	The remaining supply quantity of all types when we match the pair (i,j) to the maximum extent the top-down matching procedure
$V_t(\mathbf{x},\mathbf{y})$	:=	Optimal expected total discounted reward from period t to period T, under the state $(\mathbf{x},\mathbf{y})$ in the beginning of period t
$V_t^g(\mathbf{x},\mathbf{y})$	:=	The expected total discounted reward from period t to period T, given that the state $(\mathbf{x},\mathbf{y})$ in the beginning of period t and greedy matching is enforced from period t to period T
x_i	:=	Available quantity of type i demand in the beginning of a period
$\mathbf{x}$	:=	$(x_1,\ldots,x_m)$
$\tilde{x}_i$	:=	$\sum_{k=1}^{i} x_k$, for $i=1,\ldots,m$
$\tilde{\mathbf{x}}$	:=	$(\tilde{x}_1,\ldots,\tilde{x}_m)$
$\breve{x}_i$	:=	The quantity of available type i demand immediately prior to matching type i demand with type j supply, under the match-down-to heuristic for the horizontal model
y_j	:=	Available quantity of type j supply in the beginning of a period
$\mathbf{y}$	:=	$(y_1,\ldots,y_n)$
$\tilde{y}_j$	:=	$\sum_{k=1}^{j} y_k$, for $j=1,\ldots,n$
$\tilde{\mathbf{y}}$	:=	$(\tilde{y}_1,\ldots,\tilde{y}_n)$
$\breve{y}_j$	:=	The quantity of available type j supply immediately prior to matching type i demand with type j supply, under the match-down-to heuristic for the horizontal model
$\mathbf{z}$	:=	$(z_1,z_2)=(x_1-y_1,y_2-x_2)$, the transformed state in the 2×2 horizontal model