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Stability in a general oligopoly model

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Abstract

We analyze a general but parsimonious price competition model for an oligopoly in which each firm offers any number of products. The demand volumes are general piecewise affine functions of the full price vector, generated as the "regular" extension of a base set of affine functions. The model specifies a *product assortment*, along with their prices and demand volumes, in contrast to most commonly used demand models. We identify a *fully best* response operator which is monotonically increasing so that the market converges to a Nash equilibrium, when firms dynamically adjust their prices, as best responses to their competitors' prices, at least when starting in one of two price regions. Moreover, geometrically fast convergence to a common equilibrium can be guaranteed for an arbitrary starting point, under an additional condition for the price sensitivity matrix.

KEYWORDS

best response, contraction mapping, equilibrium stability, oligopoly, price competition, product assortment

1 | INTRODUCTION AND SUMMARY

We address a general model for oligopolistic competition, in which each of I independent retailers indexed by $i = 1, \dots, I$ offers a distinct set of products $\mathcal{N}(i)$ such that $\mathcal{N}(1) \cup \mathcal{N}(2) \cup \cdots \cup \mathcal{N}(I) = \mathcal{N}$, with $|\mathcal{N}| = N$; and with consumer demand for the N products specified by a set of *piecewise affine* demand functions. These demand functions are, parsimoniously, generated from a *single* set of affine functions $q(p) = a - Rp$ that prevails on the price polyhedron P on which these functions predict non-negative demand volumes.

Along with variants of the MultiNomial Logit (MNL) model (e.g., mixed or nested MNL models), the most frequently used demand model in operations management, marketing, and industrial organization studies employs affine demand functions. This applies both to theoretical and empirical papers. However, the affine structure cannot be assumed to prevail on the complete price space: after all, outside of the special polyhedron P, the affine demand functions predict negative demand volumes for at least some of the products.

The extension of these demand functions, beyond P to $\mathbb{R}_+^N$, the full positive orthant of the N-dimensional price space, is guided by a simple and intuitive regularity condition introduced by Shubik and Levitan (1980): under any given price vector, when some product experiences zero demand, that is, when it is priced out of the market, any *increase* of its own price keeps the product itself out of the market and has *no* impact on any of the demand volumes. While seemingly innocuous and allowing for many specifications, Soon, Zhao, and Zhang (2009) showed that this regularity condition is satisfied by a *unique* set of (piecewise linear) demand functions.

Indeed, the demand volumes resulting from an arbitrary price vector p are obtained as the unique solution of a linear complementarity problem (LCP). Geometrically, the demand volumes are obtained by applying the base affine vector function $q(\cdot)$ to the (unique) *projection* of the price vector p onto the polyhedron P.

One of the major advantages of this demand model is the fact that the product assortment sold in the market varies with the price vector chosen by the competing firms. This, in contrast to more popular demand models that are based on, say, the MNL consumer choice model, or any of its generalizations: nested MNL models, see for example Ben-Akiva (1973) and Gallego and Wang (2014), or Mixed MNL models, apparently introduced by Boyd and Mellman (1980) and Cardell and Dunbar (1980) and the most common workhorse model in marketing and economics, since Steckel and Vanhonacker (1988) and Berry, Levinsohn, and Pakes (1995), say. In all of these MNL-based demand models, *every* product maintains a positive market share irrespective of how the products are priced. This property is, in practice, usually violated. In particular, when looking for an oligopoly model in which both prices and product assortments may vary, the above model is, to the best of our knowledge, the only one available, in particular when allowing for an arbitrary number of competitors each offering an arbitrary set of potential products with demand functions that (may) depend on the full price vector.

Starting with Muto (1993), several authors have analyzed the competition model that arises under a general linear cost structure. Muto (1993) started with a symmetric duopoly in which each firm sells a single product. Farahat and Perakis (2010) addressed the general model with an arbitrary number of competitors each offering an arbitrary collection of products, however with a *symmetric* price sensitivity matrix R. The authors showed that a *unique* Nash equilibrium exists if the vector of cost rates resides in the interior of the above mentioned price polyhedron P. Federgruen and Hu (2015, 2018) characterized the equilibrium behavior in the fully general model, with a general possibly asymmetric price sensitivity matrix R and an arbitrary vector of cost rates. See the latter for a comprehensive review of various other publications addressing special cases of the model.

Federgruen and Hu (2015) proved that a pure Nash equilibrium always exists; however, there may be multiple, even infinitely many such equilibria, depending on the choice of the cost rate vector. Moreover, only part of the N products may survive the competition and be part of the equilibrium assortment.

Depending on the level of sophistication of the firms in the industry and the amount of market intelligence that is available, it may be unrealistic to assume the comprehensive knowledge of the set of all demand functions pertaining to all products on the market, the foundation of any such game-theoretical model. Federgruen and Hu (2018) therefore focus on a more simplistic, but sometimes more realistic, best response operator, where each firm restricts its price choices to a given polyhedron on which the demand volumes of its products vary with the prices in accordance with simple affine functions. Application of *this* best response operator, referred to as the *robust* best response operator, only requires *local* information, that is, a firm only needs to know the demand functions of its *own* products.

In general, a firm's *fully best* response vector fails to be unique. However, we identify a specific fully best response operator which is monotonically increasing. This implies that the dynamic adjustment process generated by these fully best responses converges to a Nash equilibrium of the competition game, at least when the starting price vector is in one of two price regions. We then show that *global* stability, that is, convergence for *any* arbitrary starting point and to the *same* equilibrium, can be guaranteed, under an additional property of the price sensitivity matrix. This additional assumption requires this matrix to be (row-wise) diagonally dominant, with the relative magnitude of the sum of the off-diagonal elements in each row, staying below a given threshold value. (The row-wise diagonal dominance, per se, is, a rather innocuous assumption: it is equivalent to assuming that a uniform price increase of all products cannot result in an increase of any product's sales volume.) Under this additional assumption, we, in fact, prove that the selected fully best response operator is a *contraction mapping*, so that the dynamic adjustment process converges to the same Nash equilibrium, irrespective of the market's starting point and this at a geometrically fast rate.

Contraction mappings have been used in a variety of game-theoretical models. In the context of Cournot, quantity-setting games, it was used in Szidarovszky and Yakowitz (1977), Gaudet and Salant (1991) and Van Long and Soubeyran (2000). In their survey paper, Cachon and Netessine (2006) provide a sufficient condition for the best response mapping to be a contraction mapping. They confine themselves, however, to settings where this mapping is unique and differentiable everywhere. Neither one of these conditions is satisfied in our model. See Altman, Hordijk, and Spieksma (1997) for an application in the area of multiperiod Markov games.

Finally, a word about Uri Rothblum of blessed memory to whom this article is dedicated. Among his many outstanding contributions to the field of operations research, is his creative and deep usage of advanced linear algebra and piecewise affine mappings to so many operational problems. See his biography in Loewy (2012).

We use some properties of square matrices of special structure.

Definition 1 (Z-matrix). A square matrix whose off-diagonal entries are nonpositive is called a Z-matrix.

Definition 2 (P-matrix). A square matrix whose principal minors are all positive is called a P-matrix.

Definition 3 (ZP-matrix). A matrix that is both a Z-matrix and a P-matrix is called a ZP-matrix.

It is well known that all positive definite matrices are P-matrices, see, for example, Cottle et al. (1992, chap. 3). However, the class of P-matrices is significantly broader.

For any pair of vectors x, y in the same Euclidean space, we write $x \le y$ if and only if each component of x is dominated by the corresponding component of y. We also use various matrix norms. A matrix norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$ assigns to any $n\times n$-matrix A, a number $\|A\|$ with the following properties:

i. $\|\lambda A\| = |\lambda|\,\|A\|$, for all $\lambda \in \mathbb{R}$.

ii. $\|A+B\| \leq \|A\| + \|B\|$,
for any $n \times n$ matrix B.

iii. $\|A\| \geq 0$ and $\|A\| = 0$ if and only if $A = 0_{n,n}$.

A matrix norm is *induced by a vector norm* $\|\cdot\|_v$ on $\mathbb{R}^n$ if

$$\|A\| = \sup\left\{ \frac{\|Ax\|_v}{\|x\|_v} \mid x \in \mathbb{R}^n \quad \text{with } x \neq 0 \right\}.$$

A vector norm $\|\cdot\|$ on $\mathbb{R}^n$ is *absolute* if $\|x\| = \| |x| \|$ where $|x| \equiv (|x_1|, |x_2|, \ldots, |x_n|)$. It is well known, see, for example, Johnson and Nylen (1991), that any absolute vector norm is *monotone*, that is,

$$\|x\| \leq \|y\|, \quad \text{if } |x| \leq |y| .$$

In fact, monotonicity and absoluteness are equivalent properties for vector norms, see Johnson and Nylen (1991). Lastly, $\mathbb{R}_+ \equiv \{x \mid x \geq 0\}$ and $\mathbb{R}_{++} \equiv \{x \mid x > 0\}$.

The remainder of this paper is organized as follows. The model and notation are specified in Section 2. This section also reviews the results needed in the remainder of this paper. Global stability under fully best responses is shown in Section 3. Section 4 ends the paper with concluding comments. We refer the readers to Federgruen and Hu (2018) for a detailed survey of the literature that is related to the above competition model.

2 | THE COMPETITION MODEL

We consider an industry with I competing retailers, each with a set of *potential* products it may offer in the market. As mentioned, we let $\mathcal{N}(i)$ denote this set of products, for retailer $i = 1, \ldots, I$, and let $\mathcal{N} = \cup_{i=1}^{I} \mathcal{N}(i)$ denote the set of *all* products that are potentially sold in the market. The sets $\{\mathcal{N}(i) : i = 1, \ldots, I\}$ are mutually exclusive and each product is identified by a pair of indices: The pair (i, k) denotes the kth product in retailer i's potential assortment. Thus, products are differentiated both by their attributes and functionalities, as well as the specific retailer via which they are distributed. For example, a Samsung television set is viewed as a different product when sold via Walmart or Target, to reflect the differences in the retail experience.

Retailers receive their products from suppliers under simple (linear) wholesale price contracts. As demonstrated for example by Hwang, Bakshi, and DeMiguel (2015), such simple wholesale pricing schemes continue to be the most frequently used type of contracts. Thus, let

$$w_l = \text{the unit procurement cost for product } l,$$

henceforth referred to as the wholesale price of product l, $l = 1, \ldots, N$.

The retailer competition model that arises under a given vector $w \in W$, has the following profit functions $\{\pi_i(p)$: $i = 1, \ldots, I\}$:

$$\pi_i(p) = \sum_{l \in \mathcal{N}(i)} (p_l - w_l) d_l(p),$$

with $d_l(p)$ the demand function for product l, $l \in \mathcal{N}$.

The demand functions used in this paper are *based on* a general set of affine functions:

$$q(p) = a - Rp, \tag{1}$$

with $p = \{p_{ik} \mid i = 1, \ldots, I, k = 1, \ldots, |\mathcal{N}(i)|\}$ the vector of retail prices, R an $N \times N$-matrix and $a \geq 0$ to reflect that all products are relevant choices, at least when they are offered for free. However, the affine functions in (1) predict non-negative demand volumes, only when $p \in P$, where

$$P \equiv \{p \geq 0 \mid a - Rp \geq 0\} \tag{2}$$

denotes the effective price polyhedron. To extend the specification on the remainder of the full price space $\mathbb{R}_+^N$, that is, on $\mathbb{R}_+^N \setminus P$, we follow the suggestion initiated by Shubik and Levitan (1980), that is, to impose the following "regularity" condition.

> **Definition 4** (Regularity). A demand function $D(p) : \mathbb{R}_+^N \to \mathbb{R}_+^N$ is said to be *regular* if for any product l and any price vector p, $D_l(p) = 0$ implies that $D(p + \Delta \cdot e_l) = D(p)$ for any $\Delta > 0$, where e_l denotes the l-th unit vector.

As mentioned in the Introduction, the regularity condition is both simple and intuitive: under any given price vector, when some product experiences zero demand, that is, when it is priced out of the market, any *increase* of its own price keeps the product itself out of the market and has *no* impact on any of the demand volumes. This regularity condition is generally accepted in the recent literature, see, for example, Farahat and Perakis (2010), Soon et al. (2009), Federgruen and Hu (2015, 2016, 2018). Alternative extensions include $d(p) = q(p)^+ = [a - Rp]^+$; however, this specification results in pathological price choices, in particular when firms offer multiple products, as is, typically the case. The regularity condition is completely intuitive and appears highly innocuous, allowing for many possible specification options. However, Soon et al. (2009) showed that the regularity condition specifies the extension of the demand functions, beyond P, in a *unique* way: instead of applying the affine functions $q(\cdot)$ directly to the price vector p, it must be applied to the downward corrected vector $p' = p - t$, where the correction vector $t \geq 0$ satisfies the LCP:

$$d(p) = q(p - t) = a - R(p - t) \geq 0, \tag{3}$$

$$t^{\mathsf{T}}[a - R(p - t)] = 0, \text{ and } t \geq 0. \tag{4}$$

Geometrically, the vector $p' = p - t$ may be viewed as the *projection* $\Omega(p)$ of p onto the polyhedron P.

The LCP (3), (4) has a *unique* solution, and the projection operator Ω: $p \mapsto \Omega(p) = p - t$ is therefore well defined, under the following common assumption about the price sensitivity matrix R, see Cottle et al. (1992, Theorem 3.1.6).

Assumption (P). The matrix R is positive definite.

Various sufficient conditions and numerical procedures may be used to verify a matrix R is positive definite. These include:

$$R_{l,l} \geq \sum_{l' \neq l} |R_{l,l'}|, \quad \text{for all } l = 1, \ldots, N,$$
$$\text{or} \quad R_{l,l} \geq \sum_{l' \neq l} |R_{l',l}|, \quad \text{for all } l = 1, \ldots, N. \tag{5}$$

In addition to property (P), we assume that the matrix R is a Z-matrix, that is, has non-positive off-diagonal elements, to reflect the fact that the products are substitutes.

Assumption (Z). The matrix R is a Z-matrix.

The following properties of the projection operator $\Omega(\cdot)$ will be used repeatedly, see Lemma A.2 in the Online Appendix of Federgruen and Hu (2016).

Lemma 1 (Projection). *Assume Assumptions (P) and (Z) hold.*

a. $\Omega(p) \in P$; *if* $p \in P$, $\Omega(p) = p$.
b. If $p \notin P$, $\Omega(p)$ *is on the boundary of* P.
c. $\Omega(p)$ *may be computed by minimizing any linear objective* $\varphi^{\top} t$ *with* $\varphi > 0$ *over the polyhedron, described by (2).*
d. The projection operator $\Omega(\cdot)$ *is monotonically increasing, and each component of* $\Omega(\cdot)$ *is a jointly concave function.*

The demand functions $\{d_l(\cdot): l = 1, \ldots, N\}$ satisfy the normal monotonicity properties; see Proposition B.1 in the Online Appendix of Federgruen and Hu (2015).

Lemma 2 *For any* $l \in \mathcal{N}$, *the demand function* $d_l(p)$ *is decreasing in its own price* p_l *and increasing in the price* $p_{l'}$ *of any of its substitutes* $l' \neq l$.

An alternative approach for the regular extension of the raw affine function $q(\cdot)$ is to derive the demand functions from a quadratic utility maximization problem of a representative consumer:

$$\text{(QP)} \quad \max_{d \geq 0} (\widetilde{R}^{-1} a - p)^{\top} d - \frac{1}{2} d^{\top} \widetilde{R}^{-1} d, \tag{6}$$

once again with a positive definite matrix $\widetilde{R}$. Since $\widetilde{R}$ is positive definite, so is its inverse $\widetilde{R}^{-1}$. The utility function in (6) is therefore a general jointly concave function of the consumption/demand volumes d.

The Karush-Kuhn-Tucker conditions of this quadratic program are therefore both necessary and sufficient, and are easily seen to be equivalent to the LCP (3) and (4), with $R = (\tilde{R} + \tilde{R}^{\top})/2$ positive definite and symmetric. In other words, the utility maximization problem (QP) may be used as the foundation for the demand functions $d(p)$, but only for *symmetric* price sensitivity matrix R, that is, only when cross elasticities are symmetric. In practice, the matrix R is often asymmetric, see, for example, Vilcassim, Kadiyali, and Chintagunta (1999) and Dubé and Manchanda (2005).

Federgruen and Hu (2015) showed that the competition game always has a pure Nash equilibrium. Whether this equilibrium can be guaranteed to be unique, or not, depends on the position of the wholesale price vector w; more specifically, uniqueness is guaranteed if w resides in a polyhedron W, defined as:

$$W \equiv \{w > 0 \mid \Psi(R)a - \Psi(R)Rw \geq 0\} \text{ with}$$
$$\Psi(R) \equiv T(R)[R + T(R)]^{-1} \text{ and}$$
$$T(R) \equiv \begin{pmatrix} R^{\top}_{\mathcal{N}(1),\mathcal{N}(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R^{\top}_{\mathcal{N}(I),\mathcal{N}(I)} \end{pmatrix}.$$

The (wholesale) price sensitivity matrix $S \equiv \Psi(R)R$ is positive definite, by Proposition 2(d) in Federgruen and Hu (2015). Moreover, S is a Z-matrix and $b \equiv \Psi(R)a \geq 0$ under a minor additional condition (beyond Assumptions (P) and (Z)) for the price sensitivity matrix R. Federgruen and Hu (2015, Proposition 3) identify the progressively weaker conditions to guarantee that S is a Z-matrix and $b \geq 0$. For the sake of simplicity, we adopt the strongest of these sufficient conditions:

Assumption (IS). The matrix R is intra-firm symmetric, that is, $R_{ik,ik'} = R_{ik',ik}$ for all $i = 1, \ldots, I$ and $k, k' \in \mathcal{N}(i)$.

Intrafirm asymmetry is trivially satisfied in the special case where each firm offers a single product.

The following proposition characterizes the set of pure Nash equilibria in this competition game, see Theorem 1 in Federgruen and Hu (2018).

Proposition 1

a. The competition game has a pure Nash equilibrium.
b. At most one (pure) equilibrium is in $int(P)$.
c. If $w \in int(W)$, *the competition game has a unique equilibrium*

$$(p^*|w) = p^*(w) \equiv w + [R + T(R)]^{-1} q(w).$$

d. If $w \in \mathbb{R}^N_{++} \setminus int(W)$, $(p^*|w) = p^*(w') \in \partial P$ *is an equilibrium, where* w' *is the projection of* w *onto the polyhedron* W, *which is defined as in the LCP (3), (4) with* a *and* R *replaced by* b *and* S.

We conclude this section with a preliminary lemma.

Lemma 3 *Let* A *be a ZP-matrix and* B *be a Z-matrix such that* $A \leq B$, *that is,* $B - A \geq 0$. *Then*

a. A^{-1} *exists and* $A^{-1} \geq 0$.
b. B *is a ZP-matrix and* $B^{-1} \leq A^{-1}$.
c. AB^{-1} *and* $B^{-1}A$ *are ZP-matrices.*

Let $Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *represent a block decomposition of the matrix Y, with A and D square matrices. The Schur complement of A in Y is defined as* $(D - CA^{-1}B)$.

d. *If Y is a ZP-matrix, so are A and its Schur complement* $(D - CA^{-1}B)$.
e. *If A is nonsingular and Y is the inverse of a ZP-matrix, then the Schur complement of A,* $(D - CA^{-1}B)$ *is the inverse of a ZP-matrix, itself, and hence* $(D - CA^{-1}B) \geq 0$.
f. *Let X be the inverse of a symmetric ZP-matrix* $Y = \begin{pmatrix} A & B \\ B^\top & D \end{pmatrix}$, *with A a square* $|\bar{S}| \times |\bar{S}|$ *matrix and D a square* $|S| \times |S|$ *matrix. Thus,* $X_{\bar{S},S}X_{S,S}^{-1}X_{S,\bar{S}} \geq 0$.
g. *If* $A = I - Q$ *is a ZP-matrix, with* $Q \geq 0$, *then* $\rho(Q) < 1$, *with* $\rho(Q)$ *the spectral radius of Q.*

3 | STABILITY UNDER FULLY BEST RESPONSES

In this section, we characterize the behavior of an iterative best response process, also referred to as a tatônnement process.

The best response operator for a given firm i is, in principle, defined as follows:

$$FB_i(p_{-\mathcal{N}(i)}) = \arg\max_{p_{\mathcal{N}(i)} \geq 0} \pi_i(p_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) = \arg\max_{p_{\mathcal{N}(i)} \geq 0} (p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^\top d_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}). \tag{7}$$

We use the notation $FB_i(\cdot)$, to emphasize that a *fully best* (FB) set of prices is chosen by firm i among *all* possible price vectors. This is in contrast to the so-called *robust* best response operator $RB_i(\cdot)$ characterized in Federgruen and Hu (2018). The definition in (7) is inadequate since multiple price vectors $p_{\mathcal{N}(i)}$ may achieve the maximum in (7), that is, the (fully) best response fails to be unique. Thus, depending on the specific selection of a best response price vector, at every stage of the tatônnement scheme, a different process of price vectors may be generated. However, we identify a specific FB response mapping for any given firm $i = 1, \ldots, I$, which is a relatively simple analytical function of the firm's wholesale prices and the retail prices chosen by its competitors, as well as *monotonically increasing*, that is, for all $i = 1, \ldots, I$, when $p_{-\mathcal{N}(i)} \leq \hat{p}_{-\mathcal{N}(i)}$, $FB_i(p_{-\mathcal{N}(i)}) \leq FB_i(\hat{p}_{-\mathcal{N}(i)})$. Then, assuming that in each iteration of the dynamic adjustment process, each firm chooses this FB response, we establish that this dynamic adjustment process converges to an equilibrium of the competition model, at least when the market starts in specific regions of the price space. This is identical to what is obtained in a supermodular game, see, for example, Theorem 2.10 in Vives (2001). For example, the process is stable, that is, converges to an equilibrium when the initial price vector $p^0 = w$ or $p^0 = p^{\max}$, with $p^{\max}$ any vector of sufficiently large prices.

This, of course, falls short of *global* stability where the dynamic adjustment process converges to a Nash equilibrium of the competition game, *regardless* of the market's starting point. In Section 3.2, we obtain this strong stability property under an additional condition for the matrix R, requiring it to be (row-wise) diagonally dominant, see (5), and the relative magnitude of the sum of the off-diagonal elements below a given threshold. To this end, we introduce a measure Δ, which characterizes the *degree* of *product substitutability* and increases as the sum of absolute values of the off-diagonal elements in any row of R comes close to its diagonal element, see (5).

3.1 | Characterization of fully best responses

We start with the following propositions.

Proposition 2 (Non-negative profit margins in best responses). *Fix* $w \geq 0$ *and i. For any price choices* $p_{-\mathcal{N}(i)}$ *by retailer i's competitors, there exists a best response* $p^*_{\mathcal{N}(i)}(p_{-\mathcal{N}(i)}) \geq w_{\mathcal{N}(i)}$.

Proof of Proposition 2 Assume for some product (i, k), $\hat{p}_{ik}(p_{-\mathcal{N}(i)}) < w_{ik}$. Increasing $\hat{p}_{ik}$ to a value $\geq w_{ik}$ improves the profit earned for *this* product, while, by Lemma 2, increasing the sales volume and hence the profit earned for all other products sold by retailer i with a non-negative profit margin. Thus, sequentially increasing each of the prices $\hat{p}_{ik} < w_{ik}$ to the w_{ik}-level results in a profit improvement while ensuring that all profit margins are non-negative. ∎

Proposition 3 (Sufficient condition for a fully best response). *Fix* $i = 1, \ldots, N$ *and a vector* $p_{-\mathcal{N}(i)}$ *of the competitors' prices. If there exists* $p^*_{\mathcal{N}(i)}$ *and* $t_{-\mathcal{N}(i)} \geq 0$ *such that*

i. $p^*_{\mathcal{N}(i)}$ *is the best response to* $p_{-\mathcal{N}(i)} - t_{-\mathcal{N}(i)}$ *among all* $p_{\mathcal{N}(i)}$ *such that* $(p_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)} - t_{-\mathcal{N}(i)}) \in P$;
ii. $t^\top_{-\mathcal{N}(i)} q_{-\mathcal{N}(i)}(p^*_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)} - t_{-\mathcal{N}(i)}) = 0$;
iii. $p^*_l \geq w_l$ *for all* $l \in \mathcal{N}(i)$ *such that* $q_l(p^*_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)} - t_{-\mathcal{N}(i)}) > 0$, *then* $p_{\mathcal{N}(i)} = \max(p^*_{\mathcal{N}(i)}, w_{\mathcal{N}(i)})$ *is a fully best response to* $p_{-\mathcal{N}(i)}$.

Proof of Proposition 3 Let $p^o \equiv (p_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)})$. By the definition of the projection operator Ω, there exists a unique solution $t' \geq 0$ to the LCP $0 \leq d(p^o) = a - R\Omega(p^o) = a - R(p^o - t')$

and $t'^{\top}[a - R(p^o - t')] = 0$. Since

$$t'_l = \begin{cases} 0 & \text{if } p^*_l \geq w_l \\ w_l - p^*_l & \text{otherwise} \end{cases} \quad \text{for} \quad l \in \mathcal{N}(i),$$

$$t'_{-\mathcal{N}(i)} = t_{-\mathcal{N}(i)}$$

is a solution to the LCP,

$$\hat{p} = (p^*_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)} - t_{-\mathcal{N}(i)}) = \Omega(p^o) = p^o - t' \leq p^o. \quad (8)$$

By Federgruen and Hu (2015, Proposition 1(b)),

$$0 \leq d(p^o) = q(\hat{p}) = d(\hat{p}).$$

Then for retailer i,

$$\begin{aligned}\pi_i(p^o) &= (p^o_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^{\top} d_{\mathcal{N}(i)}(p^o) \\ &= (p^o_{\mathcal{N}(i)} - t'_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^{\top} d_{\mathcal{N}(i)}(p^o) \\ &= (\hat{p}_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^{\top} d_{\mathcal{N}(i)}(\hat{p}) = \pi_i(\hat{p}), \end{aligned} \quad (9)$$

where the second equality is due to the complementarity of t' and $d(p^o)$. By Proposition 2, for given $\hat{p}_{-\mathcal{N}(i)}$, there exists a best response $\overline{p}_{\mathcal{N}(i)} \geq 0$ to $\hat{p}_{-\mathcal{N}(i)}$ such that $\overline{p}_{\mathcal{N}(i)} \geq w_{\mathcal{N}(i)}$. Then

$$\begin{aligned}\pi_i(\hat{p}) &\leq \max_{p_{\mathcal{N}(i)} \geq 0} [(p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^{\top} d_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}, \hat{p}_{-\mathcal{N}(i)})] \\ &= (\overline{p}_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^{\top} d_{\mathcal{N}(i)}(\overline{p}_{\mathcal{N}(i)}, \hat{p}_{-\mathcal{N}(i)}) \\ &\leq (\overline{p}_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^{\top} d_{\mathcal{N}(i)}(\overline{p}_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \\ &\leq \max_{p_{\mathcal{N}(i)} \geq 0} [(p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^{\top} d_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)})] \\ &= \pi_i(p^o) = \pi_i(\hat{p}), \end{aligned} \quad (10)$$

where the second inequality is due to $\overline{p}_{\mathcal{N}(i)} \geq w_{\mathcal{N}(i)}$ and Lemma 2; the latter guarantees that $0 \leq d_{\mathcal{N}(i)}(\overline{p}_{\mathcal{N}(i)}, \hat{p}_{-\mathcal{N}(i)}) \leq d_{\mathcal{N}(i)}(\overline{p}_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)})$, since $\hat{p}_{-\mathcal{N}(i)} \leq p_{-\mathcal{N}(i)}$, by (8). The last equality follows from (9). Thus, all inequalities in (10) hold as equalities and in particular, $\pi_i(\hat{p}) = \max_{p_{\mathcal{N}(i)} \geq 0}[(p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^{\top} d_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}, \hat{p}_{-\mathcal{N}(i)})]$. Hence $\hat{p}_{\mathcal{N}(i)} = p^*_{\mathcal{N}(i)}$ is a best response to $\hat{p}_{-\mathcal{N}(i)}$. But $(p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^{\top} d_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}, \hat{p}_{-\mathcal{N}(i)}) = (p^*_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^{\top} d_{\mathcal{N}(i)}(p^*_{\mathcal{N}(i)}, \hat{p}_{-\mathcal{N}(i)})$, since, by assumption, $p_{\mathcal{N}(i)}$ can be altered from $p^*_{\mathcal{N}(i)}$ by increasing the prices of only those products l for which $d_l(\hat{p}) = 0$. However, by the regularity condition, such price increases have no impact on *any* of the demand volumes while increasing the profit margins of only the products with zero demands. Thus $p_{\mathcal{N}(i)}$ is a specific choice for a best response $\overline{p}_{\mathcal{N}(i)} \geq w_{\mathcal{N}(i)}$ to $\hat{p}_{-\mathcal{N}(i)}$ in the second line of (10). Since all inequalities in (10) hold as equalities, $p_{\mathcal{N}(i)}$ is a best response to $p_{-\mathcal{N}(i)}$. ■

There are, typically, many, in fact infinitely many, such FB responses. This implies, a fortiori, that there are *infinitely many* dynamic adjustment processes in which, in each iteration, a firm selects a FB response to the prices chosen by its competitors. To avoid this ambiguity, we identify a specific FB response, with a relatively simple analytical characterization and characterize the dynamic adjustment process in the market, under the convention that firms select this specific FB response.

We now derive this fairly simple *analytical* characterization of a FB response vector, for any firm $i = 1, \dots, N$, to its competitor's prices $p_{-\mathcal{N}(i)}$.

For the *unconstrained* optimization problem:

$$\max_{p_{\mathcal{N}(i)}} (p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^{\top}(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p_{-\mathcal{N}(i)} - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)}), \quad (11)$$

the optimal solution is in the form (see the expression on top of Eq. (6) in Federgruen and Hu (2015)):

$$p^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \equiv [2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1} \cdot (a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p_{-\mathcal{N}(i)}) + \frac{1}{2} w_{\mathcal{N}(i)}. \quad (12)$$

Substituting (12) into $a - Rp$, we obtain:

$$a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),\mathcal{N}(i)} p^*_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p_{-\mathcal{N}(i)} = \frac{1}{2}[a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p_{-\mathcal{N}(i)} - R_{\mathcal{N}(i),\mathcal{N}(i)} w_{\mathcal{N}(i)}], \quad (13)$$

and

$$\begin{aligned} &a_{-\mathcal{N}(i)} - R_{-\mathcal{N}(i),-\mathcal{N}(i)} p_{-\mathcal{N}(i)} - R_{-\mathcal{N}(i),\mathcal{N}(i)} p^*_{\mathcal{N}(i)} \\ &= a_{-\mathcal{N}(i)} - R_{-\mathcal{N}(i),-\mathcal{N}(i)} p_{-\mathcal{N}(i)} - R_{-\mathcal{N}(i),\mathcal{N}(i)} \cdot \left\{ [2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1}(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p_{-\mathcal{N}(i)}) + \frac{1}{2} w_{\mathcal{N}(i)} \right\} \\ &= a_{-\mathcal{N}(i)} - R_{-\mathcal{N}(i),\mathcal{N}(i)} \frac{1}{2}(R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} a_{\mathcal{N}(i)} + w_{\mathcal{N}(i)}) \\ &\quad -(R_{-\mathcal{N}(i),-\mathcal{N}(i)} - R_{-\mathcal{N}(i),\mathcal{N}(i)}[2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1} \cdot R_{\mathcal{N}(i),-\mathcal{N}(i)}) p_{-\mathcal{N}(i)}. \end{aligned}$$

We define the following polyhedron

$$H = \left\{ \begin{pmatrix} w_{\mathcal{N}(i)} \\ p_{-\mathcal{N}(i)} \end{pmatrix} \geq 0 \,\middle|\, L \begin{pmatrix} w_{\mathcal{N}(i)} \\ p_{-\mathcal{N}(i)} \end{pmatrix} \equiv f - G \begin{pmatrix} w_{\mathcal{N}(i)} \\ p_{-\mathcal{N}(i)} \end{pmatrix} \geq 0 \right\},$$

where

$$f = \begin{pmatrix} a_{\mathcal{N}(i)} \\ 2a_{-\mathcal{N}(i)} - R_{-\mathcal{N}(i),\mathcal{N}(i)} R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} a_{\mathcal{N}(i)} \end{pmatrix}$$

and

$$G = \begin{pmatrix} R_{\mathcal{N}(i),\mathcal{N}(i)} & R_{\mathcal{N}(i),-\mathcal{N}(i)} \\ R_{-\mathcal{N}(i),\mathcal{N}(i)} & 2R_{-\mathcal{N}(i),-\mathcal{N}(i)} - R_{-\mathcal{N}(i),\mathcal{N}(i)} R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} R_{\mathcal{N}(i),-\mathcal{N}(i)} \end{pmatrix}.$$

The following lemma shows that $f \geq 0$ and G is a positive-definite Z-matrix.

Lemma 4

G is a positive-definite Z-matrix and $f \geq 0$.

The analytical characterization of a FB response is most easily established when $(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \in H$.

Proposition 4 *Fix* $i = 1, \ldots, I$, *a vector of wholesale prices* $w_{\mathcal{N}(i)} \geq 0$ *and a vector of the competitors' retail prices* $p_{-\mathcal{N}(i)}$. *If* $(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \in H$, $p^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)})$ *is a fully best response to* $p_{-\mathcal{N}(i)}$, *and*

$$p^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \geq w_{\mathcal{N}(i)}. \tag{14}$$

Proof of Proposition 4 We verify the conditions in Proposition 3. With $t_{-\mathcal{N}(i)} = 0$, condition (ii) of Proposition 3 is satisfied.

Since $(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \in H$, we have $(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \geq 0$ and $f - G\begin{pmatrix} w_{\mathcal{N}(i)} \\ p_{-\mathcal{N}(i)} \end{pmatrix} \geq 0$. The latter is equivalent to

$$a - R\begin{pmatrix} p^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \\ p_{-\mathcal{N}(i)} \end{pmatrix} \geq 0.$$

Given $(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \geq 0$, we have by (12), $p^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \geq 0$ because $R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} \geq 0$ and $R_{\mathcal{N}(i),-\mathcal{N}(i)} \leq 0$. Therefore,

$$(p^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}), p_{-\mathcal{N}(i)}) \in P.$$

Since $p^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)})$ is obtained from the unconstrained optimization problem (11), condition (i) of Proposition 3 is satisfied.

With (14) verified, condition (iii) of Proposition 3 is satisfied. To prove (14), note, by (12), that

$$\begin{aligned} &p^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) - w_{\mathcal{N}(i)} \\ &\quad = [2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1}(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)}p_{-\mathcal{N}(i)}) - \frac{1}{2}w_{\mathcal{N}(i)} \\ &\quad = [2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1}(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)}p_{-\mathcal{N}(i)} \\ &\qquad - R_{\mathcal{N}(i),\mathcal{N}(i)}w_{\mathcal{N}(i)}) \geq 0, \end{aligned}$$

where the inequality is due to $R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} \geq 0$ and $a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)}p_{-\mathcal{N}(i)} - R_{\mathcal{N}(i),\mathcal{N}(i)}w_{\mathcal{N}(i)} \geq 0$ since $(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \in H$. ■

The following Theorem 1 completes the characterization of a FB response when $(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \notin H$. Let

$$\begin{aligned} &P^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \\ &\quad \equiv [2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1}(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)}p'_{-\mathcal{N}(i)}) + \frac{1}{2}w_{\mathcal{N}(i)}, \end{aligned}$$

where $(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) = \Lambda(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)})$ is the projection of $(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)})$ onto the polyhedron H. Assuming each firm selects this (specific) FB response, we get that, in each iteration of the dynamic adjustment process, a retail price vector $p \in \mathbb{R}^N_+$ is transformed into:

$$\begin{aligned} FB(p) = (&\max\{P^*_{\mathcal{N}(1)}(w_{\mathcal{N}(1)}, p_{-\mathcal{N}(1)}), w_{\mathcal{N}(1)}\}, \ldots, \\ &\max\{P^*_{\mathcal{N}(I)}(w_{\mathcal{N}(I)}, p_{-\mathcal{N}(I)}), w_{\mathcal{N}(I)}\}). \end{aligned} \tag{15}$$

Theorem 1

a. *Fix* $i = 1, \ldots, I$, *a vector of wholesale prices* $w_{\mathcal{N}(i)} \geq 0$ *and a vector of the competitors' retail prices* $p_{-\mathcal{N}(i)}$. *Then* $\max\{P^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}), w_{\mathcal{N}(i)}\}$ *is a fully best response to* $p_{-\mathcal{N}(i)}$.
b. *The FB response operator* $FB(\cdot)$ *is monotonically increasing.*
c. *Let* $P^+ = \{p \geq 0 \mid FB(p) \geq p\}$ *and* $P^- = \{p \geq 0 \mid FB(p) \leq p\}$. *For any* $p \in P^+(P^-)$, $\lim_{n\to\infty} FB^{(n)}(p) = p^{**}$, *with* p^{**} *an equilibrium of the competition game and* $FB^{(n)}(\cdot)$ *the n-fold application of the* $FB(\cdot)$ *operator.*

Proof of Theorem 1 (a) If $(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \in H$, $P^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) = p^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \geq w_{\mathcal{N}(i)}$, by (14), so $\max\{P^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}), w_{\mathcal{N}(i)}\} = p^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)})$ which is a best response to $p_{-\mathcal{N}(i)}$ by Proposition 4.

Thus, assume $(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \notin H$. By Lemma 4, $(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) \geq 0$, so that $(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) \in H$. Thus, we have $(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) \geq 0$ and

$$f - G\begin{pmatrix} w'_{\mathcal{N}(i)} \\ p'_{-\mathcal{N}(i)} \end{pmatrix} \geq 0.$$

The latter is equivalent to

$$a - R\begin{pmatrix} p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) \\ p'_{-\mathcal{N}(i)} \end{pmatrix} \geq 0.$$

Given $(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) \geq 0$, we have, by (12), $p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) \geq 0$ because $R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} \geq 0$ and $R_{\mathcal{N}(i),-\mathcal{N}(i)} \leq 0$. Therefore, $(p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}), p'_{-\mathcal{N}(i)}) \in P$.

Note that $(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) = \Lambda(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)})$ with $(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) \leq (w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)})$ and

$$\left[L\begin{pmatrix} w'_{\mathcal{N}(i)} \\ p'_{-\mathcal{N}(i)} \end{pmatrix}\right]^\top [(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) - (w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)})] = 0. \tag{16}$$

By the proof of Proposition 4, $p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)})$ is a best response to $p'_{-\mathcal{N}(i)}$ among all $p_{\mathcal{N}(i)} \geq 0$ under the wholesale price vector $w'_{\mathcal{N}(i)}$.

Thus,

$$\begin{aligned} &\pi_i(p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}), p'_{-\mathcal{N}(i)}; w_{\mathcal{N}(i)}) \\ &\quad = [p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) - w_{\mathcal{N}(i)}]^\top \\ &\quad d_{\mathcal{N}(i)}(p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}), p'_{-\mathcal{N}(i)}) \\ &\quad = [p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) - w_{\mathcal{N}(i)}]^\top \\ &\quad q_{\mathcal{N}(i)}(p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}), p'_{-\mathcal{N}(i)}) \end{aligned}$$

$$= [p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) - w_{\mathcal{N}(i)}]^\top L_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)})$$
$$= [p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) - w_{\mathcal{N}(i)} + (w_{\mathcal{N}(i)} - w'_{\mathcal{N}(i)})]^\top L_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)})$$
$$= [p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) - w'_{\mathcal{N}(i)}]^\top d_{\mathcal{N}(i)}(p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}), p'_{-\mathcal{N}(i)})$$
$$= \pi_i(p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}), p'_{-\mathcal{N}(i)}; w'_{\mathcal{N}(i)}). \quad (17)$$

The second equality follows from $(p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}), p'_{-\mathcal{N}(i)}) \in P$, since $(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) \in H$. The third equality follows from that $(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \in H$ if and only if $(p^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}), p_{-\mathcal{N}(i)}) \in P$. The fourth equality follows from $(w_{\mathcal{N}(i)} - w'_{\mathcal{N}(i)})^\top L_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) = 0$. Then

$$\pi_i(p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}), p'_{-\mathcal{N}(i)}; w_{\mathcal{N}(i)})$$
$$\leq \max_{p_{\mathcal{N}(i)} \geq 0} [(p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^\top d_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)})]$$
$$\leq \max_{p_{\mathcal{N}(i)} \geq 0} [(p_{\mathcal{N}(i)} - w'_{\mathcal{N}(i)})^\top d_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)})]$$
$$= \pi_i(p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}), p'_{-\mathcal{N}(i)}; w'_{\mathcal{N}(i)}), \quad (18)$$

where the second inequality is due to $0 \leq w' \leq w$ and the last equality follows from that $p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)})$ is a best response to $p'_{-\mathcal{N}(i)}$ among all $p_{\mathcal{N}(i)} \geq 0$ under the wholesale price vector $w'_{\mathcal{N}(i)}$. By Equation (17), all inequalities in (18) hold as equalities and in particular,

$$\pi_i(p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}), p'_{-\mathcal{N}(i)}; w_{\mathcal{N}(i)}) = \max_{p_{\mathcal{N}(i)} \geq 0} [(p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^\top d_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)})].$$

That is, $p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)})$ is a best response to $p'_{-\mathcal{N}(i)} (\leq p_{-\mathcal{N}(i)})$ among all $p_{\mathcal{N}(i)} \geq 0$ under the wholesale price vector $w_{\mathcal{N}(i)}$, verifying condition (i) of Proposition 3 with $t_{-\mathcal{N}(i)} = p_{-\mathcal{N}(i)} - p'_{-\mathcal{N}(i)}$.

Now we show that $p^*_l(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) \geq w_l$ for all

$$l \in \mathcal{N}(i)^+ \equiv \{l \in \mathcal{N}(i) \mid d_l(p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}), p'_{-\mathcal{N}(i)}) > 0\},$$

verifying condition (iii) of Proposition 3. It follows from (14) and $(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) \in H$ that $p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) \geq w'_{\mathcal{N}(i)}$. It thus suffices to show that for all $l \in \mathcal{N}(i)^+$, $w'_l = w_l$.

Consider the setting where $w'_{\mathcal{N}(i)}$ is the vector of wholesale prices for firm i's products and $p'_{-\mathcal{N}(i)}$ the retail prices of the competitors' products. It thus follows from (13) that

$$a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),\mathcal{N}(i)} p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) - R_{\mathcal{N}(i),-\mathcal{N}(i)} p'_{-\mathcal{N}(i)}$$
$$= \frac{1}{2}[a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p'_{-\mathcal{N}(i)} - R_{\mathcal{N}(i),\mathcal{N}(i)} w'_{\mathcal{N}(i)}].$$

Thus if $d_l(p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}), p'_{-\mathcal{N}(i)}) > 0$, then $a_l - [R_{\mathcal{N}(i),-\mathcal{N}(i)} p'_{-\mathcal{N}(i)}]_l - [R_{\mathcal{N}(i),\mathcal{N}(i)} w'_{\mathcal{N}(i)}]_l > 0$ and, by the definition of the polyhedron H and the projection $\Lambda(\cdot)$: $(w_l - w'_l)\{a_l - [R_{\mathcal{N}(i),-\mathcal{N}(i)} p'_{-\mathcal{N}(i)}]_l - [R_{\mathcal{N}(i),\mathcal{N}(i)} w'_{\mathcal{N}(i)}]_l\} = 0$. This verifies that $w_l = w'_l$ for all $l \in \mathcal{N}(i)^+$.

By Proposition 3, $\max\{p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}), w_{\mathcal{N}(i)}\}$ is a best response to $p_{-\mathcal{N}(i)}$ under the wholesale price vector $w_{\mathcal{N}(i)}$, with the complementarity (16) verifying condition (ii) in Proposition 3.

For all $l \in \mathcal{N}(i)^+$,

$$w_l \leq p^*_l(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)})$$
$$= [2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1}[a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p'_{-\mathcal{N}(i)}]_l + \frac{1}{2}w'_l$$
$$= [2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1}[a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p'_{-\mathcal{N}(i)}]_l + \frac{1}{2}w_l$$
$$= P^*_l(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}),$$

since we have shown that $w_l = w'_l$ for all $l \in \mathcal{N}(i)^+$. Thus, in moving from $p^1 \equiv \max\{p^*_{\mathcal{N}(i)}(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}), w_{\mathcal{N}(i)}\}$ to $p^2 \equiv \max\{P^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}), w_{\mathcal{N}(i)}\}$, the price levels of all products in $\mathcal{N}(i)^+$ are maintained, while those in $\mathcal{N}(i) \setminus \mathcal{N}(i)^+$ are increased. By the regularity condition, this implies that $d_{\mathcal{N}(i)}(p^2, p'_{-\mathcal{N}(i)}) = d_{\mathcal{N}(i)}(p^1, p'_{-\mathcal{N}(i)})$, and

$$[p^2 - w_{\mathcal{N}(i)}]^\top d_{\mathcal{N}(i)}(p^2, p'_{-\mathcal{N}(i)}) \geq [p^1 - w_{\mathcal{N}(i)}]^\top d_{\mathcal{N}(i)}(p^1, p'_{-\mathcal{N}(i)}).$$

This implies that $p^2 \geq w_{\mathcal{N}(i)}$ is a best response to $p'_{-\mathcal{N}(i)}$ and by the proof of Proposition 3, it is a best response to $p_{-\mathcal{N}(i)}$ as well.

(b) It follows from Lemma 1(d) that the projection operator $\Lambda(\cdot)$ is monotonically increasing. (While proven there for the projection operator $\Omega(\cdot)$ onto the polyhedron P, the projection operator $\Lambda(\cdot)$ shares the same properties, since $f \geq 0$ while the matrix G is a ZP-matrix, see Lemma 4.) Since $R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} \geq 0$ and $R_{\mathcal{N}(i),-\mathcal{N}(i)} \leq 0$, it follows that for all $i = 1, \ldots, I$, $P^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)})$ is an increasing function of $p_{-\mathcal{N}(i)}$ and the same monotonicity property applies to the $FB(\cdot)$ operator.

(c) Fix $p^0 \in P^+$. By a standard argument, since $FB(p^0) \geq p^0$, we have, by part (b) that

$$p^0 \leq FB(p^0) \leq \cdots \leq FB^{(n)}(p^0).$$

Moreover, the sequence $\{FB^{(n)}(p^0)\}$ is bounded, since $(w'_{\mathcal{N}(i)}, p'_{-\mathcal{N}(i)}) = \Lambda(w_{\mathcal{N}(i)}, p_{-\mathcal{N}(i)}) \in H$, by Lemma 1(b), and H is a bounded polyhedron in $\mathbb{R}^N$. Since the sequence $\{FB^{(n)}(p^0)\}$ is increasing and bounded, it converges and its limit is, per definition, a fixed point of the $FB(\cdot)$ operator and hence an equilibrium of the competition game. If $p^0 \in P^-$, one shows, analogously that $\{FB^{(n)}(p^0)\}$ is a *decreasing* sequence that is bounded from below by the vector 0, and hence convergent; the remainder of the proof is analogous, again. ■

The (limited) stability result is reminiscent of that in supermodular games, see, for example, Theorem 2.10 in Vives (2001). Note however that we obtain the stability result, even though the game fails to be supermodular, as shown in Example 1 of Federgruen and Hu (2018). Note also that $0 \in P^+$, since $FB(0) \geq 0$. Similarly, as shown in the proof of Theorem 1(c), $Im(FB(\cdot))$, the image of the $FB(\cdot)$ operator, is a bounded subset of $\mathbb{R}_+^N$. This implies that a price vector $\overline{p}$ exists which dominates any point in this image set, component-wise; therefore $P^- \supseteq \{p \geq \overline{p}\}$.

3.2 | A fully best response mapping as a contraction mapping

In this subsection, we show, under an additional condition for the price sensitivity matrix R that the specific FB response identified in Section 3.1 generates a dynamic adjustment process that converges to an equilibrium of the competition game, regardless of the market's starting price vector. Here, we confine ourselves to a row-wise diagonally dominant price sensitivity matrix R, see (5); rather than general positive definite matrices, see Assumption (P). As mentioned, the (row-wise) diagonally dominant (5) is rather innocuous; it merely states that no product's demand volume increases due to a uniform price increase of all products by the same amount.

Moreover, we assume that the dominance of the diagonal elements in R over the off-diagonal elements is sufficiently strong. To this end, we introduce the following *measure* of product substitutability. Let

$$\Delta \equiv \frac{\max_l \left\{\sum_{l' \neq l} |R_{l,l'}|\right\}}{\min_l \left\{R_{l,l} - \sum_{l' \neq l} |R_{l,l'}|\right\}}.$$

Note that Δ is a dimensionless quantity, that is, an *index* that is invariant with respect to scaling or affine transformations of the matrix R. $\Delta > 0$ if and only if the matrix R is (row-wise) diagonally dominant. Moreover,

$$\max_l \frac{\sum_{l' \neq l} |R_{l,l'}|}{R_{l,l}} \leq \varepsilon \equiv \frac{\Delta}{\Delta + 1}.$$

This is because, for any l^o,

$$\sum_{l' \neq l^o} |R_{l^o,l'}| \leq \max_l \left\{ \sum_{l' \neq l} |R_{l,l'}| \right\}$$
$$= \Delta \cdot \min_l \left\{ R_{l,l} - \sum_{l' \neq l} |R_{l,l'}| \right\}$$
$$\leq \Delta \left(R_{l^o,l^o} - \sum_{l' \neq l^o} |R_{l^o,l'}| \right),$$

where the first equality follows from the definition of Δ. This implies that for any l,

$$\sum_{l' \neq l} |R_{l,l'}| \leq \frac{\Delta}{\Delta + 1} R_{l,l}.$$

Thus, for any l,

$$\frac{\sum_{l' \neq l} |R_{l,l'}|}{R_{l,l}} \leq \varepsilon = \frac{\Delta}{\Delta + 1}.$$

We establish global stability by showing that for row-wise diagonally dominant matrices with $\Delta < 1$, the mapping $FB(p)$ is a contraction mapping under the $\|\cdot\|_\infty$-matrix norm. For any $(n \times n)$-matrix A, $\|A\|_\infty = \max_{i=1,\ldots,n} \sum_{j=1}^n | A_{ij} |$, which is an absolute, and hence monotone norm.

We, first, need the following lemma which relates, for any pair of price vectors $p^1, p^2 \in \mathbb{R}_+^N$, $[\Lambda(p^1) - \Lambda(p^2)]$ to $(p^1 - p^2)$. (Recall, $\Lambda(\cdot)$ denotes the projection operator onto the polyhedron H.)

Lemma 5 *Fix a firm $i = 1, \ldots, I$ and let $\Lambda(\cdot)$ denote the projection operator onto the polyhedron H that is associated with firm i. Let $p^1, p^2 \in \mathbb{R}_+^N$. $\Lambda(p^1) = p^1 - t^1$, $\Lambda(p^2) = p^2 - t^2$ and assume that $\{l \in \mathcal{N} \mid t_l^1 = 0\} = \{l \in \mathcal{N} \mid t_l^2 = 0\}$. Let S denote this common product set, that is, $S \equiv \{l \mid t_l^1 = 0\} = \{l \mid t_l^2 = 0\}$. Then*

$$\Lambda(p^1) - \Lambda(p^2) = Q^S(p^1 - p^2), \tag{19}$$

where

$$Q^S = \begin{pmatrix} I & 0 \\ -G_{\overline{S},\overline{S}}^{-1} G_{\overline{S},S} & 0 \end{pmatrix}. \tag{20}$$

Theorem 2

a. *Assume R is a row-wise diagonally dominant matrix with a product substitutability degree $\Delta < 1$ (which implies that $\varepsilon < \frac{1}{2}$). The FB response mapping $FB(p)$ is a contraction mapping with respect to the $\|\cdot\|_\infty$-norm, that is, there exists a constant $0 < \frac{1}{2}\max(2\Delta, 1) \leq \gamma < 1$ such that for any pair of vectors $\widehat{p}, \widetilde{p} \in \mathbb{R}_+^N$,*

$$\| FB(\widehat{p}) - FB(\widetilde{p})\|_\infty \leq \gamma \|\widehat{p} - \widetilde{p}\|_\infty. \tag{21}$$

b. *In particular, there exists a Nash equilibrium p^{**} of the competition game such that $\|FB^{(n)}(p) - p^{**}\| \leq \gamma^n \|p - p^{**}\|$ and $\lim_{n\to\infty} FB^{(n)}(p) = p^{**}$ for all $p \in \mathbb{R}_+^N$.*

Proof of Theorem 2 We first need:

Lemma 6. *For any $p, p', w \in \mathbb{R}^N$,*

$$\|\max(p, w) - \max(p', w)\|_\infty \leq \|p - p'\|_\infty.$$

It suffices to prove part (a). As a contraction mapping, the $FB(\cdot)$ operator has a *unique* fixed point, which by the definition of a Nash equilibrium, is an equilibrium of the competition game. Part (b) then follows by setting $\widetilde{p} = p^{**}$, and iterating inequality (21) n times.

Proof of Theorem 2 part (a). From the definition of the $\|\cdot\|_\infty$-norm and that of the FB response mapping in (15), it suffices to show that for any given firm $i = 1, \ldots, I$,

$$\|\max\{P^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \widehat{p}_{-\mathcal{N}(i)}), w_{\mathcal{N}(i)}\} - \max\{P^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \widetilde{p}_{-\mathcal{N}(i)}), w_{\mathcal{N}(i)}\}\|_\infty \leq \gamma\|\widehat{p}_{-\mathcal{N}(i)} - \widetilde{p}_{-\mathcal{N}(i)}\|_\infty \leq \gamma\|\widehat{p} - \widetilde{p}\|_\infty.$$

Moreover, by Lemma 6, it further suffices to show that for any given firm $i = 1, \ldots, I$,

$$\|P^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \widehat{p}_{-\mathcal{N}(i)}) - P^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \widetilde{p}_{-\mathcal{N}(i)})\|_\infty \leq \gamma\|\widehat{p}_{-\mathcal{N}(i)} - \widetilde{p}_{-\mathcal{N}(i)}\|_\infty.$$

Consider the ray $\mathcal{R} = \{p \mid p = \alpha\widetilde{p} + (1-\alpha)\widehat{p} \text{ for some } 0 < \alpha < 1\}$, which connects $\widetilde{p}$ and $\widehat{p}$. For any $p \in \mathbb{R}^N_+$, let $S(p) \equiv \{l \in \mathcal{N} \mid [\Lambda(p)]_l = p_l\}$. Lemma 1 shows that the projection operator $\Lambda(\cdot)$ is a jointly concave and therefore *continuous* operator. It follows that the set $S(p) = S(\widetilde{p})$ for all $p \in \mathcal{R}$ with α sufficiently close to one. However, there may be a breakpoint $p^{(1)} \in \mathcal{R}$, corresponding with $\alpha = \alpha^1 > 0$, such that for $\alpha < \alpha^1$, a new set S^1 emerges. Thus, as α decreases from 1 to 0, the ray $\mathcal{R}$ may be partitioned by K breakpoints $\{p^{(0)} = \widetilde{p}, p^{(1)}, \ldots, p^{(K)} = \widehat{p}\}$ with corresponding weights $\{\alpha^0 = 1, \alpha^1, \ldots, \alpha^K = 0\}$ such that the same set $S^{(l)} = S(p)$ for all $p \in \mathcal{R}$ in between $p^{(l-1)}$ and $p^{(l)}$ $(l = 1, \ldots, K)$. Then,

$$\begin{aligned} &P^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \widehat{p}_{-\mathcal{N}(i)}) - P^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \widetilde{p}_{-\mathcal{N}(i)}) \\ &= \sum_{l=1}^K \left[P^*_{\mathcal{N}(i)}\left(w_{\mathcal{N}(i)}, p^{(l)}_{-\mathcal{N}(i)}\right) - P^*_{\mathcal{N}(i)}\left(w_{\mathcal{N}(i)}, p^{(l-1)}_{-\mathcal{N}(i)}\right)\right] \\ &= -\frac{1}{2}R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)} \sum_{l=1}^K \left[\Lambda\left(w_{\mathcal{N}(i)}, p^{(l)}_{-\mathcal{N}(i)}\right) - \Lambda\left(w_{\mathcal{N}(i)}, p^{(l-1)}_{-\mathcal{N}(i)}\right)\right] \\ &= -\frac{1}{2}R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)} \sum_{l=1}^K [Q^{S^{(l)}}]_{-\mathcal{N}(i),-\mathcal{N}(i)}\left(p^{(l)}_{-\mathcal{N}(i)} - p^{(l-1)}_{-\mathcal{N}(i)}\right). \end{aligned}$$

By the submultiplicativity of any vector-induced matrix norm,

$$\begin{aligned} &\|P^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \widehat{p}_{-\mathcal{N}(i)}) - P^*_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \widetilde{p}_{-\mathcal{N}(i)})\|_\infty \\ &\leq \frac{1}{2}\|R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}\|_\infty \sum_{l=1}^K \|[Q^{S^{(l)}}]_{-\mathcal{N}(i),-\mathcal{N}(i)}\|_\infty\|p^{(l)}_{-\mathcal{N}(i)} - p^{(l-1)}_{-\mathcal{N}(i)}\|_\infty \\ &\leq \frac{1}{2}\|R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}\|_\infty \sum_{l=1}^K \|[Q^{S^{(l)}}]_{-\mathcal{N}(i),-\mathcal{N}(i)}\|_\infty\|p^{(l)} - p^{(l-1)}\|_\infty \\ &\leq \frac{1}{2}\Delta\max(2\Delta, 1)\|\widehat{p} - \widetilde{p}\|_\infty \leq \gamma\|\widehat{p} - \widetilde{p}\|_\infty, \end{aligned}$$

where the last inequality follows from Lemma 7 parts (b) and (c) in the Appendix, and

$$\sum_{l=1}^K \|p^{(l)} - p^{(l-1)}\|_\infty = \sum_{l=1}^K \|(\alpha^l - \alpha^{l-1})(\widetilde{p} - \widehat{p})\|_\infty = \sum_{l=1}^K (\alpha^l - \alpha^{l-1})\|\widetilde{p} - \widehat{p}\|_\infty = \|\widetilde{p} - \widehat{p}\|_\infty,$$

where the second equality follows from $\|\beta x\| = |\beta|\,\|x\|$ for any norm $\|\cdot\|$ and constant β. ■

It shall be noted that the equilibrium point $p^{**} \neq (p^*|w)$ may occur. By the specification of the $FB(\cdot)$ operator, $p^{**} \geq w$; for certain choices of the wholesale price vector w, this implies that $p^{**} \notin P$ so that $p^{**} \neq (p^*|w)$. This is in contrast to the Robust Best response operator which converges to $(p^*|w)$ as shown in Federgruen and Hu (2018). The fact that every retail price response is capped from below by the vector of wholesale prices, is necessitated to ensure that the response vector is indeed a *best* response, see Proposition 3. Without this capping provision, the response may fail to be "best." The fact that the $FB(\cdot)$ operator is a contraction mapping, implies that convergence is *geometrically* fast so that the number of iterations required to approach the equilibrium within an arbitrary ε-ball is a *logarithmic* function of the original distance $\|p^0 - p^{**}\|_\infty$.

4 | CONCLUSION

We have analyzed a general but parsimonious price competition model for an oligopoly in which each firm offers any number of products. The demand volumes are general piecewise affine functions of the full price vector, generated as the "regular" extension of a base set of affine functions. The model specifies a *product assortment*, along with their prices and demand volumes, in contrast to most commonly used demand models. Depending on the choice of the cost rate vector, the model may have a unique Nash equilibrium, or multiple such equilibria.

As our main result, we have identified a *fully best* response operator which is monotonically increasing so that the market converges to a Nash equilibrium, at least when starting in one of two price regions. Moreover, geometrically fast convergence to an equilibrium can be guaranteed for an arbitrary starting point, under an additional condition for the price sensitivity matrix.

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APPENDIX A: PROOFS OF LEMMAS

Proof of Lemma 3 Parts (a)-(d) can be found in Horn and Johnson (1991, Section 2.5). (Horn and Johnson refer to *ZP*-matrices as *M*-matrices.) Part (e) follows from Imam (1984).

Part (f). Since Y is symmetric, $X = Y^{-1}$ is symmetric. With $Y = \begin{pmatrix} A & B \\ B^\top & D \end{pmatrix}$. Then

$$X = Y^{-1} = \begin{pmatrix} (A - BD^{-1}B^\top)^{-1} & -A^{-1}B(D - B^\top A^{-1}B)^{-1} \\ -(D - B^\top A^{-1}B)^{-1}B^\top A^{-1} & (D - B^\top A^{-1}B)^{-1} \end{pmatrix} \geq 0,$$

see Horn and Johnson (1985, Section 0.7.3), and hence,

$$X_{\overline{S},S}X_{S,S}^{-1}X_{S,\overline{S}} = A^{-1}B(D-B^{\top}A^{-1}B)^{-1}(D-B^{\top}A^{-1}B) \cdot (D-B^{\top}A^{-1}B)^{-1}B^{\top}A^{-1} = A^{-1}B(D-B^{\top}A^{-1}B)^{-1}B^{\top}A^{-1} \geq 0,$$

where the inequality is due to Y being a ZP-matrix, hence $B \leq 0$, $A^{-1} \geq 0$ and $(D-B^{\top}A^{-1}B)^{-1} \geq 0$ (since A and its Schur complement $D-B^{\top}A^{-1}B$ are ZP-matrices, by part (d)).

Part (g) follows from Horn and Johnson (1991, Lemma 2.5.2.1). ■

Proof of Lemma 4 Since R is a ZP-matrix, $R_{-\mathcal{N}(i),\mathcal{N}(i)} \leq 0$ and $R_{\mathcal{N}(i),\mathcal{N}(i)}$ is a ZP-matrix resulting in $R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} \geq 0$. Moreover, since $a \geq 0$, we have $f \geq 0$.

Note that $G = R + R'$, where

$$R' = \begin{pmatrix} 0 & 0 \\ 0 & R_{-\mathcal{N}(i),-\mathcal{N}(i)} - R_{-\mathcal{N}(i),\mathcal{N}(i)}R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)} \end{pmatrix}.$$

Since $R_{-\mathcal{N}(i),-\mathcal{N}(i)} - R_{-\mathcal{N}(i),\mathcal{N}(i)}R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} R_{\mathcal{N}(i),-\mathcal{N}(i)}$ is a Schur complement, it is positive definite. Hence R' is positive semi-definite and further, since R is positive definite, $G = R + R'$ is positive definite. Moreover, Since $R_{-\mathcal{N}(i),-\mathcal{N}(i)}$ is a Z-matrix and because $R_{-\mathcal{N}(i),\mathcal{N}(i)} \leq 0, R_{\mathcal{N}(i),-\mathcal{N}(i)} \leq 0$ and $R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} \geq 0$, $R_{-\mathcal{N}(i),-\mathcal{N}(i)} - R_{-\mathcal{N}(i),\mathcal{N}(i)}R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}$ is a Z-matrix. Then both R and R' are Z-matrices, and hence, $G = R + R'$ is a Z-matrix. ■

Proof of Lemma 5 Recall first from Lemma 4 that G is a ZP-matrix, and so is $G_{\overline{S},\overline{S}}$. This implies that $G^{-1}_{\overline{S},\overline{S}}$ exists. Recall that for $r = 1, 2$, $\Lambda(p^r) = p^r - t^r$ is the unique solution to the LCP: $f - G(p^r - t^r) \geq 0$, $(t^r)^{\top}[f - G(p^r - t^r)] = 0$, $t^r \geq 0$. Thus,

$$\Lambda(p^1)_S - \Lambda(p^2)_S = p^1_S - p^2_S. \tag{A1}$$

In addition, $t^1_{\overline{S}} > 0$ and $t^2_{\overline{S}} > 0$. By the complementarity condition, $0 = [f - G(p^1 - t^1)]_{\overline{S}} = f_{\overline{S}} - G_{\overline{S},S}p^1_S - G_{\overline{S},\overline{S}}(p^1_{\overline{S}} - t^1_{\overline{S}})$. We solve $p^1_{\overline{S}} - t^1_{\overline{S}} = G^{-1}_{\overline{S},\overline{S}}(f_{\overline{S}} - G_{\overline{S},S}p^1_S)$. Then $\Lambda(p^1)_{\overline{S}} = p^1_{\overline{S}} - t^1_{\overline{S}} = G^{-1}_{\overline{S},\overline{S}}(f_{\overline{S}} - G_{\overline{S},S}p^1_S)$. Similarly, $\Lambda(p^2)_{\overline{S}} = p^2_{\overline{S}} - t^2_{\overline{S}} = G^{-1}_{\overline{S},\overline{S}}(f_{\overline{S}} - G_{\overline{S},S}p^2_S)$. As a result,

$$\Lambda(p^1)_{\overline{S}} - \Lambda(p^2)_{\overline{S}} = -G^{-1}_{\overline{S},\overline{S}}G_{\overline{S},S}(p^1_S - p^2_S). \tag{A2}$$

Combining (A1), (A2), we obtain (19). ■

Proof of Lemma 6 From the definition of the matrix-norm $\|\cdot\|_\infty$, it suffices to show that $|\max(p, w) - \max(p', w)| \leq |p - p'|$, that is, for all $l = 1, \ldots, N$, $|\max(p_l, w_l) - \max(p'_l, w_l)| \leq |p_l - p'_l|$. The latter is easily verified, assuming, without loss of generality, that $p_l \leq p'_l$ and considering all three possible rankings: (1) $p_l \leq p'_l \leq w_l$, (2) $p_l \leq w_l < p'_l$, and (3) $w_l < p_l \leq p'_l$. ■

Lemma 7 *Let R be a row-wise, (strictly) diagonally dominant matrix.*

a. $$\|R^{-1}\|_\infty \leq \frac{1}{\min_l \left\{R_{l,l} - \sum_{l' \neq l}|R_{l,l'}|\right\}}.$$

b. For any $i = 1, \ldots, I$,
$$\left\|R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}\right\|_\infty \leq \Delta.$$

c. For any $i = 1, \ldots, I$ *and any* $S \subseteq \mathcal{N}$, $\|Q^S\|_\infty \leq \max(2\Delta, 1)$, *where* Q^S *is specified by (20).*

Proof of Lemma 7 Part (a) follows from Corollary 1 in Varah (1975).

Part (b). First, by part (a),

$$\|R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}\|_\infty \leq \frac{1}{\min_{l \in \mathcal{N}(i)}\left\{R_{l,l} - \sum_{l' \in \mathcal{N}(i), l' \neq l}|R_{l,l'}|\right\}} \leq \frac{1}{\min_{l \in \mathcal{N}(i)}\left\{R_{l,l} - \sum_{l' \neq l}|R_{l,l'}|\right\}} \leq \frac{1}{\min_l\left\{R_{l,l} - \sum_{l' \neq l}|R_{l,l'}|\right\}}.$$

Moreover,

$$\|R_{\mathcal{N}(i),-\mathcal{N}(i)}\|_\infty = \max_{l \in \mathcal{N}(i)} \sum_{l' \in -\mathcal{N}(i)} |R_{l,l'}| \leq \max_{l \in \mathcal{N}(i)} \sum_{l' \neq l} |R_{l,l'}| \leq \max_l \sum_{l' \neq l} |R_{l,l'}|.$$

Therefore, by the submultiplicativity of any vector-induced matrix-norm, in particular the $\|\cdot\|_\infty$-norm,

$$\|R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}\|_\infty \leq \|R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}\|_\infty \|R_{\mathcal{N}(i),-\mathcal{N}(i)}\|_\infty \leq \frac{\max_l \sum_{l' \neq l}|R_{l,l'}|}{\min_l\left\{R_{l,l} - \sum_{l' \neq l}|R_{l,l'}|\right\}} = \Delta.$$

Part (c). Recall that

$$G = \begin{pmatrix} R_{\mathcal{N}(i),\mathcal{N}(i)} & R_{\mathcal{N}(i),-\mathcal{N}(i)} \\ R_{-\mathcal{N}(i),\mathcal{N}(i)} & 2R_{-\mathcal{N}(i),-\mathcal{N}(i)} - R_{-\mathcal{N}(i),\mathcal{N}(i)} R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} R_{\mathcal{N}(i),-\mathcal{N}(i)} \end{pmatrix}$$
$$= R + \begin{pmatrix} 0 & 0 \\ 0 & R_{-\mathcal{N}(i),-\mathcal{N}(i)} - R_{-\mathcal{N}(i),\mathcal{N}(i)} R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} R_{\mathcal{N}(i),-\mathcal{N}(i)} \end{pmatrix}$$
$$= R + \begin{pmatrix} 0 & 0 \\ 0 & SC_{-\mathcal{N}(i),-\mathcal{N}(i)} \end{pmatrix},$$

where $SC_{-\mathcal{N}(i),-\mathcal{N}(i)} = R_{-\mathcal{N}(i),-\mathcal{N}(i)} - R_{-\mathcal{N}(i),\mathcal{N}(i)} R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} R_{\mathcal{N}(i),-\mathcal{N}(i)}$ is the Schur complement of $R_{\mathcal{N}(i),\mathcal{N}(i)}$ in R.

By Liu et al. (2010, Theorem 1), for $l \in -\mathcal{N}(i)$,

$$SC_{l,l} - \sum_{l' \neq l, l' \in -\mathcal{N}(i)} |SC_{l,l'}| \geq R_{l,l} - \sum_{l' \neq l, l' \in \mathcal{N}} |R_{l,l'}|. \quad \text{(A3)}$$

That is, the degree of row diagonal dominance is larger in the Schur complement than in its original matrix.

Recall G is a positive-definite Z-matrix, see Lemma 4. If R is row-wise diagonally dominant, so is G. This is because,

$$G_{l,l} - \sum_{l' \neq l} |G_{l,l'}| = R_{l,l} - \sum_{l' \neq l} |R_{l,l'}| > 0, \qquad l \in \mathcal{N}(i), \quad \text{(A4)}$$

and

$$G_{l,l} - \sum_{l' \neq l} |G_{l,l'}| = R_{l,l} - \sum_{l' \neq l} |R_{l,l'}| + SC_{l,l} - \sum_{l' \neq l, l' \in -\mathcal{N}(i)} |SC_{l,l'}|$$
$$\geq 2\left(R_{l,l} - \sum_{l' \neq l} |R_{l,l'}|\right) > R_{l,l} - \sum_{l' \neq l} |R_{l,l'}| > 0, \qquad l \in -\mathcal{N}(i), \quad \text{(A5)}$$

where the first inequality is by (A3). Combining (A4) and (A5), we have

$$G_{l,l} - \sum_{l' \neq l} |G_{l,l'}| \geq R_{l,l} - \sum_{l' \neq l} |R_{l,l'}|, \qquad \text{for all } l. \quad \text{(A6)}$$

Again, by (A3),

$$\sum_{l' \neq l, l' \in -\mathcal{N}(i)} |SC_{l,l'}| \leq (SC_{l,l} - R_{l,l}) + \sum_{l' \neq l, l' \in \mathcal{N}} |R_{l,l'}| \leq \sum_{l' \neq l} |R_{l,l'}|,$$

where the last inequality is due to the fact that $SC_{l,l} \leq R_{l,l}$ because $SC_{-\mathcal{N}(i),-\mathcal{N}(i)} = R_{-\mathcal{N}(i),-\mathcal{N}(i)} - R_{-\mathcal{N}(i),\mathcal{N}(i)} R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} R_{\mathcal{N}(i),-\mathcal{N}(i)}$ and $R_{-\mathcal{N}(i),\mathcal{N}(i)} R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} R_{\mathcal{N}(i),-\mathcal{N}(i)} \geq 0$. Thus, we have

$$\sum_{l' \neq l} |G_{l,l'}| = \sum_{l' \neq l} |R_{l,l'}| + \sum_{l' \neq l, l' \in -\mathcal{N}(i)} |SC_{l,l'}| \leq 2 \sum_{l' \neq l} |R_{l,l'}|, \qquad l \in -\mathcal{N}(i), \quad \text{(A7)}$$

and

$$\sum_{l' \neq l} |G_{l,l'}| = \sum_{l' \neq l} |R_{l,l'}|, \qquad l \in \mathcal{N}(i). \quad \text{(A8)}$$

Combining (A7) and (A8), we have

$$\sum_{l' \neq l} |G_{l,l'}| \leq 2 \sum_{l' \neq l} |R_{l,l'}|, \quad \text{for all } l. \quad \text{(A9)}$$

Thus, by part (b), for any $\overline{S} \subseteq \mathcal{N}$,

$$\|-G^{-1}_{\overline{S},\overline{S}} G_{\overline{S},S}\|_\infty \leq \frac{\max_l \left\{\sum_{l' \neq l} |G_{l,l'}|\right\}}{\min_l \left\{G_{l,l} - \sum_{l' \neq l} |G_{l,l'}|\right\}} \leq \frac{\max_l \left\{2\sum_{l' \neq l} |R_{l,l'}|\right\}}{\min_l \left\{R_{l,l} - \sum_{l' \neq l} |R_{l,l'}|\right\}} = 2\Delta,$$

where the second inequality is due to (A9) and (A6).

As a consequence,

$$\|Q^S\|_\infty = \left\| \begin{pmatrix} \mathrm{I} & 0 \\ -G^{-1}_{\overline{S},\overline{S}} G_{\overline{S},S} & 0 \end{pmatrix} \right\|_\infty \leq \max(2\Delta, 1). \qquad \blacksquare$$