Online Appendix to "Modified Echelon (r, Q) Policies with Guaranteed Performance Bounds for Stochastic Serial Inventory Systems"

A. The Multiple-Stage Model

In this appendix, we extend the analytic results of the two-stage system in Section 3 to the multiple-stage setting with more than two stages. We strive to keep the same notation and extend it naturally to the multiple-stage cases. We first provide an upper bound on the systemwide cost of system $(\mathscr{B})$ under any given modified echelon $(\mathbf{r}, \mathbf{Q})$ policy, and then evaluate the performance of a specific modified echelon $(\mathbf{r}, \mathbf{Q})$ policy.

A.1. An Upper Bound

We assume that the original system operates under a given modified echelon $(\mathbf{r}, \mathbf{Q})$ policy, where $\mathbf{r} = (r_1, \ldots, r_N)$ and $\mathbf{Q} = (Q_1, \ldots, Q_N)$. Namely, Stage i, $i = 1, 2, \ldots, N$, implements the modified echelon (r_i, Q_i) policy (see Definition 1). We denote by $C(\mathbf{r}, \mathbf{Q})$ the long-run average system-wide cost of system $(\mathscr{B})$ under the given modified echelon $(\mathbf{r}, \mathbf{Q})$ policy. Clearly, $C^*_{\mathscr{B}} \le C(\mathbf{r}, \mathbf{Q})$. Again, we use the same cost allocation as the one adopted in the two-stage model (see Definition 2).

A.1.1. Cycles and Regular and Irregular Shipment Periods. The approach of constructing an upper bound is similar to the two-stage case.

Definition A1 (Cycle). For any Stage $i = 1, 2, \ldots, N$, we call $[T_i^j, T_i^{j+1})$, for any $j \in \mathbb{N}$, the j-th cycle, where T_i^j is the time epoch of the 1-st unit, contained in the j-th order of Stage $i+1$, being sent to Stage i.

For $i = 1, 2, \ldots, N$, let M_i^j denote the number of shipments from Stage $i+1$ to Stage i over $[T_i^j, T_i^{j+1})$. Let $T_i^{j,l}$ be the time of the l-th shipment over $[T_i^j, T_i^{j+1})$, where $l = 1, 2, \ldots, M_i^j$. By definition, $T_i^{j,1} \equiv T_i^j$ and we have $[T_i^j, T_i^{j+1}) = \bigcup_{l=1}^{M_i^j} [T_i^{j,l}, T_i^{j,l+1})$, where $T_i^{j, M_i^j+1} \equiv T_i^{j+1}$.

We call $[T_i^{j,l}, T_i^{j,l+1})$ the l-th *shipment period* over the cycle $[T_i^j, T_i^{j+1})$. Depending on the inventory positions at the beginning and the end of a shipment period, we categorize shipment periods of Stage i into the following two types.

Definition A2 (Regular and Irregular Shipment Period). For any shipment period $[T_i^{j,l}, T_i^{j,l+1})$, $l = 1, \ldots, M_i^j$, if $IP_i(T_i^{j,l}) = r_i + Q_i$ and $IP_i^-(T_i^{j,l+1}) = r_i$, we call it a *regular* shipment period; otherwise, we call it an *irregular* shipment period.

Observation A5. *For any stage, there exists at most one irregular shipment period in any cycle.*

A.1.2. Cost Assessment in Cycles. We denote by $\Phi_i(IP_i(t))$ the total expected cost rate from Stage i to Stage 1 at time t, *excluding* the setup costs incurred at Stage i when the inventory position of Stage i at time t is $IP_i(t)$. By definition, we know that $\Phi_1(IP_1(t)) = G_1(IP_1(t))$. Denote by $\Psi_i(r_i, Q_i)$ the expected cost rate incurred at Stage i and its successors over the regular shipment period of Stage i.

OBSERVATION A6 (COST IN REGULAR AND IRREGULAR SHIPMENT PERIOD). (i) *The expected cost rate incurred at Stage i and its successors over the regular shipment period of Stage i is*

$$\Psi_i(r_i, Q_i) = \frac{1}{Q_i}\left[\lambda K_i + \int_{r_i}^{r_i+Q_i} \Phi_i(y)dy\right]. \tag{A1}$$

(ii) *For any time t in a non-empty irregular shipment period, the expected cost incurred at Stage i and its successors, excluding the setup costs incurred at Stage i, accrues at a rate equal to* $\Phi_i(IP_i(t))$. *There is exactly one fixed setup cost,* K_i*, incurred for the entire irregular shipment period.*

A.1.3. Cost Upper Bound. Observation A5 implies that the setup cost associated with an irregular shipment period is incurred at most once in one cycle;that leads to the following observation.

OBSERVATION A7. *For any time* $t \in [T_i^j, T_i^{j+1}) \neq \emptyset$ *and Stage i, setup costs for irregular shipment periods accrue at a rate that is no more than* $K_i/(T_i^{j+1} - T_i^j)$.

Define

$$\theta_i \equiv \begin{cases} 1 & \text{if } \; i = N, \\ \prod_{\tau=i}^{N-1} \lceil \dfrac{Q_{\tau+1}}{Q_\tau} \rceil & \text{if } \; i = 1, 2, \dots, N-1, \end{cases} \tag{A2}$$

where $\lceil x \rceil$ is the ceiling function that returns the smallest integer no less than x.

LEMMA A1 (CYCLE LENGTH). *For each Stage* $i = 1, 2, \dots, N-1$*, the long-run average expected cycle length satisfies* $\lim_{j\to\infty} \mathbb{E}[(T_i^{j+1} - T_i^1)]/j \geq Q_N/(\theta_{i+1}\lambda)$.

Proof of Lemma A1. To handle the technical difficulty that cycles may have a zero length, we first introduce the concept of a "longer" cycle, which consists of multiple cycles and is guaranteed to have a positive length.

For any vector $\mathbf{Q} \in \mathbb{N}^N$, let $m(\mathbf{Q})$ be the least common multiple of entries in a vector $\mathbf{Q}$. Then, there exists $n_i \in \mathbb{N}$ such that $m(\mathbf{Q}) = n_i Q_i$ for $i = 1, 2, \dots, N$. We view n_N orders of Stage N, with $n_N Q_N = m(\mathbf{Q})$ units in total, as a "long" cycle and track them across all stages. For any Stage i and $j \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, we denote by $\mathfrak{T}_i^j$ the time of the 1-st unit (i.e., Unit $(1 + j \cdot n_N, 1)$), contained in the $(1+j\cdot n_N)$-th order of Stage N, being sent to Stage i. We call $[\mathfrak{T}_i^j, \mathfrak{T}_i^{j+1})$ the $(j+1)$-th long cycle for Stage i. Clearly, for each Stage i, the entire time horizon is a union of time intervals $[\mathfrak{T}_i^j, \mathfrak{T}_i^{j+1})$ for all $j \in \mathbb{N}_0$. The following lemma provides an exact value of their long-run average length.

LEMMA A2. *Under any modified echelon* $(\mathbf{r}, \mathbf{Q})$ *policy, for Stage* $i = 1, 2, \ldots, N$, $\lim_{j\to\infty} \mathbb{E}[\mathfrak{T}_i^{j+1} - \mathfrak{T}_i^1]/j = m(\mathbf{Q})/\lambda$.

Proof of Lemma A2. We prove the result by induction. By the definition of $\mathfrak{T}_N^j$, there are a total of $m(\mathbf{Q})$ units ordered by Stage N over any $[\mathfrak{T}_N^j, \mathfrak{T}_N^{j+1})$. That is, $m(\mathbf{Q})$ units of demand are realized over any $[\mathfrak{T}_N^j, \mathfrak{T}_N^{j+1})$. Thus, $\mathbb{E}[\mathfrak{T}_N^{j+1} - \mathfrak{T}_N^j] = m(\mathbf{Q})/\lambda$. Hence the result holds for Stage N. Suppose that it also holds for Stage $i+1$. That is, $\lim_{j\to\infty} \mathbb{E}[\mathfrak{T}_{i+1}^{j+1} - \mathfrak{T}_{i+1}^1] = m(\mathbf{Q})/\lambda$. Now, consider Stage i. Define Δt_i^j such that $\mathfrak{T}_i^j = \mathfrak{T}_{i+1}^j + L_{i+1} + \Delta t_i^j$. By the same logic as that of Lemma 5, one can readily prove that $0 \le \Delta t_i^j \le t((r_{i+1} - r_i)^+)$ for all $j \in \mathbb{N}_0$. Therefore, we find that $\lim_{j\to\infty} \mathbb{E}[\mathfrak{T}_i^{j+1} - \mathfrak{T}_i^1]/j = \lim_{j\to\infty}(\mathbb{E}[\mathfrak{T}_{i+1}^{j+1} - \mathfrak{T}_{i+1}^1]/j + \mathbb{E}[\Delta t_i^{j+1} - \Delta t_i^1]/j) = \lim_{j\to\infty} \mathbb{E}[\mathfrak{T}_{i+1}^{j+1} - \mathfrak{T}_{i+1}^1]/j = m(\mathbf{Q})/\lambda$. Therefore, the result holds for Stage i and the induction is completed. □

For $i = 1, 2, \ldots, N$, let $\mathfrak{M}_i^j$ denote the number of shipments from Stage $i+1$ to Stage i over $[\mathfrak{T}_i^j, \mathfrak{T}_i^{j+1})$. We denote by $\lceil x \rceil$ the ceiling function, which returns the smallest integer no less than x. The following lemma provides an upper bound on the random variable $\mathfrak{M}_i^j$.

LEMMA A3. *For any* $j \in \mathbb{N}_0$, $\mathfrak{M}_i^j \le n_N \theta_i$, *where* θ_i *is defined in (A2).*

Proof of Lemma A3. We prove the result by induction. By the definition of $\mathfrak{T}_N^j$, it is obvious that the outside supplier delivers n_N shipments to Stage N over $[\mathfrak{T}_N^j, \mathfrak{T}_N^{j+1})$, i.e., $\mathfrak{M}_N^j = n_N$. Hence, the result holds for $i = N$.

Suppose the result holds for $i+1$, i.e., $\mathfrak{M}_{i+1}^j \le n_N \prod_{\tau=i+1}^{N-1} \lceil Q_{\tau+1}/Q_\tau \rceil$ for any j. Note that after each of these shipments arrives at Stage $i+1$, it will be shipped to Stage i through at most $\lceil Q_{i+1}/Q_i \rceil$ shipments. Therefore, the total number of shipments to Stage i over $[\mathfrak{T}_i^j, \mathfrak{T}_i^{j+1})$ for any j is at most $n_N \prod_{\tau=i}^{N-1} \lceil Q_{\tau+1}/Q_\tau \rceil$, i.e., $\mathfrak{M}_i^j \le n_N \prod_{\tau=i}^{N-1} \lceil Q_{\tau+1}/Q_\tau \rceil$ for any j. Hence the induction is completed. □

Here, we call $[T_i^j, T_i^{j+1})$ a "short" cycle, as opposed to the "long" cycle $[\mathfrak{T}_i^j, \mathfrak{T}_i^{j+1})$. Note that the long cycle consists of multiple short cycles. For convenience, we label short cycles according to the long cycle they belong to. For any $j \in \mathbb{N}_0$ and $i = 1, 2, \ldots, N$, we denote by ${}^l T_i^j$ the time of the 1-st unit, contained in the l-th order of Stage $i+1$ over long cycle $[\mathfrak{T}_i^j, \mathfrak{T}_i^{j+1})$, being sent to Stage i. Note that Stage $i+1$ receives at most $\mathfrak{M}_{i+1}^j$ shipments from Stage $i+2$ over $[\mathfrak{T}_{i+1}^j, \mathfrak{T}_{i+1}^{j+1})$.

Therefore, we know that $[\mathfrak{T}_i^j, \mathfrak{T}_i^{j+1}) = \bigcup_{l=1}^{\mathfrak{M}_{i+1}^j} [{}^l T_i^j, {}^{l+1} T_i^j)$. In other words, the long cycle $[\mathfrak{T}_i^j, \mathfrak{T}_i^{j+1})$ contains at most $\mathfrak{M}_{i+1}^j$ short cycles $[T_i^j, T_i^{j+1})$. Therefore, we have $\lim_{j\to\infty} \mathbb{E}[(T_i^{j+1} - T_i^1)]/j \geq \lim_{j\to\infty} \mathbb{E}[(\mathfrak{T}_i^{j+1} - \mathfrak{T}_i^1)]/(j \cdot \mathfrak{M}_{i+1}^j) = m(\mathbf{Q})/(\lambda \cdot \mathfrak{M}_{i+1}^j) \geq Q_N/(\theta_{i+1}\lambda)$, where the first equality follows from Lemma A2 and the second inequality follows from Lemma A3. This completes the proof of Lemma A1. □

Combining Observation A7 and Lemma A1, we immediately have the following observation.

OBSERVATION A8. *The long-run average setup cost for Stage $i = 1, 2, \ldots, N-1$ associated with irregular shipment periods has an upper bound $\theta_{i+1}\lambda K_i / Q_N$.*

We denote by $\hat{\Gamma}_i(IL_{i+1}(t))$ the expected cost rate from Stage i to Stage 1, *excluding* the setup costs for irregular shipment periods incurred at Stage i, at any time $t \in [T_i^j, T_i^{j+1}) \neq \emptyset$, when the echelon inventory level of Stage $i+1$ is $IL_{i+1}(t)$.

LEMMA A4. *For any time t, we have for $i = 1, 2, \ldots, N-1$,*

$$\hat{\Gamma}_i(IL_{i+1}(t)) \leq \begin{cases} \Phi_i(IL_{i+1}(t)) & \text{if } IL_{i+1}(t) \leq r_i, \\ \max\{\Phi_i(\omega_i), \Psi_i(r_i, Q_i)\} & \text{otherwise,} \end{cases}$$

where

$$\omega_i \equiv \arg\max_{r_i < z \leq r_i + Q_i} \Phi_i(z). \tag{A3}$$

Proof of Lemma A4. By the definition of modified (r_i, Q_i) policy, if $IL_{i+1}(t) \leq r_i$, then $IP_i(t) = IL_{i+1}(t) \leq r_i$, which implies that Stage i must be in an irregular shipment period. Therefore, we charge the expected cost rate, excluding setup costs for irregular shipment periods, $\Phi_i(IP_i(t)) = \Phi_i(IL_{i+1}(t))$.

If $IL_{i+1}(t) > r_i$, the inventory position of Stage i must be in $(r_i, r_i + Q_i]$, i.e., $r_i < IP_i(t) \leq r_i + Q_i$. In this case, it is possible that Stage i is either in a regular or in an irregular shipment period. Hence, we obtain a cost upper bound by charging the larger one of the expected cost rates in a regular shipment period and in an irregular shipment period: $\hat{\Gamma}_i(IL_{i+1}(t)) \leq \max\{\Psi_i(r_i, Q_i), \Phi_i(IP_i(t))\} \leq \max\{\Psi_i(r_i, Q_i), \Phi_i(\omega_i)\}$, where the second inequality follows from the definition (A3) of ω_i. □

As in the two-stage case, we define $\Lambda_1(y) \equiv G_1(y)$, $\hat{C}_1(r_1, Q_1) \equiv C_1(r_1, Q_1)$ and for $i = 2, 3, \ldots, N$,

$$\begin{aligned} \hat{G}_{i-1}(y) &\equiv \begin{cases} \Lambda_{i-1}(y) - \hat{C}_{i-1}(r_{i-1}, Q_{i-1}) & \text{if } y \leq r_{i-1}, \\ \max\{0, \Lambda_{i-1}(\omega_{i-1}) - \hat{C}_{i-1}(r_{i-1}, Q_{i-1})\} & \text{otherwise,} \end{cases} \\ \Lambda_i(y) &\equiv \mathbb{E}[h_i(y - D_i) + \hat{G}_{i-1}(y - D_i)], \\ \hat{C}_i(r_i, Q_i) &\equiv \frac{1}{Q_i}\left[\lambda K_i + \int_{r_i}^{r_i+Q_i} \Lambda_i(y)dy\right], \end{aligned} \tag{A4}$$

where D_i is the random lead-time demand over $[0, L_i)$.

Theorem A1 (An Upper Bound). (i) *For $i = 1, 2, \ldots, N$ and any time t,*

$$\Phi_i(IP_i(t)) \le \Lambda_i(IP_i(t)) + \sum_{e=1}^{i-1} \hat{C}_e(r_e, Q_e) + \sum_{e=1}^{i-1} \frac{\theta_{e+1}\lambda K_e}{Q_N}. \tag{A5}$$

(ii) *For any given modified echelon $(\mathbf{r}, \mathbf{Q})$ policy, the expected long-run average systemwide cost*

$$C(\mathbf{r}, \mathbf{Q}) \le \sum_{i=1}^{N} \hat{C}_i(r_i, Q_i) + \sum_{i=1}^{N-1} \frac{\theta_{i+1}\lambda K_i}{Q_N}.$$

Proof of Theorem A1. (i) We prove the result by induction. By definition, we have $\Phi_1(IP_1(t)) = G_1(IP_1(t)) = \Lambda_1(IP_1(t))$, which implies that the result holds for $i = 1$. Assume that the inequality (A5) holds for Stage $i \ge 1$. Next, we prove that the result also holds for Stage $i+1$. Let us first recall the definitions: we denote by $\Phi_i(IP_i(t))$ the total expected cost rate from Stage i to Stage 1 at time t, excluding the setup costs incurred at Stage i, when the inventory position of Stage i at time t is $IP_i(t)$, and denote by $\hat{\Gamma}_i(IL_{i+1}(t))$ the expected cost rate from Stage i to Stage 1, excluding setup costs for irregular shipment periods incurred at Stage i, at any time $t \in [T_i^j, T_{i+1}^{j+1}) \ne \emptyset$, when the echelon inventory level of Stage $i+1$ is $IL_{i+1}(t)$. Therefore, by the cost accounting scheme and Observation A8,

$$\Phi_{i+1}(IP_{i+1}(t)) \le h_{i+1}\mathbb{E}[IP_{i+1}(t) - D_{i+1}] + \mathbb{E}[\hat{\Gamma}_i(IP_{i+1}(t) - D_{i+1})] + \frac{\theta_{i+1}\lambda K_i}{Q_N}. \tag{A6}$$

By (A6) and Lemma A4, if $IP_{i+1}(t) - D_{i+1} \le r_i$,

$$\begin{aligned}
\Phi_{i+1}(IP_{i+1}(t)) &\le h_{i+1}\mathbb{E}[IP_{i+1}(t) - D_{i+1}] + \mathbb{E}[\Phi_i(IP_{i+1}(t) - D_{i+1})] + \frac{\theta_{i+1}\lambda K_i}{Q_N} \\
&\le h_{i+1}\mathbb{E}[IP_{i+1}(t) - D_{i+1}] + \mathbb{E}[\Lambda_i(IP_{i+1}(t) - D_{i+1})] + \sum_{e=1}^{i-1} \hat{C}_e(r_e, Q_e) + \sum_{e=1}^{i-1} \frac{\theta_{e+1}\lambda K_e}{Q_N} + \frac{\theta_{i+1}\lambda K_i}{Q_N} \\
&= h_{i+1}\mathbb{E}[IP_{i+1}(t) - D_{i+1}] + \mathbb{E}[\Lambda_i(IP_{i+1}(t) - D_{i+1})] - \hat{C}_i(r_i, Q_i) + \sum_{e=1}^{i} \hat{C}_e(r_e, Q_e) + \sum_{e=1}^{i} \frac{\theta_{e+1}\lambda K_e}{Q_N},
\end{aligned}$$

where the second inequality follows from that Eq. (A5) holds for Stage i. Again by Eq. (A6) and Lemma A4, if $IP_{i+1}(t) - D_{i+1} > r_i$,

$$\begin{aligned}
\Phi_{i+1}(IP_{i+1}(t)) &\le h_{i+1}\mathbb{E}[IP_{i+1}(t) - D_{i+1}] + \max\{\Psi_i(r_i, Q_i), \Phi_i(\omega_i)\} + \frac{\theta_{i+1}\lambda K_i}{Q_N} \\
&\le h_{i+1}\mathbb{E}[IP_{i+1}(t) - D_{i+1}] + \max\Big\{\hat{C}_i(r_i, Q_i), \Lambda_i(\omega_i)\Big\} + \sum_{e=1}^{i-1} \hat{C}_e(r_e, Q_e) + \sum_{e=1}^{i-1} \frac{\theta_{e+1}\lambda K_e}{Q_N} + \frac{\theta_{i+1}\lambda K_i}{Q_N} \\
&= h_{i+1}\mathbb{E}[IP_{i+1}(t) - D_{i+1}] + \max\Big\{\hat{C}_i(r_i, Q_i), \Lambda_i(\omega_i)\Big\} + \sum_{e=1}^{i-1} \hat{C}_e(r_e, Q_e) + \sum_{e=1}^{i} \frac{\theta_{e+1}\lambda K_e}{Q_N} \\
&= h_{i+1}\mathbb{E}[IP_{i+1}(t) - D_{i+1}] + \max\Big\{0, \Lambda_i(\omega_i) - \hat{C}_i(r_i, Q_i)\Big\} + \sum_{e=1}^{i} \hat{C}_e(r_e, Q_e) + \sum_{e=1}^{i} \frac{\theta_{e+1}\lambda K_e}{Q_N},
\end{aligned}$$

where the second inequality follows from that Eq. (A5) holds for Stage i, and the definitions of $\Psi_i(r_i, Q_i)$ (see Eq. (A1)) and $\hat{C}_i(r_i, Q_i)$ (see Eq. (A4)). Moreover, by the definitions of $\hat{G}_i(y)$ and $\Lambda_{i+1}(y)$ (see Eq. (A4)), $\Phi_{i+1}(IP_{i+1}(t)) \leq h_{i+1}\mathbb{E}[IP_{i+1}(t) - D_{i+1}] + \mathbb{E}[\hat{G}_i(IP_{i+1}(t) - D_{i+1})] + \sum_{e=1}^{i} \hat{C}_e(r_e, Q_e) + \sum_{e=1}^{i} \theta_{e+1}\lambda K_e/Q_N = \Lambda_{i+1}(IP_{i+1}(t)) + \sum_{e=1}^{i} \hat{C}_e(r_e, Q_e) + \sum_{e=1}^{i} \theta_{e+1}\lambda K_e/Q_N$. Therefore, the induction is completed.

(ii) By part (i), the total expected cost rate of N stages except for the setup costs incurred at Stage N is bounded as follows: $\Phi_N(IP_N(t)) \leq \Lambda_N(IP_N(t)) + \sum_{i=1}^{N-1} \hat{C}_i(r_i, Q_i) + \sum_{i=1}^{N-1} \theta_{i+1}\lambda K_i/Q_N$. Under the given modified echelon $(\mathbf{r}, \mathbf{Q})$ policy, $IP_N(t)$, the inventory position of Stage N is uniformly distributed over $\{r_N + 1, \ldots, r_N + Q_N\}$. Therefore, the long-run average system-wide cost is bounded as follows:

$$
\begin{aligned}
C(\mathbf{r}, \mathbf{Q}) &= \frac{1}{Q_N}\left[\lambda K_N + \int_{r_N}^{r_N+Q_N} \Phi_N(y)dy\right] \\
&\leq \frac{1}{Q_N}\left[\lambda K_N + \int_{r_N}^{r_N+Q_N} [\Lambda_N(y) + \sum_{i=1}^{N-1} \hat{C}_i(r_i, Q_i) + \sum_{i=1}^{N-1} \frac{\theta_{i+1}\lambda K_i}{Q_N}]dy\right] = \sum_{i=1}^{N} \hat{C}_i(r_i, Q_i) + \sum_{i=1}^{N-1} \frac{\theta_{i+1}\lambda K_i}{Q_N}.
\end{aligned}
$$

□

A.2. Heuristic: Effectiveness and Asymptotic Optimality

With the upper-bound result, we are ready to investigate the effectiveness of a specific modified echelon $(\mathbf{r}, \mathbf{Q})$ policy which is selected as follows:

$$
\hat{r}_i = r_i^* \text{ and } \hat{Q}_i = Q_i^*, \quad \text{for } i = 1, 2, \ldots, N. \tag{A7}
$$

Namely, we select those critical points in a modified echelon $(\mathbf{r}, \mathbf{Q})$ heuristic from the decomposed single-stage problems that lead to the induced-penalty lower bound (see section 2.2.2). This adoption is slightly different from the two-stage case. For the purpose of optimality gap analysis, we have some freedom in selecting Q_N, the order quantity at the most upstream stage. In the two-stage model, we are able to make an improvement by identifying a heuristic that slightly deviates from Q_2^*. However, we find that such an adjustment does not greatly improve the performance bound for the multiple-stage model and that it complicates the analysis and leads to a less clear-cut result.

For the purpose of presenting the performance guarantee, we define

$$
\theta_i^* \equiv \begin{cases} 1 & \text{if } i = N, \\ \prod_{\tau=i}^{N-1} \lceil \frac{Q_{\tau+1}^*}{Q_\tau^*} \rceil & \text{if } i = 1, 2, \ldots, N-1, \end{cases}
$$

and

$$
\beta^* \equiv \min_{i=1,2,\ldots,N-1} \left\{ \frac{Q_N^*}{Q_i^* \theta_{i+1}^*} \right\}. \tag{A8}
$$

Theorem A2 (Performance Bound). (i) *The absolute gap between* $C^*_{\mathscr{B}}$ *and* $C(\hat{\mathbf{r}},\hat{\mathbf{Q}})$ *is bounded as follows:* $0 \le C(\hat{\mathbf{r}},\hat{\mathbf{Q}}) - C^*_{\mathscr{B}} \le \sum_{i=1}^{N-1} \dfrac{\theta^*_{i+1} Q^*_i C^*_i}{2Q^*_N}$.

(ii) *The relative gap is bounded as follows:* $1 \le \dfrac{C(\hat{\mathbf{r}},\hat{\mathbf{Q}})}{C^*_{\mathscr{B}}} \le 1 + \dfrac{1}{2\beta^*}$. *That is, the modified echelon* $(\hat{\mathbf{r}},\hat{\mathbf{Q}})$ *policy is at least* $(1+\dfrac{1}{2\beta^*})$*-optimal.*

(iii) *The modified echelon* $(\hat{\mathbf{r}},\hat{\mathbf{Q}})$ *policy is asymptotically optimal, as* Q^*_{i+1}/Q^*_i *approaches to infinity for any* $i = 1,2,\ldots,N-1$.

Proof of Theorem A2. (i) We first prove by induction that $\hat{C}_i(r^*_i, Q^*_i) = C_i(r^*_i, Q^*_i) = C^*_i$ (see Eq. (7)), for any $i = 1,2,\ldots,N$. By (A4), $\hat{C}_1(r^*_1, Q^*_1) = C_1(r^*_1, Q^*_1) = C^*_1$ and $\Lambda_1(y) = G_1(y)$. That is, the result holds for $i = 1$. We assume that the result also holds for $i = j$, i.e., $\hat{C}_j(r^*_j, Q^*_j) = C_j(r^*_j, Q^*_j) = C^*_j$ and $\Lambda_j(y) = G_j(y)$. Then, By (A3) and Lemma 2(i), $\Lambda_j(\omega_i) - \hat{C}_j(r^*_j, Q^*_j) = G_j(\omega_i) - C^*_j \le 0$. Therefore, we have $\hat{G}_j(y) = \bar{G}_j(y)$, which results in that $\Lambda_{j+1}(y) = G_{j+1}(y)$. Consequently, $\hat{C}_{j+1}(r^*_{j+1}, Q^*_{j+1}) = C_{j+1}(r^*_{j+1}, Q^*_{j+1}) = C^*_{j+1}$, which completes the induction. Therefore, by Theorem A1 and Lemma 3(ii), $0 \le C(\hat{\mathbf{r}},\hat{\mathbf{Q}}) - C^*_{\mathscr{B}} \le C(\hat{\mathbf{r}},\hat{\mathbf{Q}}) - \sum_{i=1}^{N} C^*_i \le \sum_{i=1}^{N-1} \theta^*_{i+1}\lambda K_i/Q^*_N \le \sum_{i=1}^{N-1} \theta^*_{i+1} Q^*_i C^*_i/(2Q^*_N)$, where the last inequality is due to Lemma 2(ii).

(ii) By part (i), $C(\hat{\mathbf{r}},\hat{\mathbf{Q}}) \le C^*_{\mathscr{B}} + \sum_{i=1}^{N-1} \theta^*_{i+1} Q^*_i C^*_i/(2Q^*_N) \le C^*_{\mathscr{B}} + \sum_{i=1}^{N-1} C^*_i/(2\beta^*) \le C^*_{\mathscr{B}} + \sum_{i=1}^{N} C^*_i/(2\beta^*) \le (1 + 1/(2\beta^*))C^*_{\mathscr{B}}$, where the second inequality follows from (A8) and the last inequality follows from Lemma 3(ii).

(iii) For $i = 1,2,\ldots,N-1$, we define $\beta^*_i = Q^*_{i+1}/Q^*_i$. Then, we have $\theta^*_i \le \prod_{\tau=i}^{N-1}(\beta^*_\tau + 1)$ for $i = 1,2,\ldots,N-1$, and $Q^*_N = Q^*_i \prod_{\tau=i}^{N-1} \beta^*_\tau$. Hence, $\beta^* = \min_{i=1,2,\ldots,N-1}\left\{\frac{Q^*_N}{Q^*_i \theta^*_{i+1}}\right\} \ge \min_{i=1,2,\ldots,N-1}\left\{\beta^*_i \prod_{\tau=i+1}^{N-1} \frac{\beta^*_\tau}{\beta^*_\tau + 1}\right\}$. Therefore, when Q^*_{i+1}/Q^*_i for any i goes to infinity, so do β^*_i for any i and β^*. Then, the modified echelon $(\hat{\mathbf{r}},\hat{\mathbf{Q}})$ policy is asymptotic optimal. □

Theorem A2 shows that the performance of the modified echelon $(\hat{\mathbf{r}},\hat{\mathbf{Q}})$ heuristic policy depends on the ratios between the single-stage optimal order quantities of two consecutive stages when the system is decomposed into single-stage problems as Chen and Zheng (1994b). Moreover, in order to obtain asymptotic optimality for multiple-stage systems, similar conditions like those in Theorem 4 should hold for each stage. These conditions are indeed satisfied for the special case with zero fixed setup costs at all downstream stages, except the most upstream Stage N, i.e., $K_i = 0$ for $i = 1,2,\ldots,N-1$.

A.3. Numerical Results

Lastly, we illustrate numerically how the number of stages, N, affects the effectiveness of the heuristic that $\hat{r}_i = r^*_i$ and $\hat{Q}_i = Q^*_i$, $i = 1,2,\ldots,N$. To eliminate the influence of stage-specific parameters, we set equal values to the parameters of each stage as follows: $h_i = 1$, $L_i = 1$, and $K_i = 10$. The rest of the primitives are fixed as shown in Table 3. From Table A1, we can observe that the effectiveness decreases as N increases, with the marginal effectiveness decay rate decreasing in N.

Table A1 The Impact of Stage Number

N	2	3	4	10	20	40
$\xi(\hat{\mathbf{r}}, \hat{\mathbf{Q}})(\%)$	8.62	14.46	15.96	20.22	21.42	24.31