Online Appendix to "Demand Pooling in Omnichannel Operations"

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A. Proofs for the Main Results

Proof of Lemma 1. As both D_s and D_m follow the normal distributions, the optimal solutions of (2) under the two scenarios can be expressed as $q_s = \mu_s + \sigma_s z^*$ and $q_{sm} = \mu_s + \mu_m + \sqrt{\sigma_s^2 + \sigma_m^2 + 2\rho\sigma_s\sigma_m} z^*$, where $z^* = \Phi^{-1}(1 - \frac{c_b}{p_b})$. Consequently, the corresponding fill rates can be written as

$$\zeta_s = \frac{\mathbb{E}[\min(D_s, q_s)]}{\mathbb{E}[D_s]} = 1 + \frac{\sigma_s}{\mu_s}\mathbb{E}[\min\{z^*, \epsilon\}],$$
$$\zeta_{sm} = \frac{\mathbb{E}[\min(D_s + D_m, q_{sm})]}{\mathbb{E}[D_s + D_m]} = 1 + \frac{\sqrt{\sigma_s^2 + \sigma_m^2 + 2\rho\sigma_s\sigma_m}}{\mu_s + \mu_m}\mathbb{E}[\min\{z^*, \epsilon\}],$$

where ϵ follows the standard normal distribution.

As $\sigma_m/\mu_m = \sigma_s/\mu_s$, we can obtain

$$\frac{\sqrt{\sigma_s^2 + \sigma_m^2 + 2\rho\sigma_s\sigma_m}}{\mu_s + \mu_m} = \frac{\sigma_s}{\mu_s}\frac{\sqrt{1 + x^2 + 2\rho x}}{1 + x} \leq \frac{\sigma_s}{\mu_s},$$

where $x = \mu_s/\mu_m = \sigma_s/\sigma_m$, and the inequality holds because the correlation coefficient $\rho \leq 1$.

As $\mathbb{E}[\min\{z^*, \epsilon\}] \leq \mathbb{E}[\epsilon] = 0$, the above inequality implies that $\zeta_{sm} \geq \zeta_s$. $\square$

Proof of Proposition 1. Let $\hat{\zeta}$ be the customers' belief on the fill rate and $(\hat{\phi}_s, \hat{\phi}_m)$ be the retailer's belief on the proportions of store-only customers and omni-customers who purchase offline. We first characterize customers' channel choices and retailer's optimal decision in terms of these beliefs.

Customer Choice. Given customers' belief on the fill rate, let $(\bar{\phi}_s, \bar{\phi}_m)$ be the resulting proportions of omni-customers and store-only customers who purchase offline. Customers' utilities from purchasing online and offline channels are given by (1). Then, customers' channel choices are as follows: 1) store-only customers purchase the product if $U_{s,b}^B \geq 0$; otherwise, exit the market, or 2) omni-customers purchase the product from the offline channel if and only if $U_{m,b}^B \geq \max(0, U_{m,o}^B)$ and from the online channel if and only if $U_{m,o}^B \geq \max(0, U_{m,b}^B)$. In particular, we can further characterize customers' choices, $(\bar{\phi}_s, \bar{\phi}_m)$, in different parameter regions as follows.

(i) If $\hat{\zeta}u_b - k \geq (u_o - t)$, then both store-only customers and omni-customers will choose the offline channel, i.e., $(\bar{\phi}_s, \bar{\phi}_m) = (1, 1)$.

(ii) If $u_o - t \geq \hat{\zeta}u_b - k \geq 0$, then store-only customers will go to the B&M store, whereas omni-customers will choose the online channel. That is, $(\bar{\phi}_s, \bar{\phi}_m) = (1, 0)$.

(iii) If $u_o - t \geq 0 > \hat{\zeta} u_b - k$, then store-only customers will exit the market, and omni-customers will choose the online channel. That is, neither will go to the B&M store, i.e., $(\bar{\phi}_s, \bar{\phi}_m) = (0, 0)$.

(iv) If $\hat{\zeta} u_b - k < 0$ and $u_o - t < 0$, then both types of customers will exit the market, i.e., $(\bar{\phi}_s, \bar{\phi}_m) = (0, 0)$.

Therefore, given customers' belief on the fill rate $\hat{\zeta}$, the resulting market segmentation $(\bar{\phi}_s, \bar{\phi}_m)$ can be expressed as

$$(\bar{\phi}_s, \bar{\phi}_m) = \begin{cases} (1,1) \text{ if } \hat{\zeta} \geq \frac{u_o - t + k}{u_b}, \\ (1,0) \text{ if } \frac{k}{u_b} \leq \hat{\zeta} < \frac{u_o - t + k}{u_b}, \\ (0,0) \text{ if } \hat{\zeta} < \frac{k}{u_b}. \end{cases} \tag{A1}$$

Retailer's Optimal Decision. Given retailer's beliefs $(\hat{\phi}_s, \hat{\phi}_m)$ on customers' choices, let $\bar{q}$ and $\bar{\zeta}$ be the optimal order quantity for the B&M store and the corresponding resulting fill rate, respectively. As customers are homogeneous, there are only three possible values of $(\hat{\phi}_s, \hat{\phi}_m)$: $(0, 0)$, $(1, 0)$, and $(1, 1)$. We now analyze the retailer's optimal decision for the three cases.

(i) If $(\hat{\phi}_s, \hat{\phi}_m) = (1, 1)$, the retailer's expected profit is $\pi(q) = p_b \mathbb{E}[\min(D_m + D_s, q)] - c_b q$. Then, the optimal order quantity is $\bar{q} = q_{sm}$, and the corresponding fill rate is $\bar{\zeta} = \zeta_{sm}$.

(ii) If $(\hat{\phi}_s, \hat{\phi}_m) = (1, 0)$, the retailer's expected profit is $\pi(q) = p_b \mathbb{E}[\min(D_s, q)] - c_b q + (p_o - c_o)\mathbb{E}[D_m \mathbf{1}_{\{t \leq u_o\}}]$. Consequently, the optimal order quantity is $\bar{q} = q_s$, and the corresponding fill rate is $\bar{\zeta} = \zeta_s$.

(iii) If $(\hat{\phi}_s, \hat{\phi}_m) = (0, 0)$, the retailer's expected profit is $\pi(q) = (p_o - c_o)\mathbb{E}[D_m \mathbf{1}_{\{t \leq u_o\}}]$. Therefore, the retailer will order nothing from the B&M store and the resulting fill rate is zero, i.e., $\bar{q} = 0$ and $\bar{\zeta} = 0$.

In summary, given the belief $(\hat{\phi}_s, \hat{\phi}_m)$, the retailer's optimal decision of $\bar{q}$ and the corresponding fill rate $\bar{\zeta}$ can be characterized as

$$(\bar{q}, \bar{\zeta}) = \begin{cases} (q_{sm}, \zeta_{sm}) & \text{if } (\hat{\phi}_s, \hat{\phi}_m) = (1,1), \\ (q_s, \zeta_s) & \text{if } (\hat{\phi}_s, \hat{\phi}_m) = (1,0), \\ (0, 0) & \text{if } (\hat{\phi}_s, \hat{\phi}_m) = (0,0). \end{cases} \tag{A2}$$

RE Equilibrium. Next, we analyze the RE equilibrium, denoted by $(q^B, \phi_m^B, \phi_s^B)$, by connecting customers' choices with the retailer's optimal decision. The concept of RE equilibrium states 1) customers' belief on the fill rate is exactly the realized fill rate, and 2) retailer's belief on customers' choices is consistent with the realized one. Specifically, we define the RE equilibrium as follows.

Definition A1 (RE Equilibrium Under BASE). A RE equilibrium $(\bar{\zeta}, \hat{\zeta}, \bar{\phi}_m, \bar{\phi}_s, \hat{\phi}_m, \hat{\phi}_s, \bar{q})$ should satisfy the following conditions:

(i) Given $\hat{\zeta}$, $(\bar{\phi}_s, \bar{\phi}_m)$ satisfies Equation (A1);

(ii) Given $(\hat{\phi}_s, \hat{\phi}_m)$, $(\bar{q}, \bar{\zeta})$ satisfies Equation (A2);

(iii) $\bar{\zeta} = \hat{\zeta}$ and $(\bar{\phi}_s, \bar{\phi}_m) = (\hat{\phi}_s, \hat{\phi}_m)$.

By Definition A3, we know that there exist only three potential RE equilibria, i.e., $(q^B, \phi_s^B, \phi_m^B) = (q_{sm}, 1, 1)$, $(q_s, 1, 0)$, and $(0, 0, 0)$. In the following cases, we first verify whether these potential equilibria satisfy the conditions of RE equilibrium. When there exist multiple RE equilibria, we choose the one which is Pareto-dominant.

Case 1: $k \leq \zeta_{sm} u_b - (u_o - t)$. In this case, we first show that $(q_{sm}, 1, 1)$ and $(0, 0, 0)$ are RE equilibria, whereas the existence of $(q_s, 1, 0)$ depends on whether $\zeta_s u_b \geq k > \zeta_s u_b - (u_o - t)$ or not. We take the equilibrium $(q_{sm}, 1, 1)$ as an example to illustrate how to justify a RE equilibrium. Specifically, if the retailer expects that both store-only and omni-customers purchase offline, he will stock q_{sm} in the B&M store by Equation (A2). Meanwhile, if both store-only and omni-customers believe that the retailer allocates q_{sm} in the B&M store, then they anticipate that the fill rate is ζ_{sm}. As $\zeta_{sm} u_b - k \geq (u_o - t) \geq 0$, they will purchase from the B&M store (see Equation (A1)). In other words, all beliefs are consistent with the actual outcomes and thus $(q_{sm}, 1, 1)$ is a RE equilibrium. A similar argument can be applied to the equilibrium $(0, 0, 0)$.

We now discuss the possibility of equilibrium $(q_s, 1, 0)$. If the retailer expects that only store-only customers purchase offline, he will stock q_s in the B&M store by Equation (A2). If customers believe that the retailer allocates q_s in the B&M store, then their expected fill rate is ζ_s. If $0 \leq \zeta_s u_b - k < (u_o - t)$, store-only customers purchase offline, while it is not the case for omni-customers by Equation (A1), i.e., the beliefs are indeed consistent with actual outcomes. However, if $\zeta_s u_b - k \geq (u_o - t)$ (or $\zeta_s u_b - k < 0$), both store-only and omni-customers will (or not) go to the B&M store and customers' choice would be different from the retailer's expectation.

We next show that the equilibrium $(q_{sm}, 1, 1)$ always Pareto-dominates others. Clearly, the equilibrium $(0, 0, 0)$ is dominated, as it generates zero payoff for both retailer and customers. Now, we compare the equilibria $(q_{sm}, 1, 1)$ and $(q_s, 1, 0)$. On the retailer side, we have

$$
\begin{aligned}
& \pi_B(q_s|(1,0)) - \pi_B(q_{sm}|(1,1)) \\
= {} & (p_b - c_b)\mu_s + p_b \sigma_s \int_{-\infty}^{\epsilon^*} \epsilon d\Phi(\epsilon) + (p_o - c_o)\mu_m \\
& - \left[(p_b - c_b)(\mu_s + \mu_m) + p_b \sqrt{\sigma_s^2 + \sigma_m^2 + 2\rho\sigma_s\sigma_m} \int_{-\infty}^{\epsilon^*} \epsilon d\Phi(\epsilon) \right] \\
= {} & (p_o - c_o)\mu_m - (p_b - c_b)\mu_m - p_b \left(\sqrt{\sigma_s^2 + \sigma_m^2 + 2\rho\sigma_s\sigma_m} - \sigma_s\right) \int_{-\infty}^{\epsilon^*} \epsilon d\Phi(\epsilon) \\
= {} & \mu_m((p_o - c_o) - (p_b - c_b) - \Delta) \leq 0,
\end{aligned}
$$

where the last inequality holds by Assumption (M).

Thus, retailer's profit under equilibrium $(q_{sm},1,1)$ is larger than that under $(q_s,1,0)$. Meanwhile, by the condition $k \le \zeta_{sm}u_b-(u_o-t)$, omni-customers' utility under equilibrium $(q_{sm},1,1)$, i.e., $\zeta_{sm}u_b-k$, dominates that under $(q_s,1,0)$, i.e., (u_o-t), whereas store-only customers' utility under equilibrium $(q_{sm},1,1)$ also dominates as the fill rate is higher, i.e., $\zeta_{sm}\ge\zeta_s$. Therefore, the Pareto-dominant equilibrium is $q^B=q_{sm}$, and $(\phi_s^B,\phi_m^B)=(1,1)$.

Case 2: $\zeta_s u_b \ge k > \zeta_{sm}u_b-(u_o-t)$. In this case, omni-customers will never choose the offline channel as $(u_o-t)>\zeta_{sm}u_b-k$. This implies that $(q_{sm},1,1)$ is not a RE equilibrium. Similar to Case 1, one can show that both $(q_s,1,0)$ and $(0,0,0)$ are equilibria and that $(q_s,1,0)$ is Pareto-dominant. Thus, $q^B=q_s$ and $(\phi_s^B,\phi_m^B)=(1,0)$.

Case 3: $k>\zeta_s u_b$ and $k>\zeta_{sm}u_b-(u_o-t)$. In this case, one can verify that there exists just one equilibrium, $(q^B,\phi_s^B,\phi_m^B)=(0,0,0)$.

By combining the above cases, we can obtain the desired results. □

Proof of Lemma 2. We first show that $q_{sm}\ge q_p$. By (4), we have

$$\frac{\partial \pi_P(q)}{\partial q}-\frac{\partial \pi_B(q)}{\partial q}\bigg|_{(\hat{\phi}_s,\hat{\phi}_m)=(1,1)}=(c_o-p_b)\frac{\partial \mathbb{E}[\min(D_m,(q-D_s)^+)]}{\partial q}\le 0, \tag{A3}$$

where the inequality holds as $p_b\ge c_o$ and the term $\mathbb{E}[\min(D_m,(q-D_s)^+)]$ is increasing in q.

As $\pi_P(q)$ is convex, Equation (A3) implies $q_{sm}\ge q_p$. We now prove that $q_p\ge q_s$. Referring to (2), q_s is the minimizer of $\pi_B(q)$ with $\hat{\phi}_m=0$ and $\hat{\phi}_s=1$. Then, we can show that

$$\frac{\partial \pi_B(q)}{\partial q}\bigg|_{(\hat{\phi}_s,\hat{\phi}_m)=(0,1)}-\frac{\partial \pi_P(q)}{\partial q}=-c_o\frac{\partial \mathbb{E}[\min(D_m,(q-D_s)^+)]}{\partial q}\le 0,$$

which implies that $q_p\ge q_s$.

Combining the above results, we can prove that $q_{sm}\ge q_p\ge q_s$. Recall that $\zeta_s=\mathbb{E}[\min(D_s,q_s)]/\mathbb{E}[D_s]$, $\zeta_p=\mathbb{E}[\min(D_s,q_p)]/\mathbb{E}[D_s]$ and $\zeta_{sm}=\mathbb{E}[\min(D_m+D_s,q_{sm})]/\mathbb{E}[D_m+D_s]$.

As the fill rates ζ_s and ζ_p are increasing in the order quantity, one can easily show that $\zeta_p\ge\zeta_s$. We next show that ζ_{sm} could be higher or less than ζ_p. Actually, if c_o approaches 0, ζ_p converges to ζ_s, which is no higher than ζ_{sm} (see Lemma 1). Moreover, we can see that ζ_p is increasing in c_o. We just need to use a counter example to show that it is possible that $\zeta_{sm}<\zeta_p$. Let c_o approach p_b and $D_s=D_m$ (i.e., $\rho=1$, $\mu_s=\mu_m$), then, q_p converges to q_{sm} and

$$\zeta_p\to\frac{\mathbb{E}[\min(q_{sm},D_s)]}{\mathbb{E}[D_s]}=\frac{\mathbb{E}[\min(2q_{sm},2D_s)]}{\mathbb{E}[2D_s]}>\frac{\mathbb{E}[\min(q_{sm},D_s+D_m)]}{\mathbb{E}[D_s+D_m]}=\zeta_{sm}.$$

This completes the proof. □

Proof of Proposition 2. Similar to the proof logic of Proposition 1, we characterize the equilibria as follows.

Customer Choice. We first investigate customers' choices among different channels in the presence of BOPS.

- **Omni-customers.** With the BOPS channel, omni-customers have four alternative choices: exiting the market, purchasing offline, online, or through BOPS. By Assumption (U) (i.e., $t \le u_o$), omni-customers will always gain a positive utility from purchasing online, and thus, will never choose to exit the market. Moreover, as $u_o \ge u_b$, omni-customers always prefer the BOPS channel to the option of purchasing offline. Therefore, omni-customers will choose between the online and BOPS channels. Specifically, omni-customers purchase from the BOPS channel if and only if $U^P_{m,ob} \ge U^P_{m,o}$ and the online channel if and only if $U^P_{m,o} \ge U^P_{m,ob}$.
- **Store-only customers.** Store-only customers' utilities from purchasing offline are given by $U^P_{s,b} = \hat{\zeta} u_b - k$. Then, store-only customers purchase the product if $U^P_{s,b} \ge 0$; otherwise, exit the market.

Based on the analysis on customers' choice, we can derive the resulting market segmentation as follows.

(i) If $k \le t$ and $k \le \hat{\zeta} u_b$, store-only customers will go to the B&M store, and omni-customers will choose the BOPS channel. This means that the total demand for the BOPS channel is D_m, and that for the B&M store is D_s.

(ii) If $k \le t$ and $k > \hat{\zeta} u_b$, store-only customers will choose to exit the market, whereas omni-customers choose the BOPS channel. Then, the total demand for the BOPS channel is D_m and that for the B&M store is 0.

(iii) If $k > t$ and $k \le \hat{\zeta} u_b$, store-only customers will go to the B&M store, whereas omni-customers will go to the online channel. Thus, the total demand for the online channel is D_m, and that for the B&M store is D_s.

(iv) If $k > t$ and $k > \hat{\zeta} u_b$, store-only customers will exit the market, and omni-customers will choose the online channel. This means that the total demand for the online channel is D_m and that for the B&M store is 0.

We denote by ϕ_m the proportion of omni-customers who choose to purchase from BOPS and by $\bar{\phi}_s$ the proportion of store-only customers who purchase from the B&M store. Therefore, given customers' belief on the fill rate $\hat{\zeta}$, the resulting market segmentation $(\bar{\phi}_s, \phi_m)$ is given by

$$(\bar{\phi}_s, \phi_m) = \begin{cases} (1,1) \text{ if } \hat{\zeta} \geq \frac{k}{u_b} \text{ and } k \leq t, \\ (0,1) \text{ if } \hat{\zeta} < \frac{k}{u_b} \text{ and } k \leq t, \\ (1,0) \text{ if } \hat{\zeta} \geq \frac{k}{u_b} \text{ and } k > t, \\ (0,0) \text{ if } \hat{\zeta} < \frac{k}{u_b} \text{ and } k > t. \end{cases} \tag{A4}$$

Retailer's Optimal Decision. We now analyze the retailer's optimal decision given his beliefs on the market segmentation $(\hat{\phi}_s, \phi_m)$. Let $\bar{q}$ and $\bar{\zeta}$ be retailer's optimal order quantity and the corresponding resulting fill rate given his belief, respectively. From the above analysis on the market segmentation, we restrict retailer's belief $(\hat{\phi}_s, \phi_m)$ within the set of $(1,1)$, $(1,0)$, $(0,1)$, and $(0,0)$. Notice that those omni-customers who choose the BOPS channel, but if not satisfied, will switch to the online channel. We analyze the retailer's optimal decision in the following four cases.

(i) If $(\hat{\phi}_s, \phi_m) = (1,1)$, the retailer's expected profit is $\pi(q) = p_b\mathbb{E}[\min(D_s, q)] - c_b q + c_o\mathbb{E}[\min(D_m, (q-D_s)^+)] + (p_o - c_o)\mathbb{E}[D_m]$. Then, the retailer's optimal order quantity is $\bar{q} = q_p$ and the corresponding fill rate is $\bar{\zeta} = \zeta_p$.

(ii) If $(\hat{\phi}_s, \phi_m) = (0,1)$, the retailer's expected profit is $\pi(q) = c_o\mathbb{E}[\min(D_m, q)] - c_b q + (p_o - c_o)\mathbb{E}[D_m]$. The optimal order quantity is $\bar{q} = 0$, as

$$\frac{\partial \pi(q)}{\partial q} = c_o(1 - \Phi(\frac{q - \mu_m}{\sigma_m})) - c_b \leq 0,$$

where the inequality holds by $c_o \leq c_b$. Thus, the corresponding fill rate is $\bar{\zeta} = 0$.

(iii) If $(\hat{\phi}_s, \phi_m) = (1,0)$, the retailer's expected profit is $\pi(q) = p_b\mathbb{E}[\min(D_s, q)] - c_b q + (p_o - c_o)\mathbb{E}[D_m]$. The optimal order quantity is $\bar{q} = q_s$ and the corresponding fill rate is $\bar{\zeta} = \zeta_s$.

(iv) If $(\hat{\phi}_s, \phi_m) = (0,0)$, the retailer's expected profit is $\pi(q) = (p_o - c_o)\mathbb{E}[D_m]$. The resulting order quantity and fill rate are $\bar{q} = 0$ and $\bar{\zeta} = 0$, respectively.

Therefore, given retailer's belief $(\hat{\phi}_s, \phi_m)$, the optimal order quantity $\bar{q}$ and the corresponding fill rate $\bar{\zeta}$ are given by

$$(\bar{q}, \bar{\zeta}) = \begin{cases} (q_p, \zeta_p) \text{ if } (\hat{\phi}_s, \phi_m) = (1,1), \\ (q_s, \zeta_s) \text{ if } (\hat{\phi}_s, \phi_m) = (1,0), \\ \quad (0,0) \quad \text{if } \hat{\phi}_s = 0. \end{cases} \tag{A5}$$

RE Equilibrium. We now turn to analyze the RE equilibrium, which is denoted by (q^P, ϕ_s^P).

Definition A2 (RE Equilibrium Under BOPS).

A RE equilibrium $(\bar{\zeta}, \hat{\zeta}, \bar{\phi}_s, \hat{\phi}_s, \bar{q})$ should satisfy the following conditions:

(i) Given $\hat{\zeta}$ and ϕ_m, $\bar{\phi}_s$ satisfies Equation (A4);

(ii) Given $\hat{\phi}_s$ and ϕ_m, $(\bar{q}, \bar{\zeta})$ satisfies Equation (A5);

(iii) $\bar{\zeta} = \hat{\zeta}$ and $\bar{\phi}_s = \hat{\phi}_s$.

According to the above definition, we consider the following four cases.

Case 1: $k \le t$ and $k \le \zeta_p u_b$. By (A4), we have $\phi_m = 1$, and thus by (A5), we have two potential equilibria, i.e., $(q_p, 1)$ and $(0,0)$. Similar to Proposition 1, one can verify that both are RE equilibria and that $(q_p, 1)$ Pareto-dominates $(0,0)$. Thus, $q^P = q_p$ and $\phi_s^P = 1$.

Case 2: $k \le t$ and $k > \zeta_p u_b$. Again, we have two potential equilibria, i.e., $(q_p, 1)$ and $(0,0)$. However, in this case, $(q_p, 1)$ is not a RE equilibrium. Thus, there exists only a single equilibrium, i.e., $q^P = 0$ and $\phi_s^P = 0$. As the retailer does not store any inventory in the B&M store, omni-customers will choose the online channel.

Case 3: $k > t$ and $k \le \zeta_s u_b$. In this case, $\phi_m = 0$ and there are two equilibria, i.e., $(q^P, \phi_s^P) = (q_s, 1)$ or $(0,0)$. The former is Pareto-dominant and thus $q^P = q_s$ and $\phi_s^P = 1$.

Case 4: $k > t$ and $k > \zeta_s u_b$. In this case, $\phi_m = 0$, and the only equilibrium is $(q^P, \phi_s^P) = (0,0)$. As $\zeta_p > \zeta_s$ (see Lemma 2), summarizing the above four cases leads us to the proposition. □

The proofs for Propositions 3, 4 and 5 can be derived by comparing retailer's expected profits in various areas, as illustrated in Figure 3. We now illustrate the proofs as follows.

Proof of Propositions 3. We analyze this case by considering the following two cases.

- $\zeta_{sm} u_b - u_o + t < k \le \min(t, \zeta_s u_b)$ (i.e., Area (I-2)). By Proposition 1(ii) and Proposition 2(i), the retailer's expected profits in the BASE and BOPS models are respectively given by

$$\begin{aligned}
\Pi_B &= \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q + (p_o - c_o)\mathbb{E}[D_m]\};\\
\Pi_P &= \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q + c_o \mathbb{E}[\min(D_m, (q - D_s)^+)] + (p_o - c_o)\mathbb{E}[D_m]\}.
\end{aligned}$$

Clearly, $\Pi_B \le \Pi_P$.

- $\max(\zeta_{sm} u_b - u_o + t, \zeta_s u_b) < k \le \min(t, \zeta_p u_b)$ (i.e., Area (I-3)). By Proposition 1(iii) and Proposition 2(i), the retailer's expected profits in the BASE and BOPS models are given by

$$\begin{aligned}
\Pi_B &= \max_q \{(p_o - c_o)\mathbb{E}[D_m]\};\\
\Pi_P &= \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q + c_o \mathbb{E}[\min(D_m, (q - D_s)^+)] + (p_o - c_o)\mathbb{E}[D_m]\}.
\end{aligned}$$

Then, it directly follows that $\Pi_B \le \Pi_P$.

Combining the above two cases will lead to the result. This is due to the fact that,

$$[\max(\zeta_{sm} u_b - u_o + t, \min(t, \zeta_p u_b)] = [\zeta_{sm} u_b - u_o + t, \min(t, \zeta_s u_b)] \bigcup [\max(\zeta_{sm} u_b - u_o + t, \zeta_s u_b), \min(t, \zeta_p u_b)],$$

as $\zeta_s \le \zeta_p$. □

Proof of Proposition 4. When $\zeta_p u_b < k \le \zeta_{sm} u_b - u_o + t$ (i.e., Area (II-1)), we have that $\zeta_{sm} u_b - u_o + t > \zeta_p u_b$ if the region is not empty. As $\zeta_{sm} u_b \le u_o$, we have $t > \zeta_p u_b$, and the condition of Proposition 2(iii) becomes $k > \max\{\zeta_p u_b, \zeta_s u_b\} = \zeta_p u_b$. This means that the case corresponds to Proposition 2(iii) under the BOPS setting.

By Proposition 1(i) and Proposition 2(iii), we have

$$
\begin{aligned}
\Pi_B &= \max_q \{p_b \mathbb{E}[\min(D_s + D_m, q)] - c_b q\} \\
&= (p_b - c_b)(\mu_s + \mu_m) + p_b \sqrt{\sigma_s^2 + \sigma_m^2 + 2\rho\sigma_s\sigma_m} \int_{-\infty}^{\epsilon^*} \epsilon d\Phi(\epsilon) \\
&= (p_b - c_b)(\mu_s + \mu_m) + \mu_m \Delta + p_b \sigma_s \int_{\infty}^{\epsilon^*} \epsilon d\Phi(\epsilon) \\
&= (p_b - c_b)\mu_s + p_b \sigma_s \int_{\infty}^{\epsilon^*} \epsilon d\Phi(\epsilon) + (p_b - c_b + \Delta)\mu_m, \\
&\geq (p_b - c_b)\mu_s + p_b \sigma_s \int_{\infty}^{\epsilon^*} \epsilon d\Phi(\epsilon) + (p_o - c_o)\mu_m, \\
&= \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q\} + (p_o - c_o)\mu_m, \\
&\geq (p_o - c_o)\mathbb{E}[D_m] = \Pi_P,
\end{aligned}
$$

where the first inequality holds by Assumption (M). Therefore, BOPS hurts the retailer. $\square$

Proof of Proposition 5. When $k \leq \min(\zeta_p u_b, \zeta_{sm} u_b - u_o + t)$ (i.e., Area (I-1)), as $u_o \geq u_b$, we have $k \leq \min(\zeta_p u_b, \zeta_{sm} u_b - u_o + t) \leq \min(\zeta_p u_b, t)$. By Proposition 2(i) and Equation (4), the retailer's expected profit in the BOPS model is given by

$$
\Pi_P = \max_q \left\{p_b \mathbb{E}[\min(D_s, q)] - c_b q + c_o \mathbb{E}[\min(D_m, (q - D_s)^+)] + (p_o - c_o)\mathbb{E}[D_m]\right\}.
$$

Meanwhile, by Proposition 1(i) and Equation (2), the retailer's expected profit in the BASE model can be written as

$$
\begin{aligned}
\Pi_B &= \max_q \{p_b \mathbb{E}[\min(D_m + D_s, q)] - c_b q\} \\
&= \max_q \left\{p_b \mathbb{E}[\min(D_s, q)] + p_b \mathbb{E}[\min(D_m, (q - D_s)^+)] - c_b q\right\}.
\end{aligned}
$$

Then, when $p_o \to c_o$, we have $\pi_B \geq \pi_P$ as

$$
\begin{aligned}
\Pi_P &= \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q + c_o \mathbb{E}[\min(D_m, (q - D_s)^+)]\} \\
&\leq \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q + p_b \mathbb{E}[\min(D_m, (q - D_s)^+)]\} = \Pi_B,
\end{aligned}
$$

in which the inequality holds by $c_o \leq c_b \leq p_b$.

Note that

$$
\max_q \{p_b \mathbb{E}[\min(D_m + D_s, q)] - c_b q\} - \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q\}
$$

$$
\begin{aligned}
&= p_b\mathbb{E}[\min(D_m + D_s, \mu_m + \mu_s + \sqrt{\sigma_s^2 + \sigma_m^2 + 2\rho\sigma_s\sigma_m}\epsilon^*)] - c_b(\mu_m + \mu_s + \sqrt{\sigma_s^2 + \sigma_m^2 + 2\rho\sigma_s\sigma_m}\epsilon^*) \\
&\quad - p_b\mathbb{E}[\min(D_s, \mu_s + \sigma_s\epsilon^*)] - c_b(\mu_s + \sigma_s\epsilon^*) \\
&= (p_b - c_b)\mu_m + p_b(\sqrt{\sigma_s^2 + \sigma_m^2 + 2\rho\sigma_s\sigma_m} - \sigma_s)\int_{-\infty}^{\epsilon^*} \epsilon d\Phi(\epsilon) \\
&= (p_b - c_b + \Delta)\mu_m. \qquad \text{(A6)}
\end{aligned}
$$

If $p_o \to p_b - c_b + c_o + \Delta$, we have $\pi_B < \pi_P$ as

$$
\begin{aligned}
\Pi_P &= \max_q\{p_b\mathbb{E}[\min(D_s, q)] - c_b q + c_o\mathbb{E}[\min(D_m, (q - D_s)^+)] + (p_b - c_b + \Delta)\mu_m\} \\
&= \max_q\{p_b\mathbb{E}[\min(D_s, q)] - c_b q + c_o\mathbb{E}[\min(D_m, (q - D_s)^+)]\} \\
&\quad + \max_q\{p_b\mathbb{E}[\min(D_m + D_s, q)] - c_b q\} - \max_q\{p_b\mathbb{E}[\min(D_s, q)] - c_b q\} \\
&\geq \max_q\{p_b\mathbb{E}[\min(D_m + D_s, q)] - c_b q\} = \Pi_B,
\end{aligned}
$$

where the second equality follows from Equation (A6).

Thus, as Π_P is increasing in p_o, there exists a threshold $\delta_S(c_o) \in [c_o, p_b - c_b + c_o + \Delta]$ such that when $p_o < \delta_S(c_o)$, $\Pi_B \geq \Pi_P$, and when $p_o \geq \delta_S(c_o)$, $\Pi_B < \Pi_P$. □

B. High Online Waiting Time ($t > u_o$)

The results in the main body of the paper rely on the assumption that $t \leq u_o$. This assumption ensures that omni-customers can always gain a positive utility from the online channel and thus, will not exit the market. In this section, we consider the opposite case, i.e., $t > u_o$, to show that our main insights are robust. Our analysis only focuses on the case with high online base surpluse (i.e., $u_o \geq u_b$); the main results also carry over to the case with $u_o < u_b$.

The analysis on customers' choice and retailer's decision is almost the same as that of the case with $t \leq u_o$. The only difference is that under the assumption of $t > u_o$, omni-customers will never choose the online channel, but will choose BOPS, the offline channel or exit the market. See the proof of Proposition A1 for a detailed description of customers' choice and retailer's decision, as well as the equilibrium analysis. We first characterize the RE equilibrium for the BASE model.

Proposition A1 (REE under BASE with High Online Waiting Cost). *At the RE equilibrium, the inventory level in the B&M store and customers' channel choices in the offline channel are as follows:*

(*i*) *if* $k \leq \zeta_{sm}u_b$, *then* $q^B = q_{sm}$ *and* $(\phi_s^B, \phi_m^B) = (1, 1)$;

(*ii*) *if* $\zeta_{sm}u_b < k \leq \zeta_s u_b$, *then* $q^B = q_s$ *and* $(\phi_s^B, \phi_m^B) = (1, 0)$;

(*iii*) *if* $k > \max(\zeta_s u_b, \zeta_{sm}u_b)$, *then* $q^B = 0$ *and* $(\phi_s^B, \phi_m^B) = (0, 0)$.

No omni-customers will purchase through the online channel.

Consider a scenario where omni-customers definitely choose the offline channel. Then, the retailer's objective function can be rewritten as

$$\pi_P(q) = p_b\mathbb{E}[\min(D_s, q)] - c_b q + p_o\mathbb{E}[\min(D_m, (q - D_s)^+)]. \tag{A7}$$

We denote by $q_P^{(1)}$ the optimal order quantity of (A7). Then, the corresponding fill rate is given by $\zeta_p^{(1)} = \mathbb{E}[\min(q_P^{(1)}, D_s)]/\mathbb{E}[D_s]$. Let q_m be the optimal order quantity of the newsvendor problem $\pi(q) = p_o\mathbb{E}[\min(D_m, q)] - c_b q$, where the retail price is the online price, but the supply cost is the offline cost. The following proposition characterizes the equilibrium.

Proposition A2 (REE under BOPS with High Online Waiting Cost). *At the RE equilibrium, the retailer's optimal inventory level in the B&M store and customers' optimal channel choices are as follows:*

(i) if $k \le \zeta_p^{(1)} u_b$, then $q^P = q_P^{(1)}$, $\phi_s^P = 1$, and omni-customers choose BOPS;

(ii) if $\zeta_p^{(1)} u_b < k \le u_o$, then $q^P = q_m$, $\phi_s^P = 0$, and omni-customers choose BOPS.

Moreover, those unsatisfied omni-customers by BOPS will be lost.

When the store visiting cost is low, in equilibrium, store-only customers choose to visit the B&M store, whereas omni-customers choose to purchase the product via the BOPS channel. In contrast, when the store visiting cost is high but not higher than u_o, in equilibrium, store-only customers exit the market, whereas omni-customers choose BOPS. In this situation, the local inventory is only held for omni-customers. Recall that with low online waiting cost ($t \le u_o$), omni-customers unsatisfied by BOPS will switch to the online channel. However, in the case of a high online waiting cost ($t > u_o$), those omni-customers will be lost as the utility of purchasing online becomes negative.

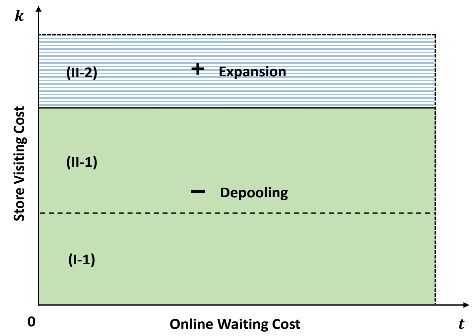
We now compare BOPS and BASE strategies for $t > u_o$ which is also illustrated by Figure A1.

Theorem A1 (BOPS vs. BASE with High Online Waiting Cost). *(i) If $\zeta_{sm} u_b \le k \le u_o$, BOPS benefits the retailer.*

(ii) If $k < \zeta_{sm} u_b$, BOPS hurts the retailer.

When the store visiting cost is relatively high ($\zeta_{sm} u_b \le k \le u_o$), no customer would purchase the product in the BASE model. As BOPS offers an attractive online price and replaces the high online waiting cost by the intermediate store visiting cost, it incentivizes omni-customers to purchase the product via the BOPS channel. Such a *market expansion effect* benefits the retailer (see a similar effect in Gao and Su 2017b attributing to the so-called "convenience" offered by BOPS). As there is no demand coming from the offline channel due to the high store visiting cost, the B&M store now only serves as a pickup location for the omni-customers.

Figure A1 **BOPS over BASE when $t > u_o$**



As the store visiting cost decreases to an intermediate level ($\zeta_p^{(1)} u_b \leq k < \zeta_{sm} u_b$), both types of customers choose the offline channel in the BASE model. With BOPS, omni-customers will migrate from the offline channel to the BOPS channel (i.e., demand depooling). The migration of omni-customers will lead to a lower fill rate that, in turn, leaves no incentive for the store-only customers to visit the B&M store. Hence, this demand depooling effect hurts the retailer. When the store visiting cost further drops to a low level ($k < \zeta_p^{(1)} u_b$), store-only customers remain in the offline channel, and omni-customers switch from the offline channel to the BOPS channel. However, omni-customers' migration still results in a profit decrease as they now pay a lower online price and are lost in the event of stockout at the B&M store (due to $t \leq u_o$). In summary, if $t > u_o$ and the store visiting cost is sufficiently low ($k < \zeta_{sm} u_b$), BOPS hurts the retailer. Therefore, although Nash (2016) recommended Walmart to sell its groceries online, the firm should be careful in this initiative because, in the case of groceries (as time-sensitive products), when the store visiting cost is low (e.g., due to the high density of Walmart stores), introducing BOPS may migrate some existing customers from the high-margin store purchase to a potentially lower-margin BOPS purchase. Due to the depooling effect, the store may no longer find it profitable to maintain a high service level, which, in turn, can hurt both the store-only customers and the retailer.

C. Low Online Base Surplus ($u_o < u_b$)

As the utility from the offline channel is always higher than that from BOPS, omni-customers never choose the BOPS channel and instead choose either the online or offline channels. Specifically, when $U_{m,o}^P \geq U_{m,b}^P$ (i.e., $u_o - t \geq u_b - k$), omni-customers will choose the online channel. If $U_{m,o}^P < U_{m,b}^P$ (i.e., $u_o - t < u_b - k$), they prefer the offline channel and choose BOPS merely for inventory information disclosure: they first check the inventory availability in store via BOPS; if the local inventory is available, they will purchase from the B&M store, and otherwise, choose the online channel. The retailer cannot differentiate store-only customers and omni-customers, when both choose to purchase offline. Hence, the inventory is allocated to those offline customers (possibly from two streams) proportionally.

Define ϕ_m as the proportion of omni-customers who prefer the offline channel. As discussed above, it can be expressed as

$$\phi_m = \begin{cases} 1 & \text{if } u_o - t < u_b - k; \\ 0 & \text{if } u_o - t \geq u_b - k. \end{cases}$$

Notice that the retailer can anticipate omni-customers' decisions ϕ_m, because it only depends on parameters which are assumed to be common knowledge. With a belief $\hat{\phi}_s$ on the proportion of store-only customers who purchase through the offline channel, the retailer anticipates there are a total of $\hat{\phi}_s D_s + \phi_m D_m$ offline customers. As the inventory is allocated to those customers proportionally, the unsatisfied BOPS demand is $\phi_m D_m (1 - \frac{q}{\hat{\phi}_s D_s + \phi_m D_m})^+$ and will switch to the online channel. Therefore, the retailer's expected profit can be expressed as

$$\begin{aligned} \pi_P(q) = & \underbrace{p_b \mathbb{E}[\min(\hat{\phi}_s D_s + \phi_m D_m, q)] - c_b q}_{\text{Profit from Offline Channel}} \\ & + \underbrace{(p_o - c_o)\mathbb{E}[(1-\phi_m)D_m + \phi_m D_m (1 - \frac{q}{\hat{\phi}_s D_s + \phi_m D_m})^+]}_{\text{Profit from Online Channel}}, \end{aligned} \tag{A8}$$

where the first term is the expected profit obtained from the offline channel, and the second term is the expected profit obtained from the online channel.

Define q_e and q_o as the optimal solution of (A8) when $(\hat{\phi}_s, \phi_m) = (1, 1)$ and $(0, 1)$, respectively. The former scenario represents both store-only customers and omni-customers choose the offline channel, while the latter represents only omni-customers purchase offline. Accordingly, we define the corresponding fill rate of the former scenario as $\zeta_e = \frac{\mathbb{E}[\min\{q_e, D_s + D_m\}]}{\mathbb{E}[D_s + D_m]}$.

Proposition A3 (REE under BOPS When $u_o < u_b$). *At the RE equilibrium, the retailer's optimal inventory level in the B&M store and customers' optimal channel choices are as follows:*

(i) if $k \leq \min(u_b - u_o + t, \zeta_e u_b)$, then $q^P = q_e$ and $(\phi_s^P, \phi_m) = (1, 1)$;

(ii) if $u_b - u_o + t \geq k > \zeta_e u_b$,

(ii-1) when $p_b - c_b > p_o - c_o$, $q^P = q_o$ and $(\phi_s^P, \phi_m) = (0, 1)$;

(ii-2) when $p_b - c_b \leq p_o - c_o$, $q^P = 0$ and $(\phi_s^P, \phi_m) = (0, 0)$;

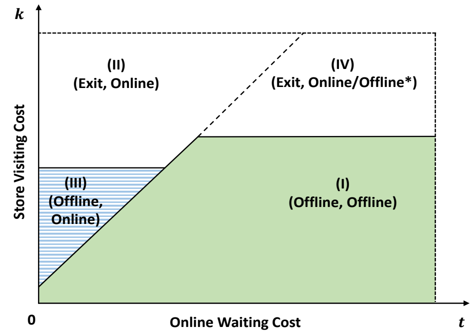
(iii) if $u_b - u_o + t < k \leq \zeta_s u_b$, then, $q^P = q_s$ and $(\phi_s^P, \phi_m) = (1, 0)$;

(iv) if $k > \max(u_b - u_o + t, \zeta_s u_b)$, then $q^P = 0$ and $(\phi_s^P, \phi_m) = (0, 0)$.

A fraction $(1 - \phi_m)$ of omni-customers will purchase through the online channel.

Figure A2 illustrates the RE equilibrium for the case with low online base surplus (i.e., $u_o < u_b$). We discuss the equilibrium by considering two cases: $k - t \geq u_b - u_o$ and $k - t < u_b - u_o$, which

Figure A2 Customer choice behavior under BOPS when $u_o < u_b$



Note. $*$In Area IV, the equilibrium is online if $p_b - c_b \leq p_o - c_o$ and otherwise, offline.

correspond to two separate regions divided by the 45° line in Figure A2. The explanation for the former case ($k - t \geq u_b - u_o$) including Areas II and III is similar to the counterpart of $u_o \geq u_b$. In the latter case ($k - t < u_b - u_o$), omni-customers always prefer the offline channel over the online channel. When the store visiting cost k is below a threshold (Area I), store-only customers will choose to shop offline. However, when the store visiting cost exceeds the threshold (Area IV), store-only customers will exit the market. Although omni-customers prefer the offline channel, the retailer can order nothing for the B&M store so as to force omni-customers to be fulfilled by DC. In particular, when $p_b - c_b > p_o - c_o$, it is more profitable for the retailer to serve omni-customers through the offline channel, and thus, order a positive quantity; otherwise, the retailer has the incentive to shut down the offline channel. This observation is slightly different from that under $u_o \geq u_b$. In the corresponding area under $u_o \geq u_b$ (see the part of Area II below the 45° line), store-only customers will also exit the market. If $p_o - c_o < p_b - c_b$, it is always more profitable for the retailer to close BOPS to force omni-customers to purchase online and be fulfilled by DC.

The following theorem presents the comparison between BOPS and BASE.

Theorem A2. (BOPS vs. BASE When $u_o < u_b$)

(i) If $\zeta_{sm} u_b - u_o + t < k \leq t + u_b - u_o$ (Areas I-2, I-3, and II-3), BOPS benefits the retailer.

(ii) If $\zeta_e u_b < k \leq \zeta_{sm} u_b - u_o + t$ (Area II-1), there exists $\delta_E(c_o)$ such that

(ii-1) if $p_o \geq \delta_E(c_o)$, BOPS benefits the retailer;

(ii-2) if $p_o < \delta_E(c_o)$, BOPS hurts the retailer.

(iii) If $k \leq \min(\zeta_e u_b, \zeta_{sm} u_b - u_o + t)$ (Area I-1), BOPS benefits the retailer.

D. Cross-Selling

In this section, we assume that there is a per unit cross-selling r for those customers visiting the B&M store. Note that customers who go to the store and experience a stockout still buy other stuff and add value by cross-buying. We incorporate this cross-selling revenue into the BASE and BOPS models and derive the corresponding equilibria as follows.

D.1. BASE Model with Cross-Selling

As the cross-selling does not directly affect customers' behavior, the corresponding analysis is identical to that in Section 4.1. Given the belief on the proportion of customers who choose to purchase offline, the retailer's expected profit function can be written as

$$\begin{aligned}\pi(q) \;=\; & p_b\mathbb{E}[\min(\hat{\phi}_m D_m + \hat{\phi}_s D_s, q)] - c_b q + (p_o - c_o)\mathbb{E}[(1-\hat{\phi}_m)D_m] \\ & + r\mathbb{E}[\hat{\phi}_m D_m + \hat{\phi}_s D_s], \end{aligned} \tag{A9}$$

where the last term represents the revenue from cross-selling in the B&M store.

Proposition A4 (REE for BASE With Cross-Selling). *At the RE equilibrium, retailer's inventory level assigned to the B&M store and customers' choices on the offline channel are as follows:*

(i) if $k < \zeta_{sm} u_b - (u_o - t)$, then $q^B = q_{sm}$ and $(\phi_s^B, \phi_m^B) = (1,1)$;

(ii) if $\zeta_{sm} u_b - (u_o - t) < k < \zeta_s u_b$, then $q^B = q_s$ and $(\phi_s^B, \phi_m^B) = (1,0)$;

(iii) if $k > \max(\zeta_s u_b, \zeta_{sm} u_b - (u_o - t))$, then $q^B = 0$ and $(\phi_s^B, \phi_m^B) = (0,0)$;

and $(1-\phi_m^B)$ proportion of omni-customers will choose the online channel.

D.2. BOPS Model with Cross-Selling

Under BOPS, the retailer's expected profit function can be expressed as

$$\begin{aligned}\pi(q) &= p_b\mathbb{E}[\min(\hat{\phi}_s D_s, q)] - c_b q + (p_o + r)\mathbb{E}[\min(\hat{\phi}_m D_m, (q - \hat{\phi}_s D_s)^+)] \\ &\quad + (p_o - c_o)\mathbb{E}[(1-\hat{\phi}_m)D_m + (\hat{\phi}_m D_m - (q - \hat{\phi}_s D_s)^+)^+] + r\mathbb{E}[\hat{\phi}_s D_s] \\ &= p_b\mathbb{E}[\min(\hat{\phi}_s D_s, q)] - c_b q + (r + c_o)\mathbb{E}[\min(\hat{\phi}_m D_m, (q - \hat{\phi}_s D_s)^+)] \\ &\quad + (p_o - c_o)\mathbb{E}[D_m] + r\mathbb{E}[\hat{\phi}_s D_s]. \end{aligned} \tag{A10}$$

One can readily verify that $\pi(q)$ is concave in q for any given $(\hat{\phi}_s, \hat{\phi}_m)$. Let $q_C(\hat{\phi}_s, \hat{\phi}_m)$ be the corresponding optimal solution of (A10). Moreover, we define $\zeta_C(\hat{\phi}_s, \hat{\phi}_m) = \frac{\mathbb{E}[q_C(\hat{\phi}_s, \hat{\phi}_m), D_s]}{\mathbb{E}[D_s]}$.

Lemma A1. *(i) $q_C(1,1) \geq q_s$ and $\zeta_C(1,1) \geq \zeta_s$.*

(ii) $q_C(0,1) \geq 0$ if and only if $r + c_o \geq c_b$.

(iii) $\zeta_C(1,1) \geq \zeta_C(0,1)$ and $\zeta_C(1,1) \geq \zeta_p$.

Given this lemma, we have the following proposition.

Proposition A5 (REE for BOPS With Cross-Selling). *At the RE equilibrium, the retailer's inventory level assigned to the B&M store and customers' choices on the offline channel are as follows:*

(i) if $k \leq \min(t, \zeta_C(1,1)u_b)$*, then* $q^P = q_C(1,1)$ *and* $(\phi_s^P, \phi_m^P) = (1,1)$*;*

(ii) if $t < k \leq \zeta_s u_b$*, then,* $q^P = q_s$ *and* $(\phi_s^P, \phi_m^P) = (1,0)$*;*

(iii) if $\zeta_C(1,1)u_b < k \leq t$*, then,*

(iii − 1) when $c_o + r > c_b$*,* $q^P = q_C(0,1)$ *and* $(\phi_s^P, \phi_m^P) = (0,1)$*;*

(iii − 2) when $c_o + r \leq c_b$*,* $q^P = 0$ *and* $(\phi_s^P, \phi_m^P) = (0,0)$*;*

(iv) if $k > \max(t, \zeta_s u_b)$*, then* $q^P = 0$ *and* $(\phi_s^P, \phi_m^P) = (0,0)$*.*

Moreover, all other unsatisfied omni-customers will buy from the online channel.

D.3. Comparison Between BASE and BOPS Models with Cross-Selling

The comparison result between BOPS and BASE is as follows.

Theorem A3. (Comparison between BOPS and BASE With Cross-Selling)

(i) If $\zeta_{sm}u_b - u_o + t < k \leq t$ *(Areas I-2, I-3 and II-2), then BOPS benefits the retailer.*

(ii) If $\zeta_C(1,1)u_b < k \leq \zeta_{sm}u_b - u_o + t$ *(Area II-1), then BOPS hurts the retailer.*

(iii) If $k \leq \min(\zeta_p u_b, \zeta_{sm}u_b - u_o + t)$ *(Area I-1), there exists a threshold* $\delta_C(c_o, r) \in [c_o, c_o + p_b - c_b + r + \Delta]$ *such that*

(iii-1) if $p_o < \delta_C(c_o, r)$*, BOPS hurts the retailer;*

(iii-2) if $p_o \geq \delta_C(c_o, r)$*, BOPS benefits the retailer.*

Moreover, $\delta_C(c_o, r)$ *is increasing in* r *and* $\delta_C(c_o, r) \geq \delta_S(c_o)$*.*

E. Recourse Behavior of Omni-Customers

In the base model, when omni-customers who visit the store experience a stock-out, they are assumed to be lost, instead of switching to the online channel. In this section, we attempt to relax this assumption by assuming that those omni-customers who experience a stock-out in store will switch to the online channel. This relaxation only influences the BASE model, but not the BOPS model. This is because under BOPS, as a feature of our model, omni-customers can always check inventory availability of the B&M store via BOPS before visiting the store and naturally switch to the online channel when facing a stock-out. We first characterize the RE equilibrium under the BASE model and then compare it with that under BOPS. Again, our analysis only focuses on the case with high online base surplus (i.e., $u_o \geq u_b$); the main results also carry over to the case with $u_o < u_b$.

BASE Model. Recall that customers who purchase via the offline channel may face a stockout risk. Without knowing the exact inventory in the B&M store, customers will first form a belief $\hat{\zeta}$ on the in-stock probability before making their purchase decisions. Based on this belief, customers' utilities from different channels can be expressed as:

$$U^B_{s,b} = \hat{\zeta} u_b - k, \quad U^B_{m,b} = \hat{\zeta} u_b - k + (1-\hat{\zeta})(u_o - t), \quad U^B_{m,o} = u_o - t. \tag{A11}$$

Different from the lost-sales case, omni-customers' utility from purchasing offline has an additional utility $u_o - t$ from the switch back to the online channel when the stock-out happens with the probability $(1-\hat{\zeta})$.

Customer Choice. Store-only customers purchase the product if $U^B_{s,b} \geq 0$; otherwise, they exit the market. For an omni-customer, she purchases the product from the offline channel if and only if $U^B_{m,b} \geq \max\{0, U^B_{m,o}\}$ and the online channel if and only if $U^B_{m,o} \geq \max\{0, U^B_{m,b}\}$.

Retailer's Decision. The retailer cannot differentiate store-only customers and omni-customers, when both choose to purchase offline. We assume that the inventory is allocated to those offline customers proportionally. Given the belief $(\hat{\phi}_s, \hat{\phi}_m)$ on the proportion of customers who choose the offline channel, the retailer's expected profit is given by

$$\begin{aligned} \bar{\pi}_B(q) = & \underbrace{p_b \mathbb{E}[\min(\hat{\phi}_m D_m + \hat{\phi}_s D_s, q)] - c_b q}_{\text{Profit from Offline Channel}} + \underbrace{(p_o - c_o)\mathbb{E}[(1-\hat{\phi}_m) D_m]}_{\text{Profit from Online Channel}} \\ & + \underbrace{(p_o - c_o)\mathbb{E}[(\hat{\phi}_m D_m - \frac{\hat{\phi}_m D_m}{\hat{\phi}_m D_m + \hat{\phi}_s D_s} q)^+]}_{\text{Profit from Online Channel}}, \end{aligned} \tag{A12}$$

where the last term represents the profit from those omni-customers who switch back to the online channel when experiencing a stock-out at store.

The optimal solution of

$$\bar{\pi}_B(q)|_{(\hat{\phi}_s, \hat{\phi}_m)=(1,1)} = p_b \mathbb{E}[\min(D_m + D_s, q)] - c_b q + (p_o - c_o)\mathbb{E}[(D_m - \frac{D_m}{D_m + D_s} q)^+],$$

is just q_e (a solution of (A8)), and the corresponding fill rate, i.e., $\mathbb{E}[\min(D_m + D_s, q_e)]/\mathbb{E}[D_m + D_s]$, is given by ζ_e, which represents the fill rate when both store-only customers and omni-customers visit the local store. Note that $\zeta_e \leq \zeta_{sm}$. However, ζ_e coulde be larger or smaller than ζ_s depending on the demand correlation.

For ease of exploration, we assume that $\zeta_e \geq \zeta_s$ (or equivalently, the demand correlation ρ is nonpositive); namely, we assume that the fill rate becomes weakly larger when the two streams of customers visit the offline store, compared to the case in which only store-only customers visit the offline store.

Proposition A6 (REE under BASE with Recourse Behavior of Omni-customers). *At the RE equilibrium, the inventory level in the B&M store and customers' channel choices in the offline channel are as follows:*

(i) if $k \leq \zeta_e(u_b - u_o + t)$*, then* $q^B = q_e$ *and* $(\phi_s^B, \phi_m^B) = (1,1)$*;*

(ii) if $\zeta_e(u_b - u_o + t) < k \leq \zeta_s u_b$*, then* $q^B = q_s$ *and* $(\phi_s^B, \phi_m^B) = (1,0)$*;*

(iii) if $k > \max(\zeta_s u_b, \zeta_e(u_b - u_o + t))$*, then* $q^B = 0$ *and* $(\phi_s^B, \phi_m^B) = (0,0)$*.*

A fraction $(1 - \phi_m^B)$ *of omni-customers will purchase through the online channel.*

When the store visiting cost is low, then both store-only customers and omni-customers will choose to purchase through the offline channel. As the store visiting cost increases to a moderate level, omni-customers will switch to the online channel because the utility of purchasing is higher online than offline, whereas store-only customers stick to the offline channel as the associated utility is still positive. When the store visiting cost is high, no customers will visit the B&M store. Moreover, those omni-customers who do not choose the offline channel will purchase online.

The following theorem demonstrates the comparison between BASE and BOPS when omni-customers who visit the store and face a stock-out will switch back to the online channel.

Theorem A4 (BOPS vs. BASE with Recourse Behavior by Omni-Customers).

(i) If $\zeta_e(u_b - u_o + t) < k \leq \min(t, \zeta_p u_b)$*, BOPS benefits the retailer.*

(ii) If $\zeta_p u_b < k \leq \zeta_e(u_b - u_o + t)$*, BOPS hurts the retailer.*

(iii) If $k \leq \min(\zeta_p u_b, \zeta_e(u_b - u_o + t))$*, there exists* $\delta_R(c_o) \geq \delta_S(c_o)$ *such that*

(iii-1) if $p_o < \delta_R(c_o)$*, BOPS hurts the retailer;*

(iii-2) if $p_o \geq \delta_R(c_o)$*, BOPS benefits the retailer.*

First, it can be seen from Theorem A4 that our main insights still hold. 1) BOPS may benefit or hurt the retailer depending on the store visiting cost and the online waiting cost. 2) The demand pooling effect still exists when the online waiting cost is relatively low and the store visiting cost is even lower. 3) In contrast, the demand depooling effect also exists when both the store visiting cost and the online waiting cost are relatively high, with the latter even higher.

Second, compared with the lost-sales case, the pooling/depooling effect could be strengthened or weakened. As the recourse behavior does not affect the BOPS model, it suffices to directly compare the BASE models with and without recourse (see Propositions 1 and A6). As the explanations for the impact of the recourse behavior on pooling and depooling effects are similar, we here only discuss how the recourse behavior influences the pooling effect.

Impact on Pooling Effect. Consider the area in which without the recourse behavior, omni-customers choose the online channel, while store-only customers exit the market in the BASE model. In this situation, without the recourse behavior, BOPS may trigger a pooling effect (the retailer would use the local inventory to serve both BOPS and store-only customers, resulting in a higher fill rate). However, the higher utility from the recourse behavior may incentivize omni-customers to switch from the online channel to the offline. That is, with the recourse behavior, both omni-customers and store-only customers can choose the offline channel in the BASE model. Then BOPS does not necessarily lead to a pooling effect. Therefore, the recourse behavior may lessen the pooling effect.

On the other hand, consider the area in which without the recourse behavior, both store-only customers and omni-customers choose the offline channel in the BASE model. In this situation, without the recourse behavior, BOPS may not trigger a pooling effect. The recourse behavior of omni-customers reduces the lost-sales cost in the B&M store. As a result, the retailer will order less inventory for the B&M store. It leads to a lower fill rate which forces omni-customers to choose the online channel in the BASE model. Consequently, the introduction of BOPS under the recourse behavior may incentivize omni-customers to visit the store, resulting in a pooling effect. That is, the recourse behavior could amplify the pooling effect.

F. Proofs of the Results in Online Appendix

Proof of Proposition A1. Let $\hat{\zeta}$ be the customers' belief on the fill rate and $(\hat{\phi}_m, \hat{\phi}_s)$ be the retailer's belief on the proportions of omni-customers and store-only customers who purchase offline. We first characterize customers' channel choices and retailer's optimal decision in terms of these private beliefs.

Customer Choice. Given customers' belief on the fill rate, let $(\bar{\phi}_m, \bar{\phi}_s)$ be the resulting proportions of omni-customers and store-only customers who purchase offline. Customers' utilities from purchasing online and offline channels are given by (2). Then, customers' channel choices are as follows: 1) store-only customers purchase the product if $U^B_{s,b} \geq 0$; otherwise, exit the market, or 2) omni-customers purchase the product from the offline channel if and only if $U^B_{m,b} \geq \max(0, U^B_{m,o})$ and from the online channel if and only if $U^B_{m,o} \geq \max(0, U^B_{m,b})$. In particular, we can further characterize customers' choices, $(\bar{\phi}_s, \bar{\phi}_m)$, in different parameter regions as follows:

(i) If $\hat{\zeta} u_b - k \geq 0$, then both store-only customers and omni-customers will choose the offline channel, i.e., $(\bar{\phi}_s, \bar{\phi}_m) = (1, 1)$.

(ii) If $\hat{\zeta}u_b - k < 0$, then both types of customers will exit the market, i.e., $(\bar{\phi}_s, \bar{\phi}_m) = (0,0)$.

Therefore, given customers' belief on the fill rate $\hat{\zeta}$, the resulting market segmentation $(\bar{\phi}_s, \bar{\phi}_m)$ is expressed as

$$(\bar{\phi}_s, \bar{\phi}_m) = \begin{cases} (1,1) \text{ if } \hat{\zeta} \geq \frac{k}{u_b}, \\ (0,0) \text{ if } \hat{\zeta} < \frac{k}{u_b}. \end{cases} \tag{A13}$$

Retailer's Optimal Decision. Given retailer's beliefs $(\hat{\phi}_m, \hat{\phi}_s)$ on customers' choices, let $\bar{q}$ and $\bar{\zeta}$ be the optimal order quantity for the B&M store and the corresponding resulting fill rate, respectively. As customers are homogeneous, there are three possible values of $(\hat{\phi}_s, \hat{\phi}_m)$: $(1,1)$, $(1,0)$, and $(0,0)$. We now analyze the retailer's optimal decision for these three cases.

(i) If $(\hat{\phi}_s, \hat{\phi}_m) = (1,1)$, the retailer's expected profit is $\pi(q) = p_b \mathbb{E}[\min(D_m + D_s, q)] - c_b q$. Then, the optimal order quantity is $\bar{q} = q_{sm}$, and the corresponding fill rate is $\bar{\zeta} = \zeta_{sm}$.

(ii) If $(\hat{\phi}_s, \hat{\phi}_m) = (1,0)$, the retailer's expected profit is $\pi(q) = p_b \mathbb{E}[\min(D_s, q)] - c_b q$. Consequently, the optimal order quantity is $\bar{q} = q_s$, and the corresponding fill rate is $\bar{\zeta} = \zeta_s$.

(iii) If $(\hat{\phi}_s, \hat{\phi}_m) = (0,0)$, the retailer's expected profit is $\pi(q) = 0$. Therefore, the retailer will order nothing from the B&M store and the resulting fill rate is zero, i.e., $\bar{q} = 0$ and $\bar{\zeta} = 0$.

In summary, given the belief $(\hat{\phi}_s, \hat{\phi}_m)$, the retailer's optimal decision of $\bar{q}$ and the corresponding fill rate $\bar{\zeta}$ can be characterized as

$$(\bar{q}, \bar{\zeta}) = \begin{cases} (q_{sm}, \zeta_{sm}) & \text{if } (\hat{\phi}_s, \hat{\phi}_m) = (1,1), \\ (q_s, \zeta_s) & \text{if } (\hat{\phi}_s, \hat{\phi}_m) = (1,0), \\ (0,0) & \text{if } (\hat{\phi}_s, \hat{\phi}_m) = (0,0). \end{cases} \tag{A14}$$

RE Equilibrium. Next, we analyze the RE equilibrium, denoted by $(q^B, \phi_m^B, \phi_s^B)$, by connecting customers' choices with the retailer's optimal decision. The concept of RE equilibrium states 1) customers' belief on the fill rate is exactly the realized fill rate, and 2) retailer's belief on customers' choices is just consistent with the realized one. Specifically, we define the RE equilibrium as follows.

Definition A3 (RE Equilibrium Under BASE). A RE equilibrium

$$(\bar{\zeta}, \hat{\zeta}, \bar{\phi}_m, \bar{\phi}_s, \hat{\phi}_m, \hat{\phi}_s, \bar{q})$$

should satisfy the following conditions:

(i) Given $\hat{\zeta}$, $(\bar{\phi}_s, \bar{\phi}_m)$ satisfies Equation (A13);

(ii) Given $(\hat{\phi}_s, \hat{\phi}_m)$, $(\bar{q}, \bar{\zeta})$ satisfies Equation (A14);

(iii) $\bar{\zeta} = \hat{\zeta}$ and $(\bar{\phi}_s, \bar{\phi}_m) = (\hat{\phi}_s, \hat{\phi}_m)$.

Similar to the proof of Proposition 1, one can obtain the desired result by checking Definition A3. $\square$

Proof of Proposition A2. As $t > u_o$, omni-customers will never choose the online channel. We use the same notation as in Proposition 2.

Customer Choice. Given the belief on the fill rate $\hat{\zeta}$, customers' channel choices are as follows: 1) store-only customers purchase the product if $U^P_{s,b} \geq 0$; otherwise, they exit the market; 2) omni-customers purchase the product from the BOPS channel if and only if $U^P_{m,ob} \geq 0$; otherwise, they exit the market. The market segmentation is characterized as follows.

(i) If $k > u_o \geq u_b$, both omni-customers and store-only customers will exit the market. That is, the demand for all channels is zero.

(ii) If $u_o \geq k > \hat{\zeta} u_b$, omni-customers will choose the BOPS channel, whereas store-only customers will exit the market. This means that the total demand for BOPS is D_m and that for the B&M store is zero.

(iii) If $\hat{\zeta} u_b \geq k$, omni-customers again choose the BOPS channel, whereas store-only customers will buy from the offline channel. Then, the total demand for BOPS is D_m and that for the B&M store is D_s.

In summary, given customers' belief on the fill rate $\hat{\zeta}$, the resulting market segmentation is given by

$$(\bar{\phi}_s, \phi_m) = \begin{cases} (1,1) \text{ if } \hat{\zeta} \geq \frac{k}{u_b}, \\ (0,1) \text{ if } \hat{\zeta} < \frac{k}{u_b} \text{ and } k \leq u_o, \\ (0,0) \text{ if } k > u_o. \end{cases} \tag{A15}$$

Retailer's Optimal Decision. Depending on the above analysis, we can restrict the retailer's belief on market segmentation $(\hat{\phi}_s, \phi_m)$ within $(0,0)$, $(0,1)$, and $(1,1)$.

(i) If $(\hat{\phi}_s, \phi_m) = (1,1)$, the retailer's expected profit is $\pi(q) = p_b \mathbb{E}[\min(D_s, q)] - c_b q + p_o \mathbb{E}[\min(D_m, (q - D_s)^+)]$. The optimal order quantity is $\bar{q} = q_P^{(1)}$, and the corresponding fill rate is $\bar{\zeta} = \zeta_p^{(1)}$.

(ii) If $(\hat{\phi}_s, \phi_m) = (0,1)$, the retailer's expected profit is $\pi(q) = p_o \mathbb{E}[\min(D_m, q)] - c_b q$. The optimal order quantity is $\bar{q} = q_m$, and the corresponding fill rate is $\hat{\zeta} = \zeta_m = \frac{\mathbb{E}[\min(D_m, q_m)]}{\mu_m}$. However, the fill rate for store-only customers is zero.

(iii) If $(\hat{\phi}_s, \phi_m) = (0,0)$, the retailer does not stock any inventory in the B&M store as there is no offline demand.

In summary, given the retailer's belief, the optimal quantity $\bar{q}$ and the corresponding fill rate $\bar{\zeta}$ are

$$(\bar{q}, \bar{\zeta}) = \begin{cases} (q_P^{(1)}, \zeta_p^{(1)}) \text{ if } (\hat{\phi}_s, \phi_m) = (1,1), \\ (q_m, \zeta_m) \text{ if } (\hat{\phi}_s, \phi_m) = (0,1), \\ (0,0) \text{ if } (\hat{\phi}_s, \phi_m) = (0,0). \end{cases} \tag{A16}$$

RE Equilibrium.

Definition A4 (RE Equilibrium Under BOPS with $t > u_o$).

A RE equilibrium $(\bar{\zeta}, \hat{\zeta}, \bar{\phi}_s, \hat{\phi}_s, \bar{q})$ should satisfy the following conditions:

(i) Given $\hat{\zeta}$ and ϕ_m, $\bar{\phi}_s$ satisfies Equation (A15);

(ii) Given $\hat{\phi}_s$ and ϕ_m, $(\bar{q}, \bar{\zeta})$ satisfies Equation (A16);

(iii) $\bar{\zeta} = \hat{\zeta}$ and $\bar{\phi}_s = \hat{\phi}_s$.

We characterize the RE equilibria by considering two cases.

Case 1: $k \le \zeta_p^{(1)} u_b$. In this case, we have $\phi_m = 1$ and two potential equilibria with $(q^P, \phi_s^P) = (q_P^{(1)}, 1)$ or $(q_m, 0)$. The equilibrium $(q_P^{(1)}, 1)$ always exists, whereas $(q_m, 0)$ exists only when $k > \zeta_m u_b$. Otherwise, when both customers believe that the retailer holds the inventory level at q_m, store-only customers will purchase as $k \le \zeta_m u_b$. Meanwhile, when the retailer expects store-only customers' purchasing, she will order $q_P^{(1)}$. Thus, the self-fulfilling prophecy fails. Nevertheless, we next show that $(q_P^{(1)}, 1)$ Pareto-dominates $(q_m, 0)$ regardless of whether $(q_m, 0)$ is an equilibrium or not.

Under $(q_P^{(1)}, 1)$, the market segmentation is $(\hat{\phi}_s, \phi_m) = (1, 1)$, whereas under $(q_m, 0)$, it is $(\hat{\phi}_s, \phi_m) = (1, 0)$. Clearly, the retailer's expected profit under the former is larger than that under the latter. Because $p_b > p_o$, we have $q_m < \mu_m + \sigma_m \Phi^{-1}(1 - \frac{c_b}{p_b}) = q_s$ and the corresponding fill rate is $\zeta_m < \zeta_s$. Thus, by Lemma 2, $\zeta_m < \zeta_p^{(1)}$, which implies that customers will gain a higher utility from the equilibrium $(q_P^{(1)}, 1)$.

Case 2: $u_o \ge k > \zeta_p^{(1)} u_b$. In this case, we can verify that $\phi_m = 1$ and there is only one equilibrium $(q^P, \phi_s^P) = (q_m, 0)$.

Combining the above cases will yield the desired results. $\square$

Proof of Theorem A1. We prove the results by comparing retailer's expected profits in various areas as illustrated in Figure A1.

- $k \le \min\{\zeta_p^{(1)}, \zeta_{sm}\} u_b$ (i.e., Area (I)). By Proposition 1(i) and Proposition 2(i), the retailer's expected profits in the BASE and BOPS models are respectively given by

$$\begin{aligned}
\Pi_B &= \max_q \{p_b \mathbb{E}[\min(D_m + D_s, q)] - c_b q\};\\
\Pi_P &= \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q + p_o \mathbb{E}[\min(D_m, (q - D_s)^+)]\}.
\end{aligned}$$

Because $p_b \ge p_o$, we have

$$\begin{aligned}
\Pi_P &= \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q + p_o \mathbb{E}[\min(D_m, (q - D_s)^+)]\}\\
&\le \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q + p_b \mathbb{E}[\min(D_m, (q - D_s)^+)]\}\\
&= \Pi_B.
\end{aligned}$$

- $\zeta_p^{(1)} u_b < k \le \zeta_{sm} u_b$ (i.e., Area (II-1)). By Proposition 1(i) and Proposition 2(ii), the retailer's expected profits in the BASE and BOPS models are respectively given by

$$\Pi_B = \max_q \{p_b \mathbb{E}[\min(D_s + D_m, q)] - c_b q\};$$
$$\Pi_P = \max_q \{p_o \mathbb{E}[\min(D_m, q)] - c_b q\}.$$

Then, $\Pi_B \ge \Pi_P$ as $p_o \le p_b$.

- $\zeta_{sm} u_b < k$ (i.e., Area (II-2)). Because $\zeta_{sm} \ge \zeta_s$, this case in the BASE model corresponds to Proposition 1(iii). Thus, $\Pi_B = 0$. Because Π_P is always nonnegative, $\Pi_B \le \Pi_P$.

Therefore, we can obtain the desired results by combing the above cases. □

Proof of Proposition A3. Similar to the proof logic of Propositions 1 and 2, we can characterize the customers' choice as follows.

Customer Choice. Given customers' belief on the fill rate $\hat{\zeta}$, we can analyze the resulting market segmentation as follows.

(i) If $k \le u_b - u_o + t$ and $k \le \hat{\zeta} u_b$, both store-only customers and omni-customers will go to the offline channel. Then, the total demand for the B&M store is $D_s + D_m$, and those omni-customers who find a stockout will switch to the online channel.

(ii) If $k \le u_b - u_o + t$ and $k > \hat{\zeta} u_b$, store-only customers will exit the market, and omni-customers will go to the offline channel. Then, the total demand for the offline channel is D_m.

(iii) If $k > u_b - u_o + t$ and $k \le \hat{\zeta} u_b$, store-only customers will go to the B&M store, whereas omni-customers will go to the online channel. Then, the total demand for the online channel is D_m and that for the offline channel is D_s.

(iv) If $k > u_b - u_o + t$ and $k > \hat{\zeta} u_b$, store-only customers will exit the market, and omni-customers will go to the online channel. Then, the total demand for the online channel is D_m and that for the offline channel is 0.

In summary, given customers' belief on the fill rate $\hat{\zeta}$, the resulting market segmentation is given by

$$(\bar{\phi}_s, \phi_m) = \begin{cases} (1,1) \text{ if } \hat{\zeta} \ge \frac{k}{u_b} \text{ and } k \le u_b - u_o + t, \\ (0,1) \text{ if } \hat{\zeta} < \frac{k}{u_b} \text{ and } k \le u_b - u_o + t, \\ (1,0) \text{ if } \hat{\zeta} \ge \frac{k}{u_b} \text{ and } k > u_b - u_o + t, \\ (0,0) \text{ if } \hat{\zeta} < \frac{k}{u_b} \text{ and } k > u_b - u_o + t. \end{cases} \tag{A17}$$

Retailer's Optimal Decision. We now analyze the retailer's optimal decision given his beliefs on the market segmentation $(\hat{\phi}_s, \phi_m)$. By the above analysis on the market segmentation, we can see that the feasible market segmentation $(\hat{\phi}_s, \phi_m)$ will be $(1,1)$, $(1,0)$, $(0,1)$, and $(0,0)$. Thus, we discuss the following four cases.

(i) If $(\hat{\phi}_s, \phi_m) = (1,1)$, the retailer's expected profit is $\pi(q) = p_b \mathbb{E}[\min(D_s + D_m, q)] - c_b q + (p_o - c_o)\mathbb{E}[(D_m - \frac{D_m}{D_s + D_m} q)^+]$. The resulting optimal order quantity is $\bar{q} = q_e$, and the corresponding fill rate is $\bar{\zeta} = \zeta_e$.

(ii) If $(\hat{\phi}_s, \phi_m) = (0,1)$, the retailer's expected profit is $\pi(q) = p_b \mathbb{E}[\min(D_m, q)] - c_b q + (p_o - c_o)\mathbb{E}[(D_m - q)^+]$. The resulting optimal order quantity is $\bar{q} = q_o$. Define the corresponding fill rate of the local store by $\zeta_o = \frac{\mathbb{E}\{D_m, q_o\}}{\mathbb{E}[D_m]}$.

(iii) If $(\hat{\phi}_s, \phi_m) = (1,0)$, the retailer's expected profit is $\pi(q) = p_b \mathbb{E}[\min(D_s, q)] - c_b q + (p_o - c_o)\mathbb{E}[D_m]$. The resulting optimal order quantity is $\bar{q} = q_s$, and the corresponding fill rate is ζ_s.

(iv) If $(\hat{\phi}_s, \phi_m) = (0,0)$, the retailer's expected profit is $\pi(q) = (p_o - c_o)\mathbb{E}[D_m]$. The optimal order quantity is $\bar{q} = 0$ and the fill rate is 0.

Therefore, given retailer's belief $(\hat{\phi}_s, \phi_m)$, the optimal order quantity $\bar{q}$ and the corresponding fill rate $\bar{\zeta}$ are given by

$$(\bar{q}, \bar{\zeta}) = \begin{cases} (q_e, \zeta_e) \text{ if } (\hat{\phi}_s, \phi_m) = (1,1), \\ (q_o, \zeta_o) \text{ if } (\hat{\phi}_s, \phi_m) = (0,1), \\ (q_s, \zeta_s) \text{ if } (\hat{\phi}_s, \phi_m) = (1,0), \\ \ (0,0) \ \text{ if } (\hat{\phi}_s, \phi_m) = (0,0). \end{cases} \tag{A18}$$

RE Equilibrium. We now turn to analyze the RE equilibrium, which is denoted by (q^P, ϕ_s^P).

Definition A5 (RE Equilibrium Under BOPS When $u_o < u_b$).

A RE equilibrium $(\bar{\zeta}, \hat{\zeta}, \bar{\phi}_s, \hat{\phi}_s, \bar{q})$ should satisfy the following conditions:

(i) Given $\hat{\zeta}$ and ϕ_m, $\bar{\phi}_s$ satisfies Equation (A17);

(ii) Given $\hat{\phi}_s$ and ϕ_m, $(\bar{q}, \bar{\zeta})$ satisfies Equation (A18);

(iii) $\bar{\zeta} = \hat{\zeta}$ and $\bar{\phi}_s = \hat{\phi}_s$.

According to Definition A5, we consider the following four cases.

Case 1: $k \le u_b - u_o + t$ and $k \le \zeta_e u_b$. By (A17), we have $\phi_m = 1$, and thus by (A18), we have three potential equilibria, i.e., (q_e, ζ_e), (q_o, ζ_o) and $(0,0)$. Similar to Proposition 1, one can verify that all are RE equilibria and that (q_e, ζ_e) is Pareto-dominant. Thus, $q^P = q_e$ and $\phi_s^P = 1$.

Case 2: $k \le u_b - u_o + t$ and $k > \zeta_e u_b$. Again, we have three potential equilibria, i.e., (q_e, ζ_e), (q_o, ζ_o) and $(0,0)$. However, as $k > \zeta_e u_b$, the store-only customers will exit the market and thus, (q_e, ζ_e) is not a RE equilibrium. Notice that the equilibrium $(0,0)$ means that the retailer does not store any inventory in the B&M store so that omni-customers have to choose the online channel. If $p_b - c_b > p_o - c_o$, it is more profitable for the retailer to serve omni-customers through the offline channel and thus, (q_o, ζ_o) dominates $(0,0)$. Thus, there exists only a unique equilibrium, i.e., $q^P = q_o$ and $\phi_s^P = 0$. On the other hand, when $p_b - c_b \le p_o - c_o$, it is more profitable for the retailer to serve omni-customers through the online channel and thus, $(0,0)$ dominates (q_o, ζ_o), i.e., the only unique equilibrium is $q^P = 0$ and $\phi_s^P = 0$.

Case 3: $k > u_b - u_o + t$ and $k \leq \zeta_s u_b$. In this case, $\phi_m = 0$ and there are two potential equilibria, i.e., (q_s, ζ_s) and $(0,0)$. Both are RE equilibria. The former is Pareto-dominant and thus $q^P = q_s$ and $\phi_s^P = 1$.

Case 4: $k > u_b - u_o + t$ and $k > \zeta_s u_b$. In this case, $\phi_m = 0$ and there are two potential equilibria, i.e., (q_s, ζ_s) and $(0,0)$. However, as $k > \zeta_s u_b$, the store-only customers will exit the market and thus, (q_s, ζ_s) is not a RE equilibrium. The only equilibrium is $(q^P, \phi_s^P) = (0,0)$.

Summarizing the above four cases leads us to the proposition. □

Proof of Theorem A2. The results can be derived by comparing retailer's expected profits in various areas. We consider the following five cases.

Case 1: $k \leq \min(\zeta_e u_b, \zeta_{sm} u_b - u_o + t)$ (i.e., Area I-1). On the one hand, by the equilibrium analysis in Online Supplement A, the retailer's expected profit in the BOPS model is given by

$$\Pi_P = \max_q \left\{ p_b \mathbb{E}[\min(D_s + D_m, q)] - c_b q + (p_o - c_o)\mathbb{E}[(D_m - \frac{D_m}{D_s + D_m} q)^+] \right\}.$$

On the other hand, the retailer's expected profit in the BASE model is

$$\Pi_B = \max_q \{ p_b \mathbb{E}[\min(D_m + D_s, q)] - c_b q \}.$$

Clearly, $\Pi_B \leq \Pi_P$.

Case 2: $\zeta_{sm} u_b - u_o + t < k \leq \min(t, \zeta_s u_b)$ (i.e., Area I-2). By Proposition 1(ii) and Proposition A3, the retailer's expected profits in the BASE and BOPS models are respectively given by

$$\begin{aligned} \Pi_B &= \max_q \{ p_b \mathbb{E}[\min(D_s, q)] - c_b q + (p_o - c_o)\mathbb{E}[D_m] \}; \\ \Pi_P &= \max_q \{ p_b \mathbb{E}[\min(D_s + D_m, q)] - c_b q + (p_o - c_o)\mathbb{E}[(D_m - \frac{D_m}{D_s + D_m} q)^+] \}. \end{aligned}$$

Clearly, $\Pi_B \leq \Pi_P$, as Assumption (M) implies that

$$\max_q \{ p_b \mathbb{E}[\min(D_s, q)] - c_b q + (p_o - c_o)\mathbb{E}[D_m] \} \leq \max_q \{ p_b \mathbb{E}[\min(D_s + D_m, q)] - c_b q \}.$$

Case 3: $\max(\zeta_{sm} u_b - u_o + t, \zeta_s u_b) < k \leq \min(t, \zeta_e u_b)$ (i.e., Area I-3). By Proposition 1(iii) and Proposition A3(i), the retailer's expected profits in the BASE and BOPS models are given by

$$\begin{aligned} \Pi_B &= \max_q \{ (p_o - c_o)\mathbb{E}[D_m] \}; \\ \Pi_P &= \max_q \{ p_b \mathbb{E}[\min(D_s + D_m, q)] - c_b q + (p_o - c_o)\mathbb{E}[(D_m - \frac{D_m}{D_s + D_m} q)^+] \}. \end{aligned}$$

Then, it directly follows that $\pi_B \leq \pi_P$ as

$$\begin{aligned}\Pi_P &= \max_q \{p_b \mathbb{E}[\min(D_s+D_m,q)] - c_b q + (p_o-c_o)\mathbb{E}[(D_m - \frac{D_m}{D_s+D_m}q)^+]\} \\ &\geq \max_q \{p_b \mathbb{E}[\min(D_m,q)] - c_b q + (p_o-c_o)\mathbb{E}[(D_m-q)^+]\} \geq (p_o-c_o)\mathbb{E}[D_m].\end{aligned}$$

Case 4: $\max(\zeta_{sm}u_b - u_o + t, \zeta_e u_b) < k \leq t + u_b - u_o$ (i.e., Area II-2). By Proposition 1(iii) and Proposition A3(ii), the retailer's expected profits in the BASE and BOPS models (when $p_b - c_b > p_o - c_o$) are given by $\Pi_B = \max_q\{(p_o-c_o)\mathbb{E}[D_m]\}$ and $\Pi_P = \max_q\{p_b\mathbb{E}[\min(D_m,q)] - c_b q + (p_o - c_o)\mathbb{E}[(D_m-q)^+]\}$, Then, it directly follows that $\Pi_B \leq \Pi_P$. On the other hand, when $p_b - c_b \leq p_o - c_o$, we have that $\Pi_B \leq \Pi_P$.

Case 5: $\zeta_e u_b < k \leq \zeta_{sm}u_b - u_o + t$ (i.e., Area II-1). By Proposition 1(i) and Proposition A3(iii), we have $\Pi_B = \max_q\{p_b\mathbb{E}[\min(D_s+D_m,q)] - c_b q\}$ and $\Pi_P = \max_q\{p_b\mathbb{E}[\min(D_m,q)] - c_b q + (p_o - c_o)\mathbb{E}[(D_m-q)^+]\}$, respectively.

We can easily verify that when $p_o \to c_o$, we have $\Pi_P \leq \Pi_B$; whereas when $p_o \to \infty$, $\Pi_P > \Pi_B$. Moreover, Π_P is increasing in p_o and Π_B is independent of p_o. Therefore, there exists a unique $\delta_E(c_o)$ such that

$$\max_q\{p_b\mathbb{E}[\min(D_s+D_m,q)] - c_b q\} = \max_q\{p_b\mathbb{E}[\min(D_m,q)] - c_b q + (\delta_E(c_o) - c_o)\mathbb{E}[(D_m-q)^+]\}.$$

If such $\delta_E(c_o)$ violates Assumption (M), we just let $\delta_E(c_o) = p_b - c_b + \Delta + c_o$. Consequently, BOPS benefits the retailer if and only if $p_o \geq \delta_E(c_o)$. $\square$

Proof of Proposition A4. Customers' choices are the same as those in Proposition 1, and thus, we obtain

$$(\bar{\phi}_s, \bar{\phi}_m) = \begin{cases} (1,1) \text{ if } \hat{\zeta} \geq \frac{u_o - t + k}{u_b}, \\ (1,0) \text{ if } \frac{k}{u_b} \leq \hat{\zeta} < \frac{u_o-t+k}{u_b}, \\ (0,0) \text{ if } \hat{\zeta} < \frac{k}{u_b}. \end{cases} \tag{A19}$$

We start the analysis with retailer's decisions as follows.

Retailer's Optimal Decisions. We have the following three cases.

(i) If $(\hat{\phi}_s, \hat{\phi}_m) = (1,1)$, the retailer's expected profit is $\pi(q) = p_b\mathbb{E}[\min(D_m + D_s, q)] - c_b q + r\mathbb{E}[D_m + D_s]$, and the resulting optimal order quantity is $\bar{q} = q_{sm}$. Then, the corresponding fill rate is $\bar{\zeta} = \zeta_{sm}$.

(ii) If $(\hat{\phi}_s, \hat{\phi}_m) = (1,0)$, the retailer's expected profit is $\pi(q) = p_b\mathbb{E}[\min(D_s,q)] - c_b q + (p_o - c_o)\mathbb{E}[D_m] + r\mathbb{E}[D_s]$. Then, the resulting optimal order quantity is $\bar{q} = q_s$, and the corresponding fill rate is $\bar{\zeta} = \zeta_s$.

(iii) If $(\hat{\phi}_s, \hat{\phi}_m) = (0,0)$, the retailer's expected profit is $\pi(q) = (p_o - c_o)\mathbb{E}[D_m]$, and the resulting fill rate is 0.

RE Equilibrium. The equilibrium analysis is similar to Proposition 1. Under the condition $p_o - c_o \leq p_b - c_b + \Delta$, we can obtain the desired equilibria by checking the definition of RE equilibrium. $\square$

Proof of Lemma A1. Part (i) is similar to Lemma 2 as $r \geq 0$. We now prove parts (ii) and (iii) as follows.

(ii) From Equation (A10), $q_C(0,1)$ satisfies the first order condition as

$$\frac{\partial \pi(q)}{\partial q} = \frac{\partial[(r+c_o)\mathbb{E}[\min(D_m, q)] - c_b q}{\partial q} = 0.$$

Clearly, $q_C(0,1) \geq 0$ if and only if $r + c_o \geq c_b$.

(iii) By the optimality condition, $q_C(1,1)$ satisfies the first order condition

$$\begin{aligned}\frac{\partial \pi(q)}{\partial q} &= \frac{\partial[p_b\mathbb{E}[\min(D_s, q)] - c_b q + (r+c_o)\mathbb{E}[\min(D_m, (q-D_s)^+)]}{\partial q} \\ &\geq \frac{\partial[(r+c_o)\mathbb{E}[\min(D_m, q)] - c_b q}{\partial q},\end{aligned}$$

which implies that $q_C(1,1) \geq q_C(0,1)$. As the fill rate increases in the order quantity, it follows that $\zeta_C(1,1) \geq \zeta_C(0,1)$.

Finally, we prove $\zeta_C(1,1) \geq \zeta_p$. Notice that $q_C(1,1)$ satisfies the first order condition

$$\frac{\partial \pi(q)}{\partial q} = \frac{\partial[\mathbb{E}[p_b \min(D_s, q)] - c_b q + (r+c_o)\mathbb{E}[\min(D_m, (q-D_s)^+)]}{\partial q} = 0,$$

and q_p satisfies

$$\frac{\partial \pi(q)}{\partial q} = \frac{\partial[\mathbb{E}[p_b \min(D_s, q)] - c_b q + c_o\mathbb{E}[\min(D_m, (q-D_s)^+)]}{\partial q} = 0.$$

Because $r \geq 0$, it is obvious that $q_C(1,1) \geq q_p$, and thus, $\zeta_C(1,1) \geq \zeta_p$. $\square$

Proof of Proposition A5. Customers' choices are the same as those under BOPS in Section 4 of **?**. We next analyze the retailer's optimal decisions.

Retailer's Optimal Decision. The feasible market segmentation $(\hat{\phi}_s, \phi_m)$ will be $(1,1)$, $(1,0)$, $(0,1)$, and $(0,0)$. Omni-customers who experience a stock-out case in the B&M store will go to the online store and unsatisfied store-only customers will be lost. Thus, we have the following four cases.

(i) If $(\hat{\phi}_s, \phi_m) = (1,1)$, the retailer's expected profit is $\pi(q) = p_b\mathbb{E}[\min(D_s, q)] - c_b q + (c_o + r)\mathbb{E}[\min(D_m, (q-D_s)^+)] + (p_o - c_o)\mathbb{E}[D_m] + r\mathbb{E}[D_s]$. The resulting optimal order quantity $\bar{q} = q_C(1,1)$, and the corresponding fill rate is $\zeta_C(1,1)$.

(ii) If $(\hat{\phi}_s, \phi_m) = (0,1)$, the retailer's expected profit is $\pi(q) = (c_o + r)\mathbb{E}[\min(D_m, q)] - c_b q + (p_o - c_o)\mathbb{E}[D_m]$. Then, when $r + c_o > c_b$, the resulting optimal order quantity $\bar{q} = q_C(0,1)$, and the corresponding fill rate is $\zeta_C(0,1)$; when $r + c_o \le c_b$, $\bar{q} = 0$ and the fill rate is zero.

(iii) If $(\hat{\phi}_s, \phi_m) = (1,0)$, the retailer's expected profit is $\pi(q) = p_b\mathbb{E}[\min(D_s, q)] - c_b q + (p_o - c_o)\mathbb{E}[D_m] + r\mathbb{E}[D_s]$. The resulting optimal order quantity is $\bar{q} = q_s$, and the corresponding fill rate is ζ_s.

(iv) If $(\hat{\phi}_s, \phi_m) = (0,0)$, the retailer's expected profit is $\pi(q) = (p_o - c_o)\mathbb{E}[D_m]$, and the resulting fill rate is 0.

RE Equilibrium. Given customers' purchasing behavior and retailer's ordering policy, by checking the definition of RE equilibrium as that for Proposition 2, we can obtain the desired result. □

Proof of Theorem A3. The results can be derived by comparing retailer's expected profits in various areas.

• $k \le \min(\zeta_C(1,1)u_b, \zeta_{sm}u_b - u_o + t)$ (i.e., Area (I-1)). Because $u_o \ge u_b$, $k \le \min(\zeta_C(1,1)u_b, \zeta_{sm}u_b - u_o + t) \le \min(\zeta_C(1,1)u_b, t)$. By Proposition A5(i), the retailer's expected profit under BOPS is given by

$$\Pi_P = \max_q \left\{ p_b\mathbb{E}[\min(D_s, q)] - c_b q + (c_o + r)\mathbb{E}[\min(D_m, (q - D_s)^+)] + (p_o - c_o)\mathbb{E}[D_m] + r\mathbb{E}[D_s] \right\}.$$

By Proposition A4(i), the retailer's expected profit in the BASE model is

$$\begin{aligned}\Pi_B &= \max_q \{p_b\mathbb{E}[\min(D_m + D_s, q)] - c_b q + r\mathbb{E}[D_s + D_m]\} \\ &= \max_q \left\{ p_b\mathbb{E}[\min(D_s, q)] + p_b\mathbb{E}[\min(D_m, (q - D_s)^+)] - c_b q + r\mathbb{E}[D_s + D_m] \right\}.\end{aligned}$$

Then, when $p_o - c_o \to 0$, i.e., $p_o \to c_o$, we have $\Pi_B \ge \Pi_P$ as

$$\begin{aligned}\Pi_P &= \max_q \{p_b\mathbb{E}[\min(D_s, q)] - c_b q + (c_o + r)\mathbb{E}[\min(D_m, (q - D_s)^+)] + r\mathbb{E}[D_s]\} \\ &\le \max_q \{p_b\mathbb{E}[\min(D_s, q)] - c_b q + p_b\mathbb{E}[\min(D_m, (q - D_s)^+)] + r\mathbb{E}[D_s + D_m]\} = \Pi_B,\end{aligned}$$

where the inequality holds by $c_o \le c_b \le p_b$ and $\mathbb{E}[\min(D_m, (q - D_s)^+)] \le \mathbb{E}[D_m]$.

If $p_o - c_o \to p_b - c_b + r + \Delta$, we have $\Pi_B < \Pi_P$ as

$$\begin{aligned}\Pi_P &= \max_q \{p_b\mathbb{E}[\min(D_s, q)] - c_b q + (c_o + r)\mathbb{E}[\min(D_m, (q - D_s)^+)] + (p_b - c_b + r + \Delta)\mu_m + r\mu_s\} \\ &= \max_q \{p_b\mathbb{E}[\min(D_s, q)] - c_b q + (c_o + r)\mathbb{E}[\min(D_m, (q - D_s)^+)]\} \\ &\quad + \max_q \{p_b\mathbb{E}[\min(D_m + D_s, q)] - c_b q\} - \max_q \{p_b\mathbb{E}[\min(D_s, q)] - c_b q\} + r(\mu_m + \mu_s)\end{aligned}$$

$$\geq \max_q \{p_b \mathbb{E}[\min(D_m + D_s, q)] - c_b q\} + r(\mu_m + \mu_s) = \Pi_B.$$

Therefore, as Π_P is increasing in p_o, there exists a threshold $\delta_C \in [c_o, c_o + p_b - c_b + r + \Delta]$ such that when $p_o < \delta_C$, $\Pi_B \geq \Pi_P$, and when $p_o \geq \delta_C$, $\Pi_B < \Pi_P$. Moreover, as $\Pi_p - \Pi_c$ is decreasing in r, we have that:

- $\zeta_{sm} u_b - u_o + t < k \leq \min(t, \zeta_s u_b)$ (i.e., Area (I-2)). By Proposition A4(ii) and Proposition A5(i), the retailer's expected profits in the BASE and BOPS models are respectively given by

$$\Pi_B = \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q + (p_o - c_o)\mathbb{E}[D_m]\} + r\mu_s;$$
$$\Pi_P = \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q + (c_o + r)\mathbb{E}[\min(D_m, (q - D_s)^+)] + (p_o - c_o)\mathbb{E}[D_m] + r\mu_s\}.$$

Clearly, $\Pi_B \leq \Pi_P$.

- $\max(\zeta_{sm} u_b - u_o + t, \zeta_s u_b) < k \leq \min(t, \zeta_C(1,1) u_b)$ (i.e., Area (I-3)). By Proposition A4(iii) and Proposition A5(i), the retailer's expected profits in the BASE and BOPS models are given by

$$\Pi_B = \max_q \{(p_o - c_o)\mathbb{E}[D_m]\};$$
$$\Pi_P = \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q + (c_o + r)\mathbb{E}[\min(D_m, (q - D_s)^+)] + (p_o - c_o)\mathbb{E}[D_m] + \mu_s\}.$$

Then, it directly follows that $\Pi_B \leq \Pi_P$.

- $\max(\zeta_{sm} u_b - u_o + t, \zeta_C(1,1) u_b) < k \leq t$ (i.e., Area (II-2)). By Proposition A4(iii) and Proposition A5(iii), the retailer's expected profits in the BASE and BOPS models are respectively given by

$$\Pi_B = \max_q \{(p_o - c_o)\mathbb{E}[D_m]\};$$
$$\Pi_P = \max_q \{(c_o + r)\mathbb{E}[\min(D_m, q)] - c_b q + (p_o - c_o)\mathbb{E}[D_m]\}.$$

Clearly, $\Pi_B \leq \Pi_P$.

- $\zeta_C(1,1) u_b < k \leq \zeta_{sm} u_b - u_o + t$ (i.e., Area (II-1)). By Propositions A4(i) and A5(iv), we have

$$\Pi_B = \max_q \{p_b \mathbb{E}[\min(D_s + D_m, q)] - c_b q\} + r(\mu_s + \mu_m)$$
$$\geq \max_q \{p_b \mathbb{E}[\min(D_s + D_m, q)] - c_b q\} \geq (p_o - c_o)\mathbb{E}[D_m] = \Pi_P.$$

Combing all above cases leads to the desired result. $\square$

Proof of Proposition A6. Let $\hat{\zeta}$ be the customers' belief on the fill rate and $(\hat{\phi}_m, \hat{\phi}_s)$ be the retailer's belief on the proportions of omni-customers and store-only customers who purchase offline. We first characterize customers' channel choices and retailer's optimal decision in terms of these private beliefs.

Customer Choice. Given customers' belief on the fill rate, let $(\bar{\phi}_m, \bar{\phi}_s)$ be the resulting proportions of omni-customers and store-only customers who purchase offline. Customers' utilities from purchasing online and offline channels are given by (A11). Then, customers' channel choices are as follows: 1) store-only customers purchase the product if $U^B_{s,b} \geq 0$; otherwise, exit the market, or 2) omni-customers purchase the product from the offline channel if and only if $U^B_{m,b} \geq \max(0, U^B_{m,o})$ and from the online channel if and only if $U^B_{m,o} \geq \max(0, U^B_{m,b})$. In particular, we can further characterize customers' choices, $(\bar{\phi}_s, \bar{\phi}_m)$, in different parameter regions as follows.

(i) If $\hat{\zeta} u_b - k + (1-\hat{\zeta})(u_o - t) \geq (u_o - t)$, then both store-only customers and omni-customers will choose the offline channel, i.e., $(\bar{\phi}_s, \bar{\phi}_m) = (1,1)$. The omni-customers will switch to the online channel when experiencing a stockout.

(ii) If $u_o - t \geq \hat{\zeta} u_b - k + (1-\hat{\zeta})(u_o - t) \geq 0$ and $\hat{\zeta} u_b - k \geq 0$, then store-only customers will go to the B&M store, whereas omni-customers will choose the online channel. That is, $(\bar{\phi}_s, \bar{\phi}_m) = (1,0)$.

(iii) If $\hat{\zeta} u_b - k < 0$, then store-only customers will exit the market, and omni-customers will choose the online channel. That is, neither type will go to the B&M store, i.e., $(\bar{\phi}_s, \bar{\phi}_m) = (0,0)$.

Therefore, given customers' belief on the fill rate $\hat{\zeta}$, the resulting market segmentation $(\bar{\phi}_s, \bar{\phi}_m)$ is expressed as

$$(\bar{\phi}_s, \bar{\phi}_m) = \begin{cases} (1,1) \text{ if } \hat{\zeta} \geq \frac{k}{u_b - (u_o - t)}, \\ (1,0) \text{ if } \frac{k}{u_b} \leq \hat{\zeta} < \frac{k}{u_b - (u_o - t)}, \\ (0,0) \text{ if } \hat{\zeta} < \frac{k}{u_b}. \end{cases} \tag{A20}$$

Retailer's Optimal Decision. Given retailer's beliefs $(\hat{\phi}_m, \hat{\phi}_s)$ on customers' choices, let $\bar{q}$ and $\bar{\zeta}$ be the optimal order quantity for the B&M store and the corresponding resulting fill rate, respectively. As customers are homogeneous, there are only three possible values of $(\hat{\phi}_s, \hat{\phi}_m)$: $(1,1)$, $(1,0)$, and $(0,0)$. We now analyze the retailer's optimal decision for the three cases.

(i) If $(\hat{\phi}_s, \hat{\phi}_m) = (1,1)$, the retailer's expected profit is $\pi(q) = p_b \mathbb{E}[\min(D_m + D_s, q)] - c_b q + (p_o - c_o)\mathbb{E}[(D_m - \frac{D_m}{D_m + D_s} q)^+]$. Then, the optimal order quantity is $\bar{q} = q_e$, and the corresponding fill rate is $\bar{\zeta} = \zeta_e$.

(ii) If $(\hat{\phi}_s, \hat{\phi}_m) = (1,0)$, the retailer's expected profit is $\pi(q) = p_b \mathbb{E}[\min(D_s, q)] - c_b q + (p_o - c_o)\mathbb{E}[D_m]$. Consequently, the optimal order quantity is $\bar{q} = q_s$, and the corresponding fill rate is $\bar{\zeta} = \zeta_s$.

(iii) If $(\hat{\phi}_s, \hat{\phi}_m) = (0,0)$, the retailer's expected profit is $\pi(q) = (p_o - c_o)\mathbb{E}[D_m]$. Therefore, the retailer will order nothing for the B&M store and the resulting fill rate is zero, i.e., $\bar{q} = 0$ and $\bar{\zeta} = 0$.

In summary, given the belief $(\hat{\phi}_s, \hat{\phi}_m)$, the retailer's optimal decision of $\bar{q}$ and the corresponding fill rate $\bar{\zeta}$ can be characterized as

$$(\bar{q}, \bar{\zeta}) = \begin{cases} (q_e, \zeta_e) \text{ if } (\hat{\phi}_s, \hat{\phi}_m) = (1,1), \\ (q_s, \zeta_s) \text{ if } (\hat{\phi}_s, \hat{\phi}_m) = (1,0), \\ \;(0,0) \;\; \text{ if } (\hat{\phi}_s, \hat{\phi}_m) = (0,0). \end{cases} \tag{A21}$$

RE Equilibrium. Next, we analyze the RE equilibrium, denoted by $(q^B, \phi_m^B, \phi_s^B)$, by connecting customers' choices with the retailer's optimal decision. The concept of RE equilibrium states 1) customers' belief on the fill rate is exactly the realized fill rate, and 2) retailer's belief on customers' choices is just consistent with the realized one. Specifically, we define the RE equilibrium as follows.

Definition A6 (RE Equilibrium Under BASE). A RE equilibrium

$$(\bar{\zeta}, \hat{\zeta}, \bar{\phi}_m, \bar{\phi}_s, \hat{\phi}_m, \hat{\phi}_s, \bar{q})$$

should satisfy the following conditions:

(i) Given $\hat{\zeta}$, $(\bar{\phi}_s, \bar{\phi}_m)$ satisfies Equation (A20);

(ii) Given $(\hat{\phi}_s, \hat{\phi}_m)$, $(\bar{q}, \bar{\zeta})$ satisfies Equation (A21);

(iii) $\bar{\zeta} = \hat{\zeta}$ and $(\bar{\phi}_s, \bar{\phi}_m) = (\hat{\phi}_s, \hat{\phi}_m)$.

By Definition A6, we know that there exist only three potential RE equilibria, i.e., $(q^B, \phi_s^B, \phi_m^B) = (q_e, 1, 1)$, $(q_s, 1, 0)$, and $(0,0,0)$. In the following cases, we first verify whether these potential equilibria satisfy the conditions of RE equilibrium. When there exist multiple RE equilibria, we choose the one which is Pareto-dominant.

Case 1: $k \leq \zeta_e(u_b - (u_o - t))$. In this case, we first show that $(q_e, 1, 1)$ and $(0,0,0)$ are RE equilibria, whereas the existence of $(q_s, 1, 0)$ as an equilibrium depends on whether $\zeta_s u_b \geq k > \zeta_s u_b - (u_o - t)$ or not. The justification of a RE equilibrium for those cases is similar to Proposition 1.

We next show that the equilibrium $(q_e, 1, 1)$ always Pareto-dominates others. Clearly, the equilibrium $(0,0,0)$ is dominated, as it generates zero payoff for both the retailer and customers. Now, we compare the equilibria $(q_e, 1, 1)$ and $(q_s, 1, 0)$. From Proposition 1, we know that $\bar{\pi}_B(q_s|(1,0)) = \pi_B(q_s|(1,0)) \leq \pi_B(q_{sm}|(1,1)) < \bar{\pi}_B(q_{sm}|(1,1))$, in which the last inequality holds due to the additional profits from omni-customers' switching behavior.

The domination of omni-customers' utility and store-only customers' utility follows the argument of Proposition 1. Therefore, the Pareto-dominant equilibrium is $q^B = q_e$, and $(\phi_s^B, \phi_m^B) = (1,1)$.

Case 2: $\zeta_s u_b \geq k > \zeta_e(u_b - (u_o - t))$. Similar to Case 1, one can show that both $(q_s, 1, 0)$ and $(0,0,0)$ are equilibria and that $(q_s, 1, 0)$ is Pareto-dominant. Thus, $q^B = q_s$ and $(\phi_s^B, \phi_m^B) = (1,0)$.

Case 3: $k > \zeta_s u_b$ and $k > \zeta_e(u_b - (u_o - t))$. In this case, one can verify that there exists just one equilibrium, $(q^B, \phi_s^B, \phi_m^B) = (0, 0, 0)$.

By combining the above cases, we can obtain the desired results. $\square$

Proof of Theorem A4. The first case is similar to the one in Theorem 1, and we omit its proof.

- $\zeta_p u_b < k \le \zeta_e(u_b - u_o + t)$. In this case, the retailer's expected profit in the BASE model can be written as

$$\begin{aligned}\bar{\Pi}_B &= \max_q \{p_b \mathbb{E}[\min(D_m + D_s, q)] - c_b q + (p_o - c_o)\mathbb{E}[(D_m - \frac{D_m}{D_m + D_s} q)^+]\} \\ &= \max_q \left\{ p_b \mathbb{E}[\min(D_s, q)] + p_b \mathbb{E}[\min(D_m, (q - D_s)^+)] - c_b q + (p_o - c_o)(\mathbb{E}[D_m] - \mathbb{E}[\min(D_m, \frac{D_m}{D_m + D_s} q)] \right\}.\end{aligned}$$

Then, we have

$$\bar{\Pi}_B \le (p_o - c_o)\mathbb{E}[D_m] = \Pi_P.$$

- $k \le \min(\zeta_p u_b, \zeta_e(u_b - u_o + t))$. Because $u_o \ge u_b$, $k \le \min(\zeta_p u_b, \zeta_e(u_b - u_o + t)) \le \min(\zeta_p u_b, t)$. By Proposition 2(i), the retailer's expected profit in the BOPS model is given by

$$\Pi_P = \max_q \left\{ p_b \mathbb{E}[\min(D_s, q)] - c_b q + c_o \mathbb{E}[\min(D_m, (q - D_s)^+)] + (p_o - c_o)\mathbb{E}[D_m] \right\}.$$

When $p_o \to c_o$, we have $\bar{\pi}_B \ge \pi_P$ as

$$\begin{aligned}\Pi_P &= \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q + c_o \mathbb{E}[\min(D_m, (q - D_s)^+)]\} \\ &\le \max_q \{p_b \mathbb{E}[\min(D_s, q)] - c_b q + p_b \mathbb{E}[\min(D_m, (q - D_s)^+)]\} = \bar{\Pi}_B,\end{aligned}$$

in which the inequality holds by $c_o \le c_b \le p_b$.

Note that for any given q, D_m and D_s, we can prove the following inequality:

$$\frac{D_m}{D_s + D_m} \min\{D_s + D_m, q\} \ge \min(D_m, (q - D_s)^+),$$

by considering the following three cases:

(a) if $q \ge D_s + D_m$, we have that $\frac{D_m}{D_s + D_m} \min\{D_s + D_m, q\} = D_m = \min(D_m, (q - D_s)^+)$;

(b) if $D_s \le q < D_s + D_m$, we have that $\frac{D_m}{D_s + D_m} \min\{D_s + D_m, q\} = \frac{D_m}{D_s + D_m} q > q - D_s = \min(D_m, (q - D_s)^+)$;

(c) if $q < D_s$, we have that $\frac{D_m}{D_s + D_m} \min\{D_s + D_m, q\} = \frac{D_m}{D_s + D_m} q > 0 = \min(D_m, (q - D_s)^+)$.

Therefore, we have $\mathbb{E}[\min(D_m, \frac{D_m}{D_m+D_s}q)] \geq \mathbb{E}[\min(D_m,(q-D_s)^+)]$. It follows that when $p_o \to p_b$. This is because

$$\begin{aligned}\bar{\Pi}_B &\leq \max_q\{p_b\mathbb{E}[\min(D_s,q)] - c_bq + (p_b - p_o + c_o)\mathbb{E}[\min(D_m,(q-D_s)^+)]\} + (p_o - c_o)\mathbb{E}D_m \\ &= \max_q\{p_b\mathbb{E}[\min(D_s,q)] - c_bq + c_o\mathbb{E}[\min(D_m,(q-D_s)^+)]\} + (p_o - c_o)\mathbb{E}D_m = \Pi_P.\end{aligned}$$

Because $\Pi_P - \bar{\Pi}_B$ is increasing in p_o, it ensures that there is a threshold in $[c_o, p_b]$, lower than which we have $\Pi_P < \bar{\Pi}_B$. Moreover, we can verify that when $p_o \to p_b - c_b + c_o + \Delta$ and $c_b - \Delta \leq c_o$, $\bar{\Pi}_B < \Pi_P$. This is because

$$\begin{aligned}\bar{\Pi}_B &\leq \max_q\{p_b\mathbb{E}[\min(D_s,q)] - c_bq + (p_b - p_o + c_o)\mathbb{E}[\min(D_m,(q-D_s)^+)]\} + (p_o - c_o)\mathbb{E}D_m \\ &= \max_q\{p_b\mathbb{E}[\min(D_s,q)] - c_bq + (c_b - \Delta)\mathbb{E}[\min(D_m,(q-D_s)^+)]\} + (p_o - c_o)\mathbb{E}D_m \\ &= \max_q\{p_b\mathbb{E}[\min(D_s,q)] - c_bq + c_o\mathbb{E}[\min(D_m,(q-D_s)^+)]\} + (p_o - c_o)\mathbb{E}D_m = \Pi_P.\end{aligned}$$

Thus, there exists a threshold $\delta_R(c_o) \in [c_o, p_b - c_b + c_o + \Delta]$ such that when $p_o < \delta_R(c_o)$, $\bar{\Pi}_B \geq \Pi_P$, and when $p_o \geq \delta_R(c_o)$, $\bar{\Pi}_B < \Pi_P$.

Thus, this completes the proof. $\square$