Online Appendix to the Paper
“No Claim? Your Gain:
Design of Residual Value Extended Warranties
under Risk Aversion and Strategic Claim Behavior”

**Lemma 1.** Given any \( x \geq y \geq 0 \), \((e^{y} - e^{x})/\gamma\) is increasing in \( \gamma \). Moreover, \((e^{y} - e^{x})/\gamma \geq x - y \) if \( \gamma > 0 \); \((e^{y} - e^{x})/\gamma \leq x - y \) if \( \gamma < 0 \).

*Proof of Lemma 1.* Notice that \((e^{y} - e^{x})/\gamma = (e^{(x-y)-1})/\gamma \cdot e^{y}\). Because both \((e^{(x-y)-1})/\gamma \) and \(e^{y}\) are non-negative for any \( \gamma \) and \( x \geq y \), and \( e^{y}\) is increasing in \( \gamma \) for any given \( y \geq 0 \), then it is sufficient to show that \((e^{(x-y)-1})/\gamma \) is increasing in \( \gamma \).

Consider its first-order derivative with respect to \( \gamma \)

\[
\frac{\partial}{\partial \gamma} \left( \frac{e^{(x-y)-1}}{\gamma} \right) = \frac{\gamma (x-y) e^{(x-y)} - (e^{(x-y)-1})}{\gamma^2}.
\]

Let \( G(\gamma) := e^{(x-y)-1} \). Then, \( \frac{\partial G(\gamma)}{\partial \gamma} \geq 0 \) and \( \frac{\partial^2 G(\gamma)}{\partial \gamma^2} \geq 0 \) for any given \( x \geq y \), so \( G(\gamma) \) is increasing convex in \( \gamma \). Note that \( G(0) = 0 \). Apparently, for any \( \gamma \geq 0 \),

\[
G(\gamma) = G(0) + \int_{0}^{\gamma} G'(\tau) d\tau \leq \gamma G'(\gamma).
\]

We note that the above inequality also holds for any \( \gamma \leq 0 \) because \( G(\gamma) = G(0) - \int_{0}^{0} G'(\tau) d\tau \leq -(0 - \gamma)G'(\gamma) = \gamma G'(\gamma) \). Thus,

\[
\frac{\partial}{\partial \gamma} \left( \frac{e^{(x-y)-1}}{\gamma} \right) = \frac{G'(\gamma) - G(\gamma)}{\gamma^2} \geq 0,
\]

and given any \( x \geq y \geq 0 \), \((e^{y} - e^{x})/\gamma\) is increasing in \( \gamma \). Consider the limit as \( \gamma \) goes to zero, \( \lim_{\gamma \to 0} (e^{y} - e^{x})/\gamma = \lim_{\gamma \to 0} (xe^{y} - ye^{x}) = x - y \). Therefore, \((e^{y} - e^{x})/\gamma \geq x - y \) if \( \gamma > 0 \); \((e^{y} - e^{x})/\gamma \leq x - y \) if \( \gamma < 0 \). Furthermore, there exist tighter bounds for \((e^{y} - e^{x})/\gamma\), e.g., for any \( x \geq y \),

\[
\min(e^{y}, e^{x}) \cdot (x - y) \leq (e^{y} - e^{x})/\gamma = \int_{y}^{x} e^{\gamma} d\tau \leq \max(e^{y}, e^{x}) \cdot (x - y).
\]

*Proof of Theorem 1.* Apparently, the optimal claim policy has a threshold structure: it is optimal for a customer with risk attitude \( \gamma \) to place a claim at time \( t \) for a failure with repair cost \( C_{t} \) if and only if \( C_{t} \geq g(t; \gamma, r) \). Moreover, it is straightforward that \( g(t; \gamma, r) \) is decreasing in \( t \) and increasing in \( r \), noting that \( t \) represents the time-to-go.

We next show the monotonic comparative statics of \( g(t; \gamma, r) \) with respect to \( \gamma \). Suppose \( g(t; \gamma, r) \leq g(t; \gamma', r) \) at time \( t \) for any \( \gamma > \gamma' \). We will next show \( g(t + \delta; \gamma, r) \leq g(t + \delta; \gamma', r) \) for a sufficiently small \( \delta > 0 \). According to the differential equation (1),

\[
g(t + \delta; \gamma, r) = g(t; \gamma, r) - \frac{\lambda_{t}}{\gamma} \left( E[e_{\gamma} \min(C_{t}, g(t; \gamma, r))] - 1 \right) + o(\delta), \quad g(t + \delta; \gamma', r) = g(t; \gamma', r) - \frac{\lambda_{t}}{\gamma'} \left( E[e_{\gamma'} \min(C_{t}, g(t; \gamma', r))] - 1 \right) + o(\delta).
\]

Then,

\[
\left( E[e_{\gamma} \min(C_{t}, g(t; \gamma, r))] - 1 \right)/\gamma - \left( E[e_{\gamma'} \min(C_{t}, g(t; \gamma', r))] - 1 \right)/\gamma' \geq \frac{1}{\gamma'} \left( E[e_{\gamma'} \min(C_{t}, g(t; \gamma, r))] - e_{\gamma} \min(C_{t}, g(t; \gamma, r)) \right) \geq E[\Theta \cdot (g(t; \gamma, r) - g(t; \gamma', r))],
\]

where \( \Theta = e_{\gamma'} \min(C_{t}, g(t; \gamma', r)) \) if \( \gamma' \geq 0 \); \( \Theta = e_{\gamma} \min(C_{t}, g(t; \gamma, r)) \) if \( \gamma \leq 0 \). The first inequality holds because \( E[e_{\gamma} \min(C_{t}, g(t; \gamma, r))] - 1 \) is increasing in \( \gamma \) by Lemma 1; the second inequality holds by a similar argument in the proof of Lemma 1; the third inequality holds because \( \min(C_{t}, x) - \min(C_{t}, y) \leq (x - y) \) for any \( x \geq y \).
Therefore, \( g(t + \delta; \gamma, r) - g(t + \delta; \gamma', r) \leq (g(t; \gamma, r) - g(t; \gamma', r)) \cdot (1 - \lambda \delta \Theta) \leq 0 \). Then, \( g(t; \gamma, r) \) is decreasing with respect to the risk attitude \( \gamma \).

The case with \( \gamma \) approaching \( -\infty \):

\[
g'(t; -\infty, r) = \lim_{\gamma \to -\infty} \frac{-\lambda t}{\gamma} \left( E\left[e^{\gamma \min(C_t, g(t; \gamma, r))}\right] - 1 \right) = 0.
\]

Then, \( g(t; -\infty, r) = g(0; -\infty, r) + \int_0^t g'(s; -\infty, r) ds = r \).

The case with \( \gamma \) approaching \( +\infty \): for any positive time \( t \)

\[
g'(t; \infty, r) = \lim_{\gamma \to +\infty} \frac{-\lambda t}{\gamma} \left( E\left[e^{\gamma \min(C_t, g(t; \gamma, r))}\right] - 1 \right) = -\lambda t E\left[\min(C_t, g(t; \infty, r)) \cdot e^{-\min(C_t, g(t; \infty, r))}\right].
\]

If \( g(t; \infty, r) > 0 \), then \( g'(t; \infty, r) = -\infty \) and \( g(t; \infty, r) = g(0; \infty, r) + \int_0^t g'(s; \infty, r) ds < 0 \) for any positive \( t \), which is impossible. Thus, the only solution to the above differential equation is \( g(t; \infty, r) = 0 \) for any positive \( t \).

**Proof of Proposition 1.** (a) Note that \( g(t; \gamma, r) \) is decreasing in \( t \) and \( g(0; \gamma, r) = r \). When the time-to-go \( t \) is very small, \( g(t; \gamma, r) \) is sufficiently close to \( r \), so it is optimal for the customer not to claim any failure since \( g(t; \gamma, r) \approx r > c \). Then, the differential equation (1) becomes \( g'(t; \gamma, r) = -\frac{\lambda t}{\gamma} (e^{\gamma t} - 1) \). Solving the above differential equation and combining with the boundary condition \( g(0; \gamma, r) = r \) yields \( g(t; \gamma, r) = r - \frac{1}{\gamma} (e^{\gamma t} - 1) \Lambda(t) \). Denote the unique solution to the equation \( r - 1/\gamma \cdot (e^{\gamma t} - 1) \Lambda(t) = c \) with respect to \( t \) by \( t^* \). Then, for any \( t \geq t^* \), it is optimal to claim all the failures so the differential equation (1) becomes \( g'(t; \gamma, r) = -\frac{\lambda t}{\gamma} (e^{\gamma (t^*+\gamma t)} - 1) \), with boundary condition \( g(t^*; \gamma, r) = c \). Similarly, the unique solution to the above differential equation is \( g(t; \gamma, r) = -\frac{1}{\gamma} \ln \left(1 - e^{-\Lambda(t^*) + \Lambda(t)} \cdot (1 - e^{-\gamma t})\right) \).

(b) Under the exponential distribution, the differential equation (1) can be rewritten as follows

\[
g'(t; \gamma, r) = -\frac{\lambda t}{\gamma} (e^{(\gamma - \mu) \min(C_t, g(t; \gamma, r))} - 1).
\]

It is straightforward to verify that function (3) satisfies the differential equation (1) and its boundary condition. In particular,

\[
\lim_{\gamma \to \mu} g(t; \gamma, r) = \lim_{\gamma \to \mu} \frac{e^{-\Lambda(t)} e^{-(\gamma - \mu) r}}{1 - e^{-\Lambda(t)} \cdot (1 - e^{-(\gamma - \mu) r})} = e^{-\Lambda(t)} r.
\]

**Proof of Proposition 2.** We first consider the case \( \gamma > 0 \). Consider the first-order derivative with respect to \( \gamma \),

\[
\frac{\partial u_{\alpha}(t; \gamma)}{\partial \gamma} = \frac{1}{\gamma^2 E[\gamma R(t)]} \left( \gamma E[R(t) e^{\gamma R(t)}] - E[e^{\gamma R(t)}] \log E[e^{\gamma R(t)}] \right)
\]

\[
\geq \frac{1}{\gamma E[e^{\gamma R(t)}]} \left( E[R(t) e^{\gamma R(t)}] - E[R(t)] E[e^{\gamma R(t)}] \right).
\]

The inequality holds because of the Jensen’s inequality: \( \log (E[e^{\gamma R(t)}]) \leq E[\log (e^{\gamma R(t)})] = \gamma E[R(t)] \). For a similar reason, \( E[R(t)] e^{\gamma R(t)} \geq E[R(t)] e^{E[R(t)]} \). We will next show a stronger result, i.e., \( E[R(t) e^{\gamma R(t)}] \geq E[R(t)] E[e^{\gamma R(t)}] \). First, suppose that \( R(t) \) takes values from the discrete set \( \{R(t)^1, \ldots, R(t)^n\} \) with respective probabilities \( \alpha_1, \ldots, \alpha_n \), where \( \alpha_1 + \cdots + \alpha_n = 1 \). Then,

\[
E[R(t) e^{\gamma R(t)}] - E[R(t)] E[e^{\gamma R(t)}] = \sum_{i=1}^n \alpha_i R(t)^i e^{\gamma R(t)^i} - \sum_{i=1}^n \alpha_i R(t)^i \sum_{j=1}^n \alpha_j e^{\gamma R(t)^j}
\]

\[
= \sum_{i=1}^n \alpha_i R(t)^i \left( e^{\gamma R(t)^i} - \sum_{j=1}^n \alpha_j e^{\gamma R(t)^j} \right) = \sum_{i=1}^n \alpha_i R(t)^i \sum_{j=1}^n \alpha_j \left( e^{\gamma R(t)^j} - e^{\gamma R(t)^i} \right)
\]

\[
= \sum_{(i,j)} \alpha_i \alpha_j (R(t)^i - R(t)^j) \left( e^{\gamma R(t)^i} - e^{\gamma R(t)^j} \right) \geq 0,
\]

where \( (i, j) \) and \( (j, i) \) are considered the same pair. The inequality holds because \( (R(t)^i - R(t)^j) \cdot \left( e^{\gamma R(t)^i} - e^{\gamma R(t)^j} \right) \geq 0 \) for each pair \( (i, j) \). Similarly, we can show that the inequality also holds when \( R(t) \) has a continuous support set. Therefore, for \( \gamma > 0 \)

\[
\frac{\partial u_{\alpha}(t; \gamma)}{\partial \gamma} \geq \frac{1}{\gamma E[e^{\gamma R(t)}]} \left( \sum_{(i,j)} \alpha_i \alpha_j (R(t)^i - R(t)^j) \left( e^{\gamma R(t)^i} - e^{\gamma R(t)^j} \right) \right) > 0.
\]
Similarly, we can show that the above inequality also holds for $\gamma < 0$. Thus, $w_{\text{rvw}}(t; \gamma)$ is increasing in $\gamma$.

We next consider the case with $\gamma$ approaching 0 as follows,

$$\lim_{\gamma \to 0} w_{\text{rvw}}(t; \gamma) = \lim_{\gamma \to 0} \frac{\partial \ln \left( E[e^{\gamma R(t)}] \right)}{\partial \gamma} = \lim_{\gamma \to 0} \frac{E[R(t)e^{\gamma R(t)}]}{E[e^{\gamma R(t)}]} = E[R(t)].$$

It completes the proof. $\square$

**Proof of Proposition 3.** Let $N_t$ be a random variable following the Poisson distribution with mean $\Lambda(t) = \int_0^t \lambda(x)dx$. Notice that $\sum_{i=1}^{N_t} C_i$ is a compound Poisson random variable. It is known that the total repair cost $R(t)$ has the same distribution as $\sum_{i=1}^{N_t} C_i$, assuming that the repair costs are i.i.d., independent of the failure process (see, e.g., Ross 1995). By equation (5),

$$w_{\text{rvw}}(t; \gamma) = \frac{1}{\gamma} \log \left( E[e^{\gamma C}] \right) = \frac{1}{\gamma} \log \left( E[e^{\gamma \sum_{i=1}^{N_t} C_i}] \right) = \frac{1}{\gamma} \log \left( E \left[ e^{\sum_{i=1}^{N_t} C_i} | N_t \right] \right)$$

$$= \frac{1}{\gamma} \log \left( E \left[ \prod_{i=1}^{N_t} e^{c_i C_i} | N_t \right] \right) = \frac{1}{\gamma} \log \left( E[M_C(\gamma)^{N_t}] \right) = \frac{1}{\gamma} \log \left( e^{\Lambda(t) - (M_C(\gamma)^{N_t})} \right) = \frac{\Lambda(t) \cdot (M_C(\gamma) - 1)}{\gamma},$$

where $M_C(\gamma)$ is the moment generating function of the repair cost $C$, i.e., $M_C(\gamma) = E[e^{\gamma C}]$. The second last equality holds because $E[e^{N_t}] = M_N(\log(x)) = e^{\Lambda(t) \cdot (x-1)}$. Apparently, for the constant repair cost

$$w_{\text{rvw}}(t; \gamma) = \frac{\Lambda(t) \cdot (M_C(\gamma) - 1)}{\gamma} = \frac{\Lambda(t)}{\mu - \gamma}.$$

Moreover, $w_{\text{rvw}}(t; \gamma) \to \Lambda(t)c$ as $\gamma \to 0$ because $\lim_{\gamma \to 0} (e^{\gamma c} - 1) / \gamma = c$.

For the exponential distributed repair cost,

$$w_{\text{rvw}}(t; \gamma) = \frac{\Lambda(t) \cdot (M_C(\gamma) - 1)}{\gamma} = \frac{\Lambda(t)}{\mu - \gamma}.$$

The last equality holds because $M_C(\gamma) = \mu / (\mu - \gamma)$ for $\gamma < \mu$. Apparently, $w_{\text{rvw}}(t; \gamma) \to \Lambda(t) / \mu$ as $\gamma \to 0$. $\square$

**Proof of Proposition 4.** The willingness-to-pay $w_{\text{rvw}}(t; \gamma, r)$ is the quality such that the utility of buying and not buying an RVW is indifferent, taking into account the possible out-of-pocket cost and refund for the RVW, i.e.,

$$E[-e^{-\gamma (v - R(t))}] = -e^{-\gamma (v - w_{\text{rvw}}(t; \gamma, r) + g(t; \gamma, r))}.$$

Combining with equation (5) yields $w_{\text{rvw}}(t; \gamma, r) = w_{\text{rvw}}(t; \gamma) + g(t; \gamma, r)$.

(a). First, we consider the risk-neutral case, i.e., $\gamma = 0$. Note that $w_{\text{rvw}}(t; 0) = E[R(t)]$ by Proposition 2, so we only need to show $h(t; 0, r) = g(t; 0, r) + E[R(t)]$. Recall that

$$\frac{\partial}{\partial t} \left( g(t; 0, r) + E[R(t)] \right) = \lambda_i E[C_i - \min(C_i, g(t; 0, r))] = \lambda_i E[(C_i - g(t; 0, r))^+].$$

Plugging $h(t; 0, r) = g(t; 0, r) + E[R(t)]$ into equation (4) results in

$$h'(t; 0, r) = \lambda_i \Pr(C_i \geq g(t; 0, r)) \{ E[C_i | C_i \geq g(t; 0, r)] - g(t; 0, r) \} = \lambda_i E[(C_i - g(t; 0, r))^+]$$

Combining it with the boundary condition $h(0; 0, r) = g(0; 0, r) + E[R(0)] = r$, we have obtained $h(t; 0, r) = g(t; 0, r) + E[R(t)]$ at any $t \geq 0$ for any given refund $r$.

Next, we consider the case of $\gamma > 0$. Suppose $h(t; \gamma, r) \geq g(t; \gamma, r) + E[R(t)]$ at some $t \geq 0$ for given $\gamma > 0$ and $r$. We will next show that for any sufficiently small $\delta > 0$, $h(t + \delta; \gamma, r) \geq g(t + \delta; \gamma, r) + E[R(t + \delta)]$. By the differential equations (1) and (4),

$$h(t + \delta; \gamma, r) - g(t + \delta; \gamma, r) - E[R(t + \delta)] = h(t; \gamma, r) - g(t; \gamma, r) - E[R(t)] + \delta \lambda_i \left( \int_{0}^{\infty} c_i dF_i(c_i) \right)$$

$$- (1 - F_i(g(t; \gamma, r))) \cdot (h(t; \gamma, r) - E[R(t)]) + (-1 + E[e^{\gamma \min(C_i, g(t; \gamma, r))}]) / \gamma - E[C_i] \right) o(\delta)$$

$$= h(t; \gamma, r) - g(t; \gamma, r) - E[R(t)] + \delta \lambda_i \left( - \int_{0}^{g(t; \gamma, r)} c_i dF_i(c_i) - (1 - F_i(g(t; \gamma, r))) \cdot (h(t; \gamma, r) - E[R(t)]) \right)$$
\[\begin{align*}
\frac{1}{\gamma} \log \left( E\left[ e^{\gamma R(t)} \right] \right) & = \frac{1}{\gamma} \log \left( E\left[ e^{\gamma \min(C_t, g(t; \gamma, r))} \right] \right) = w_{\text{tw}}(t; \gamma) + \frac{\lambda_t \delta}{\gamma} \left( E[e^{\gamma C_t}] - 1 \right) + o(\delta),
\end{align*}\]

where \( R(t, t+\delta) \) represents the total repair cost from time-to-go \( t+\delta \) to \( t \). The equality (21) holds because

\[\log \left( E\left[ e^{\gamma \min(C_t, g(t; \gamma, r))} \right] \right) = \log \left( \lambda_t \delta E[e^{\gamma C_t}] \right) + (1 - \lambda_t \delta) + o(\delta) = \log \left( 1 + \lambda_t \delta (E[e^{\gamma C_t}] - 1) + o(\delta) \right) = \lambda_t \delta (E[e^{\gamma C_t}] - 1) + o(\delta).\]

The last equality holds because of the Taylor expansion. Then, consider

\[\begin{align*}
h(t+\delta; \gamma, r) - g(t+\delta; \gamma, r) - w_{\text{tw}}(t+\delta; \gamma) & = h(t; \gamma, r) - g(t; \gamma, r) - w_{\text{tw}}(t; \gamma) + \lambda_t \int_{g(t; \gamma, r)}^{\infty} c_t dF_t(c_t) \\
& - (1 - F_t(g(t; \gamma, r))) \cdot h(t; \gamma, r) - E[R(t)] + (-1 + E[e^{\gamma \min(C_t, g(t; \gamma, r))}]) / \gamma - (E[e^{\gamma C_t}] - 1) / \gamma + o(\delta)
\end{align*}\]

The first inequality holds because \( w_{\text{tw}}(t; \gamma) \leq E[R(t)] \) for any \( t \geq 0 \) and \( \gamma < 0 \), and by Lemma 1, for any given \( \gamma < 0 \),

\[\left( E[e^{\gamma C_t}] - E[e^{\gamma \min(C_t, g(t; \gamma, r))}] \right) / \gamma \leq E[C_t] - E[\min(C_t, g(t; \gamma, r))];\]

the second inequality holds because \( h(t; \gamma, r) \geq g(t; \gamma, r) + w_{\text{tw}}(t; \gamma) \). Therefore, \( h(t; \gamma, r) \geq w_{\text{tw}}(t; \gamma, r) \) for any \( t \geq 0 \) for given \( \gamma < 0 \) and \( r \). Similarly, we can prove \( h(t; r, r) \leq w_{\text{tw}}(t; \gamma, r) \) at any \( t \geq 0 \) for given \( \gamma > 0 \) and \( r \).

(b). First, consider the case of \( \gamma < 0 \). Suppose \( h(t; \gamma, r) - g(t; \gamma, r) \leq h(t; \gamma, r') - g(t; \gamma, r') \) for some \( t \geq 0 \) and \( r > r' \). Then,
- (E[R(t)] - E[min(C_t, g(t; γ, r))]) - (F_t(g(t; γ, r)) - F_t(g(t; γ, r')))) \cdot E[R(t)] - (-1 + E[\gamma \min(C_t, g(t; γ, r'))]) / \gamma \\
= \left( (h(t; γ, r) - g(t; γ, r)) - (h(t; γ, r') - g(t; γ, r')) \right) \cdot (1 - \delta t) (1 - F_t(g(t; γ, r'))) + \delta t \left( -E[min(C_t, g(t; γ, r))] + E[\gamma \min(C_t, g(t; γ, r'))] / \gamma \right) \\
+ \lambda_t \left( -E[min(C_t, g(t; γ, r))] + E[\gamma \min(C_t, g(t; γ, r'))] / \gamma \right)
\end{align*}

\begin{align*}
+ (F_t(g(t; γ, r)) - F_t(g(t; γ, r'))) \cdot (h(t; γ, r) - g(t; γ, r) - E[R(t)])
\end{align*}

\begin{align*}
\leq \left( (h(t; γ, r) - g(t; γ, r)) - (h(t; γ, r') - g(t; γ, r')) \right) \cdot (1 - \delta t) (1 - F_t(g(t; γ, r'))) \leq 0.
\end{align*}

The first inequality holds because $h(t; γ, r) \leq g(t; γ, r) + E[R(t)]$ for $γ < 0$, and by Lemma 1,

\begin{align*}
(E[\gamma \min(C_t, g(t; γ, r'))] - E[\gamma \min(C_t, g(t; γ, r'))]) / \gamma \leq E[min(C_t, g(t; γ, r))] - E[min(C_t, g(t; γ, r'))];
\end{align*}

the second inequality holds because $h(t; γ, r) - g(t; γ, r) \leq h(t; γ, r') - g(t; γ, r')$. Therefore, $h(t; γ, r) - g(t; γ, r)$ is decreasing in $r$ for any given $γ < 0$. Similarly, we can prove that $h(t; γ, r) - g(t; γ, r)$ is increasing in $r$ for any given $γ > 0$.

Obviously, $g(t; γ, r) \to ∞$ as $r \to ∞$. We have already known that $h(t; γ, r) - g(t; γ, r)$ is monotonic in $r$ for any given $γ$ and is bounded from below and above, i.e., $w_w(t; γ) > h(t; γ, r) - g(t; γ, r) > E[R(t)]$ for any $γ > 0$ and $w_w(t; γ) < h(t; γ, r) - g(t; γ, r) < E[R(t)]$ for any $γ < 0$, so it converges to a limit as $r$ approaches infinity. To show its limit, we consider its derivative with respect to $t$. From differential equations (1) and (4),

\begin{align*}
\frac{∂}{∂t} (h(t; γ, r) - g(t; γ, r)) &= λ_t \Pr(C_t > g(t; γ, r)) \cdot \left\{ E[C_t | C_t > g(t; γ, r)] - (h(t; γ, r) - E[R(t)]) \right\} \\
&+ \frac{λ_t}{γ} \left( -1 + E[\gamma \min(C_t, g(t; γ, r'))] \right)
\end{align*}

\begin{align*}
= λ_t \frac{E[\gamma C_t] - 1}{γ} - λ_t \int_{g(t; γ, r)}^∞ \frac{e^{γ C_t}}{γ} - \frac{e^{γ g(t; γ, r)}}{γ} + (h(t; γ, r) - g(t; γ, r) - E[R(t)]) \cdot \left( (h(t; γ, r) - g(t; γ, r)) \right) dF_t(C_t)
\end{align*}

Apparently, $Pr(C_t > g(t; γ, r)) \cdot (h(t; γ, r) - g(t; γ, r) - E[R(t)]) \to 0$ as $g(t; γ, r) \to ∞$ because $h(t; γ, r) - g(t; γ, r)$ is bounded. Assume that the moment generating function is finite, i.e., $E[e^{γ C_t}]$ is finite. Then,

\begin{align*}
\lim_{g(t; γ, r) \to ∞} \int_{g(t; γ, r)}^∞ \frac{e^{γ C_t}}{γ} - \frac{e^{γ g(t; γ, r)}}{γ} - g(t; γ, r) dF_t(C_t) = 0.
\end{align*}

The equality holds because the limit of each integral in the above equation is equal to zero. Thus, it holds that

\begin{align*}
\frac{∂}{∂t} (h(t; γ, ∞) - g(t; γ, ∞)) = λ_t \frac{E[\gamma C_t] - 1}{γ}.
\end{align*}

Then, for any given $γ$, we have

\begin{align*}
h(t; γ, ∞) - g(t; γ, ∞) = \int_0^t λ_t \frac{E[\gamma C_t] - 1}{γ} dF_t(C_t) = w_w(t; γ).
\end{align*}

The second equality holds because

\begin{align*}
\frac{∂}{∂t} w_w(t; γ) &= \lim_{δ \to 0} \frac{w_w(t + δ; γ) - w_w(t; γ)}{δγ} = \lim_{δ \to 0} \frac{\log (E[e^{γ R(t; t + δ)}])}{δγ} = \lim_{δ \to 0} \frac{\log (λ_0 δ E[e^{γ C_t}] + (1 - λ_0 δ) + o(δ))}{δγ} \\
&= \lim_{δ \to 0} \frac{λ_0 δ (E[e^{γ C_t}] - 1) + o(δ)}{δγ} = λ_t (E[e^{γ C_t}] - 1).
\end{align*}

It completes the proof. □
Proof of Theorem 2. The maximum price of the TW is equal to its willingness-to-pay. Then, the profit of offering a TW is equal to

\[ w_{tw}(T; \gamma) - E[R(T)]. \]

For risk-averse customers with \( \gamma > 0 \), we have

\[ w_{rvw}(T; \gamma, r) - h(T; \gamma, r) = w_{tw}(T; \gamma) + g(T; \gamma, r) - h(T; \gamma, r) < w_{tw}(T; \gamma) - E[R(t)]. \]

The inequality holds for any \( r > 0 \) by Proposition 4. For the RVW provider with a positive refund \( r \) earns less profit than the TW, so the RVW degenerates to a TW in a homogeneous market with risk-averse customers, i.e., \( r^* = 0 \).

Similarly, for risk-seeking customers, i.e., \( \gamma < 0 \),

\[ w_{rvw}(T; \gamma, r) - h(T; \gamma, r) = w_{tw}(T; \gamma) + g(T; \gamma, r) - h(T; \gamma, r) > w_{tw}(T; \gamma) - E[R(t)]. \]

Notice that the TW loses money for \( \gamma < 0 \), so the RVW can balance the revenue and the support cost by offering a sufficiently large refund because Proposition 4 shows that \( h(T; \gamma, r) - g(T; \gamma, r) \rightarrow w_{tw}(T; \gamma) \) as \( r \rightarrow \infty \).

Proof of Theorem 3. To investigate the profitability of the RVW over the TW, we consider the following comparison

\[ \max_r \{ w_{rvw}(T; \gamma_a) + g(T; \gamma_a, r) - h(T; \gamma_a, r) \} > w_{tw}(T; \gamma_b) - E[R(T)]. \]

Or equivalently, \( \min_r \{ h(T; \gamma_a, r) - g(T; \gamma_a, r) \} < E[R(T)] \). By Theorem 4, \( h(T; \gamma_a, r) \) is first increasing in \( \gamma_a \) for \( \gamma_a \leq 0 \) and then is decreasing in it for \( \gamma_a > 0 \). Then, \( \max_r \{ h(T; \gamma_a, r) - g(T; \gamma_a, r) \} \) is also first increasing and then decreasing in \( \gamma_a \). Denote the two solutions to the following equation with respect to \( \gamma_a \),

\[ \min_r \{ h(T; \gamma_a, r) - g(T; \gamma_a, r) \} = E[R(T)] \]

by \( \gamma_a \) and \( \gamma_a' \), and \( \gamma_a \leq \gamma_a' \). Therefore, the RVW is strictly more profitable than the TW if and only if \( \gamma_a < \gamma_a' \) or \( \gamma_a > \gamma_a' \).

Proof of Proposition 5. Let \( \alpha^H (w_{tw}(T; \gamma^H) - E[R(T)]) = w_{tw}(T; \gamma^L) - E[R(T)] \). We have

\[ w_{tw}(T; \gamma^H) = \frac{w_{tw}(T; \gamma^L) - \alpha^L E[R(T)]}{\alpha^H}. \]

By Proposition 2, \( w_{tw}(T; \gamma^H) \) is increasing in \( \gamma^H \), so there exists a unique solution to equation (5) with respect to \( \gamma^H \), denoted by \( \gamma^H \). We remark that if \( \gamma^H > \gamma^L \), it is more profitable for the provider to only serve type-H customers by charging price \( w_{tw}(T; \gamma^H) \).

Proof of Theorem 4. We will prove this theorem by induction.

(a) Suppose that for a given refund \( r \), \( w_{rvw}(t; \gamma, r) \geq w_{rvw}(t; \gamma', r) \) at time-to-go \( t \) for any \( \gamma > \gamma' \). We will next show that the inequality also holds at time-to-go \( t + \delta \) for a sufficiently small \( \delta > 0 \), i.e., \( w_{rvw}(t + \delta; \gamma, r) \geq w_{rvw}(t + \delta; \gamma', r) \).

According to the differential equation (1), we have

\[ g(t + \delta; \gamma, r) = g(t; \gamma, r) - \frac{\lambda_\delta}{\gamma} \left( E[e^{\gamma \min(C_{1, Greg}(t; \gamma, r))] - 1 \right) + o(\delta). \]

Combing with equation (21), we have

\[ g(t + \delta; \gamma, r) + w_{tw}(t + \delta; \gamma) = g(t; \gamma, r) + w_{tw}(t; \gamma) + \frac{\lambda_\delta}{\gamma} \frac{E[e^{\gamma C_1}] - E[e^{\gamma \min(C_{1, Greg}(t; \gamma, r))}]}{\gamma} + o(\delta). \]

Therefore,

\[ \left( g(t + \delta; \gamma, r) + w_{tw}(t + \delta; \gamma) \right) - \left( g(t + \delta; \gamma', r) + w_{tw}(t + \delta; \gamma') \right) = \left( g(t; \gamma, r) + w_{tw}(t; \gamma) \right) - \left( g(t; \gamma', r) + w_{tw}(t; \gamma') \right) \]

\[ - \left( g(t; \gamma', r) + w_{tw}(t; \gamma') \right) + \frac{\lambda_\delta}{\gamma} \frac{E[e^{\gamma C_1}] - E[e^{\gamma \min(C_{1, Greg}(t; \gamma, r))}]}{\gamma} \]

\[ \geq \left( g(t; \gamma, r) + w_{tw}(t; \gamma) \right) - \left( g(t; \gamma', r) + w_{tw}(t; \gamma') \right) \]

\[ - \left( g(t; \gamma', r) + w_{tw}(t; \gamma') \right) + \frac{\lambda_\delta}{\gamma} \frac{E[e^{\gamma C_1}] - E[e^{\gamma \min(C_{1, Greg}(t; \gamma', r))}]}{\gamma} + o(\delta). \]
\[ + \lambda_i \delta \left( E[e^{\gamma C_i}] - E[e^{\gamma \min(C_i, g(t; \gamma, r))}] \right) - E[e^{\gamma \min(C_i, g(t; \gamma, r))}] \bigg] + o(\delta) \\
\geq \left( g(t; \gamma, r) + w_{tw}(t; \gamma) \right) - \left( g(t; \gamma', r) + w_{tw}(t; \gamma') \right) \geq 0. \]

The first inequality holds because \( g(t; \gamma, r) \leq g(t; \gamma', r) \) and \( E[e^{\gamma \min(C_i, g(t; \gamma', r))}] / \gamma' \geq E[e^{\gamma \min(C_i, g(t; \gamma', r))}] / \gamma' \); the second inequality holds because by Lemma 1,

\[ E[e^{\gamma C_i}] - E[e^{\gamma \min(C_i, g(t; \gamma, r))}] \geq E[e^{\gamma C_i}] - E[e^{\gamma \min(C_i, g(t; \gamma', r))}] ; \]

the last inequality holds because \( g(t; \gamma, r) + w_{tw}(t; \gamma) \geq g(t; \gamma', r) + w_{tw}(t; \gamma') \). Thus, for any given time-to-go \( t \) and refund \( r \), \( w_{tw}(t; \gamma, r) \) is decreasing with respect to \( \gamma \) no matter whether \( \gamma \) is positive or negative. Notice that \( w_{tw}(t; \gamma, r) = w_{tw}(t; \gamma) + g(t; \gamma, r) \) and \( g(t; \gamma, r) \) is decreasing in \( \gamma \), so \( w_{tw}(t; \gamma, r) \) is increasing at a lower rate than \( w_{tw}(t; \gamma) \).

(b) We first consider the risk-averse customers, i.e., \( \gamma > 0 \). Suppose that for a given refund \( r \), \( h(t; \gamma, r) \leq h(t; \gamma', r) \) for any \( \gamma > \gamma' > 0 \). Then, we compare \( h(t + \delta; \gamma, r) \) and \( h(t + \delta; \gamma', r) \). From the differential equation (4),

\[ h(t + \delta; \gamma, r) = h(t; \gamma, r) + \delta \lambda_i \Pr(C_i > g(t; \gamma, r)) \left\{ E[C_i | C_i > g(t; \gamma, r)] - (h(t; \gamma, r) - E[R(t)]) \right\} + o(\delta) \]

\[ = h(t; \gamma, r) + \delta \lambda_i \left( \int_{g(t; \gamma, r)}^{\infty} c_i dF_i(c_i) \right) (1 - F_i(g(t; \gamma, r))) (h(t; \gamma, r) - E[R(t)]) + o(\delta) \]

Then, we have

\[ h(t + \delta; \gamma, r) - h(t + \delta; \gamma', r) = h(t; \gamma, r) - h(t; \gamma', r) + \delta \lambda_i \left( \int_{g(t; \gamma, r)}^{\infty} c_i dF_i(c_i) + (F_i(g(t; \gamma, r)) - F_i(g(t; \gamma', r))) \right) \]

\[ \cdot \left[ (h(t; \gamma, r) - h(t; \gamma', r)) (1 - \delta \lambda_i) + \delta \lambda_i \left( (F_i(g(t; \gamma, r)) - F_i(g(t; \gamma', r))) \cdot (g(t; \gamma', r) + E[R(t)]) \right) \]

\[ + (F_i(g(t; \gamma, r)) h(t; \gamma, r) - F_i(g(t; \gamma', r)) h(t; \gamma', r)) \right) \]

\[ \leq (h(t; \gamma, r) - h(t; \gamma', r)) (1 - \delta \lambda_i) + \delta \lambda_i \left( (F_i(g(t; \gamma, r)) - F_i(g(t; \gamma', r))) \cdot (g(t; \gamma', r) + E[R(t)]) \right) \]

\[ \leq (h(t; \gamma, r) - h(t; \gamma', r)) (1 - \delta \lambda_i) \leq 0. \]

The first inequality holds because \( g(t; \gamma, r) \leq g(t; \gamma', r) \); the second inequality holds because \( h(t; \gamma, r) \leq h(t; \gamma', r) \); the third inequality holds because \( g(t; \gamma, r) + E[R(t)] - h(t; \gamma, r) \leq 0 \) for \( \gamma' > 0 \) by Proposition 4. Therefore, for any given time-to-go \( t \) and refund \( r \), \( h(t; \gamma, r) \) is decreasing in \( \gamma \) for risk-averse customers.

Next, consider the risk-seeking customers, i.e., \( \gamma < 0 \). Suppose that for a given refund \( r \), \( h(t; \gamma, r) \geq h(t; \gamma', r) \) at some \( t \geq 0 \) for any \( \gamma' < \gamma < 0 \). Similarly, we have

\[ h(t + \delta; \gamma, r) - h(t + \delta; \gamma', r) = h(t; \gamma, r) - h(t; \gamma', r) + \delta \lambda_i \left( \int_{g(t; \gamma, r)}^{\infty} c_i dF_i(c_i) + (F_i(g(t; \gamma, r)) \right) \]

\[ - F_i(g(t; \gamma, r)) E[R(t)] \]

\[ \geq (h(t; \gamma, r) - h(t; \gamma', r)) (1 - \delta \lambda_i) + \delta \lambda_i \left( (F_i(g(t; \gamma, r)) - F_i(g(t; \gamma', r))) \cdot (g(t; \gamma, r) + E[R(t)]) \right) \]

\[ + (F_i(g(t; \gamma, r)) h(t; \gamma, r) - F_i(g(t; \gamma', r)) h(t; \gamma', r)) \right) \]

\[ \geq (h(t; \gamma, r) - h(t; \gamma', r)) (1 - \delta \lambda_i) \geq 0. \]

The first inequality holds because \( g(t; \gamma, r) \leq g(t; \gamma', r) \); the second inequality holds because \( h(t; \gamma, r) \geq h(t; \gamma', r) \); the third inequality holds because \( g(t; \gamma, r) + E[R(t)] - h(t; \gamma, r) \geq 0 \) for \( \gamma < 0 \) by Proposition 4. Thus, for given time-to-go \( t \) and refund \( r \), \( h(t; \gamma, r) \) is increasing in \( \gamma \) for risk-seeking customers. \( \Box \)
Proof of Theorem 5. For \( \gamma^L < \gamma^H \leq \tilde{\gamma} \), the TW captures both type-L and type-H customers and its profit is equal to \( \bar{w}_v(T;\gamma^L) - E[R(T)] \). An RVW with refund \( r \) and price \( \bar{w}_{rvw}(T;\gamma^L,r) \) captures both market segments. Consider the following inequality

\[
\max_r \left\{ \bar{w}_{rvw}(T;\gamma^L) + g(T;\gamma^L,r) - \alpha^L h(T;\gamma^L,r) - \alpha^H h(T;\gamma^H,r) \right\} > \bar{w}_v(T;\gamma^L) - E[R(T)],
\]

or equivalently,

\[
\min_r \left\{ -g(T;\gamma^L,r) + \alpha^L h(T;\gamma^L,r) + \alpha^H h(T;\gamma^H,r) \right\} < E[R(T)]. \tag{22}
\]

Since \( h(T;\gamma^H,r) \) is decreasing in \( \gamma^H \) for any given \( r \) by Theorem 4, then \( \min_r \left\{ -g(T;\gamma^L,r) + \alpha^L h(T;\gamma^L,r) + \alpha^H h(T;\gamma^H,r) \right\} \) is also decreasing in \( \gamma^H \). Therefore, there exists a threshold \( \bar{\gamma}^H \) such that inequality (22) holds for any \( \gamma^H > \bar{\gamma}^H \). Apparently, \( \gamma^H < \tilde{\gamma}^H \).

We have already shown that the RVW is strictly more profitable than the TW for \( \gamma^H \in (\bar{\gamma}^H, \tilde{\gamma}^H) \). We remark that the RVW may be more profitable for \( \gamma^H \) varying in an even larger range. \( \square \)

Proof of Proposition 6. (a). We will use sample path argument to show the monotonic property. For a Poisson process with a stationary failure rate \( \lambda^H \), the optimal claim policy has a threshold \( g(t;\lambda^H, r) \) for each failure at time \( t \). Assume that the customer with a stationary failure rate \( \lambda^L \) adopts the same policy with threshold \( g(t;\lambda^H, r) \). Apparently, the expected refund net of out-of-pocket repair cost taking into account the risk attitude is greater than that under the failure process with a rate \( \lambda^H \) because each failure occurs with a lower probability. Therefore, \( g(t;\lambda^H, r) \), that is the net value corresponding to rate \( \lambda^L \) under the optimal claim policy is even higher.

(b). For a given refund \( r \), suppose \( \bar{w}_{rvw}(t;\lambda^H, r) \geq \bar{w}_{rvw}(t;\lambda^L, r) \) at some \( t \geq 0 \). Next, we will show that the inequality also holds at time \( t + \delta \) for a sufficiently small \( \delta > 0 \), i.e., \( \bar{w}_{rvw}(t + \delta;\lambda^H, r) \geq \bar{w}_{rvw}(t + \delta;\lambda^L, r) \). Similar to the proof of Theorem 4, we have the following equations

\[
\bar{w}_{rvw}(t + \delta;\lambda^H, r) = \bar{w}_{rvw}(t;\lambda^H, r) + \lambda^H \delta E\left[ e^{\gamma \min(C_t, g(t;\lambda^H, r))} \right] + o(\delta),
\]

\[
\bar{w}_{rvw}(t + \delta;\lambda^L, r) = \bar{w}_{rvw}(t;\lambda^L, r) + \lambda^L \delta E\left[ e^{\gamma \min(C_t, g(t;\lambda^L, r))} \right] + o(\delta).
\]

Comparing them, we have

\[
\bar{w}_{rvw}(t + \delta;\lambda^H, r) \geq \bar{w}_{rvw}(t + \delta;\lambda^L, r).
\]

The inequality holds because \( \bar{w}_{rvw}(t;\lambda^H, r) \geq \bar{w}_{rvw}(t;\lambda^L, r) \), \( \lambda^H \geq \lambda^L \) and \( g(t;\lambda^H, r) \leq g(t;\lambda^L, r) \). Notice that the net value \( g(t;\lambda, r) \) of the RVW is decreasing in failure rate \( \lambda \), so the willingness-to-pay for the RVW is increasing at a lower rate than that for the TW. \( \square \)

Proof of Theorem 6. (a). Notice that the willingness-to-pay can be expressed as follows

\[
\tilde{w}_{rvw}(T;\lambda, r) = w_v(T;\lambda) + \tilde{g}(T;\lambda, r) = \frac{\Lambda(T) \cdot (M_0(\gamma) - 1)}{\gamma} - \frac{1}{\gamma} \ln \left( (1 - e^{-\Lambda(T)}) + e^{-\gamma r - \Lambda(T)} \right).
\]

Consider its first-order derivative with respect to \( \lambda \)

\[
\frac{\partial \tilde{w}_{rvw}(T;\lambda, r)}{\partial \lambda} = \frac{T}{\gamma} \left( M_0(\gamma) - 1 - \frac{e^{-\Lambda(T)} - e^{-\gamma r - \Lambda(T)}}{(1 - e^{-\Lambda(T)}) + e^{-\gamma r - \Lambda(T)}} \right) > \frac{T}{\gamma} \left( M_0(\gamma) - 1 \right) \frac{1}{(1 - e^{-\Lambda(T)})} \geq 0.
\]

The last inequality holds because of the condition in part (a) of Theorem 6.

(b). Dividing the both sides of equation (15) by \( (e^{\min(C_t, g(t;\lambda, r))} - 1)/\gamma \), taking integrals with respect to \( t \) and combining the boundary condition yields the closed-form solution for \( \tilde{g}(T;\lambda, r) \)

\[
\tilde{g}(T;\lambda, r) = -\frac{1}{\gamma} \ln \left( (1 - e^{-\Lambda(t)}) + e^{-\gamma r - \Lambda(t)} \right).
\]

For notational convenience, let \( \Pi_{rvw}(r) = \tilde{w}_{rvw}(T;\lambda^L, r) - (\alpha^L \tilde{h}_{rvw}(T;\lambda^L, r) + \alpha^H \tilde{h}_{rvw}(T;\lambda^H, r)) \). It can be further expressed by

\[
\Pi_{rvw}(r) = w_v(T;\lambda^L) - (\alpha^L \lambda^L + \alpha^H \lambda^H) T e - \frac{1}{\gamma} \ln \left( (1 - e^{-\lambda^L T}) + e^{-\gamma r - \lambda^L T} \right) - (\alpha^L e^{-\lambda^L T} + \alpha^H e^{-\lambda^H T}) r.
\]
Consider the derivatives of $\Pi_{rvw}(r)$ with respect to $r$.

$$\frac{\partial \Pi_{rvw}(r)}{\partial r} = \frac{e^{-\gamma r - \lambda^T r}}{(1 - e^{-\lambda^T r} + e^{-\gamma r - \lambda^T r})} - (\alpha^L e^{-\lambda^T r} + \alpha^H e^{-\gamma r} - \lambda^T r),$$

$$\frac{\partial^2 \Pi_{rvw}(r)}{\partial r^2} = -\frac{\gamma (1 - e^{-\lambda^T r}) \cdot e^{-\gamma r - \lambda^T r}}{(1 - e^{-\lambda^T r} + e^{-\gamma r - \lambda^T r})^2} < 0.$$

So, the profit function $\Pi_{rvw}(r)$ is strictly concave in $r$. Solving the first-order condition $\partial \Pi_{rvw}(r)/\partial r = 0$ yields the optimal refund

$$r^* = \frac{1}{\gamma} \ln \left( \frac{e^{-\lambda^T r} \cdot (1 - (\alpha^L e^{-\lambda^T r} + \alpha^H e^{-\gamma r} - \lambda^T r))}{(1 - e^{-\lambda^T r} + e^{-\gamma r - \lambda^T r})} \right).$$

It can be verified that $r^* > 0$ because for any $\lambda^H > \lambda^L$,

$$e^{-\lambda^T r} \cdot (1 - (\alpha^L e^{-\lambda^T r} + \alpha^H e^{-\gamma r} - \lambda^T r)) > (1 - e^{-\lambda^T r}) \cdot (\alpha^L e^{-\lambda^T r} + \alpha^H e^{-\gamma r} - \lambda^T r).$$

(c). Solving the equation $w_{rvw}(T; \lambda^L) - (\alpha^L \lambda^L + \alpha^H \lambda^H) T c = \alpha^H (w_{tw}(T; \lambda^H) - \lambda^H T c)$ for $\lambda^H$ yields

$$\hat{\lambda}^H = \lambda^L \cdot \frac{(M_c(\gamma) - 1)/\gamma - \alpha^L c}{\alpha^H (M_c(\gamma) - 1)/\gamma}.$$

To show that the RVW is always strictly more profitable than the TW, we compare $\Pi(r^*)$ to the profit of the TW, which is equal to $w_{rvw}(T; \lambda^L) - (\alpha^L \lambda^L + \alpha^H \lambda^H) T c$, i.e.,

$$\Pi(r^*) > w_{tw}(T; \lambda^L) - (\alpha^L \lambda^L + \alpha^H \lambda^H) T c.$$

Notice that the RVW degenerates to the TW when the refund is equal to zero, i.e., $\Pi(0) = w_{tw}(T; \lambda^L) - (\alpha^L \lambda^L + \alpha^H \lambda^H) T c$. Because $\Pi(r)$ is strictly increasing in $r$ and $r^* > 0$, the above strict inequality holds. □

Proof of Proposition 7. (a). We can use the similar arguments as in the proof of Proposition 6 to show the monotonic property with respect to the repair cost.

(b). For a given refund $r$, suppose that $w_{rvw}(t; \gamma^H, r) > w_{rvw}(t; \gamma^L, r)$ at some $t \geq 0$. Next, we will show $w_{rvw}(t + \delta; \gamma^H, r) \geq w_{rvw}(t + \delta; \gamma^L, r)$. Similarly,

$$w_{rvw}(t + \delta; C, r) = w_{rvw}(t; C, r) + \lambda t \delta \frac{E[e^{\gamma C}] - E[e^{\gamma \min(C, g(t; C, r))}]}{\gamma} + o(\delta).$$

We next compare

$$w_{rvw}(t + \delta; C^H, r) = w_{rvw}(t; C^H, r) + \lambda t \delta \frac{E[e^{\gamma C^H}] - E[e^{\gamma \min(C^H, g(t; C^H, r))}]}{\gamma} + o(\delta),$$

$$w_{rvw}(t + \delta; C^L, r) = w_{rvw}(t; C^L, r) + \lambda t \delta \frac{E[e^{\gamma C^L}] - E[e^{\gamma \min(C^L, g(t; C^L, r))}]}{\gamma} + o(\delta).$$

Moreover, we have

$$E[e^{\gamma C^H}] - E[e^{\gamma \min(C^H, g(t; C^H, r))}] \geq E[e^{\gamma C^H}] - E[e^{\gamma \min(C^H, g(t; C^L, r))}] \geq E[e^{\gamma C^L}] - E[e^{\gamma \min(C^L, g(t; C^L, r))}].$$

The first inequality holds because $g(t; C^H, r) \leq g(t; C^L, r)$; the second inequality holds because function $(e^{x} - e^{\min(x, g)})$ is increasing in $x$ and $C^H$ is stochastically larger than $C^L$. Therefore, for any given refund $r$

$$w_{rvw}(t + \delta; C^H, r) \geq w_{rvw}(t + \delta; C^L, r),$$

which completes the proof. □

Proof of Proposition 8. For any given refund $r$, problem (17) is a linear program. From the constraints, we have $p_{rvw} < p_{tw} + g(T; \gamma^L, r) \leq w_{rvw}(T; \gamma^L) + g(T; \gamma^L, r)$. Recall that $p_{rvw} \leq w_{rvw}(T; \gamma^L, r) = w_{rvw}(T; \gamma^L) + g(T; \gamma^L, r)$. Because $w_{rvw}(T; \gamma^L) \leq w_{tw}(T; \gamma^H)$, then the maximum price of the RVW is $p_{rvw} = w_{rvw}(T; \gamma^L, r)$. Because $p_{tw} \leq p_{rvw} - g(T; \gamma^L, r)$, then the maximum price of the TW is $p_{tw} = w_{rvw}(T; \gamma^L, r) - g(T; \gamma^L, r)$. □
Proof of Theorem 7. Comparison between problems (18) and (13) yields that the warranty menu earns strictly more profit than the RVW alone for any given refund $r$, i.e.,
\[
\max_r \left\{ g(T; \gamma^L, r) - \alpha^L h(T; \gamma^L, r) - \alpha^H (g(T; \gamma^H, r) + E[R(T)]) \right\} > \max_r \left\{ g(T; \gamma^L, r) - \alpha^L h(T; \gamma^L, r) - \alpha^H h(T; \gamma^H, r) \right\}.
\]
The inequality holds because $h(T; \gamma^H, r) \geq g(T; \gamma^H, r) + E[R(T)]$ for any risk-averse customer and any positive refund $r$ by Proposition 4.
To show the profit advantage of the menu over the TW alone, consider the following inequality
\[
\max_r \left\{ w_{tv}(T; \gamma^L) + g(T; \gamma^L, r) - \alpha^L h(T; \gamma^L, r) - \alpha^H (g(T; \gamma^H, r) + E[R(T)]) \right\} > w_{tv}(T; \gamma^L) - E[R(T)],
\]
or equivalently,
\[
\min_r \left\{ -g(T; \gamma^L) + \alpha^L h(T; \gamma^L, r) + \alpha^H g(T; \gamma^H, r) \right\} < \alpha^L E[R(T)]. \tag{23}
\]
Since $g(T; \gamma^H, r)$ is decreasing in $\gamma^H$ for any given $r$ by Theorem 1, then $\min_r \left\{ -g(T; \gamma^L, r) + \alpha^L h(T; \gamma^L, r) + \alpha^H g(T; \gamma^H, r) \right\}$ is also decreasing in $\gamma^H$. Therefore, there exists a threshold $\gamma^H_1$ such that inequality (23) holds for any $\gamma^H \geq \gamma^H_1$. Comparing inequalities (22) and (23) and recalling $h(T; \gamma^H, r) > g(T; \gamma^H, r) + E[R(T)]$ for $\gamma^H > 0$ by Proposition 4, we have $\gamma^L_1 < \gamma^H$. It completes the proof. \qed