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Up Then Down: Bid-Price Trends in Revenue Management

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In the classic revenue management (RM) problem of selling a fixed quantity of perishable inventories to price-sensitive non-strategic consumers over a finite horizon, the optimal pricing decision at any time depends on two important factors: consumer valuation and bid price. The former is determined exogenously by the demand side, while the latter is determined jointly by the inventory level on the supply side and the consumer valuations in the time remaining within the selling horizon. Because of the importance of bid prices in theory and practice of RM, this study aims to enhance the understanding of the intertemporal behavior of bid prices in dynamic RM environments. We provide a probabilistic characterization of the optimal policies from the perspective of bid-price processes. We show that an optimal bid-price process has an upward trend over time before the inventory level falls to one and then has a downward trend. This *intertemporal* up-then-down pattern of bid-price processes is related to two fundamental *static* properties of the optimal bid prices: (i) At any given time, a lower inventory level yields a higher optimal bid price, which is referred to as the *resource scarcity effect*; (ii) Given any inventory level, the optimal bid price decreases with time; that is referred to as the *resource perishability effect*. The demonstrated upward trend implies that the optimal bid-price process is mainly driven by the resource scarcity effect, while the downward trend implies that the bid-price process is mainly driven by the resource perishability effect. We also demonstrate how optimal bid price and consumer valuation, as two competing forces, interact over time to drive the optimal-price process. The results are also extended to the network RM problems.

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1. Introduction

Revenue management (RM) has arisen as an important tactic in many industries, such as airlines, hotels, and retailing, that are characterized by fluctuating demand, perishable capacity and a finite selling season. Advances in information and communication technology enable sellers to easily adjust their pricing and capacity allocation decisions depending on the inventory level and remaining time in the selling season to better match supply and demand intertemporally. Talluri and van Ryzin (2004) provide a comprehensive review of industrial practices and the growing RM literature, where they classify the RM models into two categories: dynamic pricing models, which vary prices over time, and capacity rationing models, which allocate capacity to different priority-class customers.

In RM, optimal bid price (bid price, hereafter for simplicity), which is also referred to as shadow price, is a central economic concept for measuring the opportunity cost of losing one unit of the resource. In a discrete capacity/inventory system, it is calculated as the first-order difference of the value function (i.e., the expected revenue function) with respect to the inventory level at any given time. It is a crucial factor in determining the optimal policy (e.g., pricing or capacity allocation) in RM. However, there are still gaps in our understanding of how bid-price processes may evolve over time. To quote Dr. Darius Walczak, Director of Optimization, at PROS Pricing, in his personal communication with us:

> In the airline industry, many carriers do rely on bid-price control. Despite popularity of bid prices, there are still misconceptions remaining among the users: some claim that bid prices should have increasing trend, some claim the opposite, and some others believe that there is no trend to be expected.

Our paper aims to characterize the probabilistic trends of the bid-price processes and to investigate their implications for pricing and capacity allocation decisions in dynamic RM environments.

In a typical single-product setting, the dynamic pricing and capacity rationing models have two common structural properties: the expected revenue function is concave in the inventory level, and submodular in the inventory level and elapsed time; that implies that the bid price of the inventory is decreasing in the inventory level and elapsed time (see, e.g., Feng and Xiao 2000, Gallego and van Ryzin 1994, Zhao and Zheng 2000 for dynamic pricing models, and Lautenbacher and Stidham (1999), Feng and Xiao (2001) for capacity allocation models). The monotonicity in the inventory level means that the less the remaining capacity, the higher the bid price; this is called the *resource scarcity effect*. The monotonicity in time means that the more time is elapsed, the lower the bid price; this is called the *resource perishability effect*. In a dynamic pricing setting, Gallego and van Ryzin (1994) infer from these static properties that the optimal price is decreasing in the remaining inventory level at any given time, and is decreasing in the elapsed time for any given inventory level.

In a dynamic stochastic system, inventory level falls over time. The decrease in the inventory level and the elapse of time have opposite effects on the bid prices. At any point in time, the future bid prices can be either higher or lower than the current bid price. Thus, the statically monotonic structural properties cannot predict how the bid-price process will evolve. Since the optimal policy is driven by the bid prices, the static properties also cannot predict the trends of optimal decisions, for example, the optimal-price trends. Assuming a Poisson arrival process and time-invariant customer valuation distribution, Gallego and van Ryzin (1994) show that the sample path of an optimal-price process has a zig-zag pattern over time that does not exhibit monotonicity. Xu and Hopp (2009) identify sufficient conditions under which the average sample path of the optimal-price process has some monotonic trends in the probabilistic sense. They show that if the customer valuation increases rapidly, the optimal-price process is a submartingale, which implies an upward trend. If the customer valuation decreases rapidly, the optimal-price process is a supermartingale, which implies a downward trend. However, they do not study the bid-price trends under the optimal policy. In a stochastic fluid model with an iso-elastic demand function, Xu and Hopp (2006) show that the optimal-price process is a martingale. It is not hard to infer from their closed-form solutions that the bid-price process of their model is indeed a martingale as well. In a more general stochastic fluid model under the bid-price control, Akan and Ata (2009) also show that the bid-price process is a martingale. They propose an ε-optimal approximation model under which the bid-price process has the martingale property. Inspired by the interesting findings of Xu and Hopp (2006) and Akan and Ata (2009), we aim to answer the following questions: Do bid-price processes in the conventional RM models (e.g., Gallego and van Ryzin (1994), Zhao and Zheng (2000)) also evolve as martingales? In dynamic pricing models, how do the two competing forces, bid price and consumer valuation, drive the optimal-price processes?

In this study, we analyze the bid-price processes under the optimal policies for the standard RM models with discrete inventory levels. More specifically, we show that in a dynamic pricing model, the bid-price process under the optimal policy has an upward trend before the inventory level falls to one, and a downward trend afterwards. Such an up-then-down pattern implies that the bid-price processes are not martingales and moreover, that they are neither supermartingales nor submartingales. This finding implies that each of the two driving forces of the bid prices, namely, resource scarcity and resource perishability, plays a dominant role in different situations: When the inventory level is greater than one, the resource scarcity effect is the dominant driver for the bid-price trend; when only one unit of inventory is left, the resource scarcity no longer exists and the perishability effect becomes the only driver for the bid-price trends.

To explore the implications of the bid-price trends, we first investigate the trends of optimal selling prices (optimal price, hereafter for simplicity). We show that the optimal-price trends are determined jointly by the trends of bid-price and valuation processes. In particular, when the valuation distributions are stationary, the optimal-price trends may also exhibit similar up-then-down patterns: the optimal-price processes are expected to move upward before the inventory level falls to one and then to move downward. This result is different from that of Xu and Hopp (2009), who show that when the valuation increases or decreases dramatically over time, the optimal-price trends are upward or downward, respectively. We focus on the role of bid-price trends in determining the optimal-price trends, while they focus on the role of valuation trends. From this standpoint, our analysis complements theirs for an understanding of the optimal-price trends.

In the numerical study, we demonstrate the effects of valuation trends, demand/capacity ratio, and the demand and capacity scaling, on the bid-price and optimal-price trends. We observe that the bid-price trends are not sensitive to the valuation trends, whereas the optimal-price trends are. We also observe that the higher the demand/capacity ratio—hence the more stringent the capacity—the more significant are the up-then-down patterns of the bid-price and optimal-price trends. When both demand and capacity are sufficiently high with their ratio fixed, the bid-price and optimal-price processes tend to behave like martingales, an observation that is in line with the finding of Akan and Ata (2009) for stochastic fluid models.

Finally, we extend our results on the bid-price trends and optimal-price trends to the network RM problem.

In summary, our contribution is twofold. First, we provide a probabilistic characterization of the bid-price process and show that the bid-price trends have an up-then-down pattern. To the best of our knowledge, this is a new insight for RM problems. Second, we show how the two competing forces, bid-prices, and customer valuation, drive the optimal-price trend in the dynamic pricing model, a finding which complements Xu and Hopp (2009).

Throughout the rest of the study, we use "decreasing" and "increasing" in a weak sense. For any real numbers x,y, $x \wedge y = \min(x,y)$ and $x \vee y = \max(x,y)$. For any vector x, x' is its transpose.

2. Single-Product Dynamic Pricing

2.1. The Model

Consider a monopolistic firm that sells C discrete units of a perishable product over a finite horizon $[0, T]$. Customers arrive according to a non-homogeneous Poisson process with a time-varying arrival rate Λ_t, $t \in [0,T]$. At any point in time, the firm selects a selling price p from a compact set $[\underline{p},\overline{p}]$. Each arriving customer will buy one item if and only if the price p is no more than his or her valuation of the product. At the end of the horizon, all unsold items have zero salvage values. The firm aims to maximize the total expected revenue over the horizon.

Assume that $\Lambda_t > 0$ for all $t \in (0,T)$. For any $t \in [0,T)$, let $\Phi_t(\cdot) : \mathbb{R}_+ \to [0,1]$ be the cumulative distribution function of customer valuation with probability density function $\phi_t : \mathbb{R}_+ \to \mathbb{R}_+$. Define the failure (or hazard) rate of $\Phi_t(\cdot)$ as $h_t(\cdot) \triangleq \phi_t(\cdot)/\bar{\Phi}_t(\cdot)$, where $\bar{\Phi}_t(\cdot) = 1 - \Phi_t(\cdot)$. Assume that both Φ_t and ϕ_t are continuous and differentiable. Note that the failure rate is also called the *inverse Mills ratio*, where the Mills ratio is defined as $m_t(p) \triangleq 1/h_t(p)$. We assume that $\Phi_t(p)$ is strictly increasing in p with the support $[\underline{p}, \bar{p}]$.

Then, $\bar{\Phi}_t(p)$ is strictly decreasing in $[\underline{p}, \bar{p}]$, and for any $d \in [0,1]$, there is a unique p such that $d = \bar{\Phi}_t(p)$. Moreover, the strict monotonicity implies that $\bar{\Phi}_t(p)$ has an inverse function, denoted by $\psi_t(d)$, such that $\psi_t(d)$ is also a strictly decreasing function. The one-to-one mapping between p and d implies that one can treat d, the probability that customer valuation is greater than price, as the decision variable. See Gallego and van Ryzin (1994) for the same assumption of one-to-one correspondence between prices and demand rates. For convenience, we call d the demand rate in the following analysis. Then the expected revenue rate becomes $\Lambda_t r_t(d)$, where $r_t(d) \triangleq d\psi_t(d)$ is the expected revenue per arrival. Like Gallego and van Ryzin (1994), we assume that $r_t(d)$ is bounded and concave in d; that ensures that $r_t(d)$ has a bounded least maximizer $d_t^* \triangleq \inf\{d \in [0,1] : r_t(d) = \max_{\xi \in [0,1]} r_t(\xi)\}$. The concavity of $r_t(d)$ implies that the marginal value of the expected revenue rate, $r_t'(d)$, is decreasing in d, consistent with the standard economic assumption. In fact, even if $r_t(d)$ is not a concave function, there exists a *maximum increasing concave envelope* (MICE) for $r_t(d)$ such that the optimal policy can be found on the MICE (see Feng and Xiao 2000 for a detailed discussion). Thus, the concavity assumption for $r_t(d)$ is without loss of generality.

Let $\mathcal{U}$ be the set of admissible policies. A policy u in $\mathcal{U}$ can be specified as $\{d^u(t), 0 \le t \le T\}$, where $d^u(t)$ is the demand rate under policy u at time t. Define $\tau_n \triangleq \inf\{t \in [0,T] | N(t) = n\}$, which represents the hitting time when the inventory level first falls to the level n. Note that in the literature a null price is typically used to model the out-of-stock condition under which the demand is zero (see, e.g., Gallego and van Ryzin 1994). However, to analyze the price trend, such a null price may artificially distort the price trends: in practice, a product that is out of stock is more likely to be removed from the market, instead of being offered at an outrageously high price to shut down demand. Following Xu and Hopp (2009), we assume that the price process is stopped immediately after the inventory level falls to zero and let the price process take the value it had just before the occurrence of this stockout. That is, $d^u(t) = d^u(\tau_0-)$ if $t \ge \tau_0$. Correspondingly, for any $u \in \mathcal{U}$, the total number of sales up to time $t(< \tau_0)$, denoted by $A^u(t)$, is a Poisson process with arrival rate $\Lambda_t d^u(t)$ and the maximum value C. Since the sales process stops when the inventory falls to zero, we assume that $A^u(t) = A^u(\tau_0-)$ for $t \ge \tau_0$.

Suppose that the demand process is defined on a probability space $(\Omega, \mathcal{F})$. For any policy $u \in \mathcal{U}$, the associated probability measure $\mathcal{P}^u$ is defined on $(\Omega, \mathcal{F})$ such that $A^u(t)$ admits the $(\mathcal{P}^u, \mathcal{F}_t)$-intensity $\lambda^u(t) = \Lambda_t d^u(t)$, where $\mathcal{F}_t = \{A^u(s) | 0 \le s \le t\}$ is the filtration (history) generated by $A^u(t)$. The revenue rate for a unit sale at time t under the policy u is $\psi_t(d^u(t-))$, that is, the optimal price right before time t. Then the firm's expected revenue maximization problem is

$$\max_{u \in \mathcal{U}} E\left[\int_0^T \psi_s(d^u(s-))dA^u(s) | \mathcal{F}_0\right]$$

subject to $A^u(s) \le C$ for all s.

Let u^* be an optimal policy and $N(t) \triangleq C - A^{u^*}(t)$ be the inventory process under the optimal policy. Given the inventory level n at time t (i.e., $N(t) = n$), let $V(t,n)$ be the expected revenue over period $[t,T]$ under the optimal policy (namely, the value function), that is,

$$V(t,n) \triangleq E\left[\int_t^T \psi_s(d^{u^*}(s-))dA^{u^*}(s)|\mathcal{F}_t\right].$$

The boundary conditions are $V(t,0) = 0$ and $V(T, n) = 0$ for any t and n. Note that since the underlying sales process is Markovian, the dependence on the filtration $\mathcal{F}_t$ can be replaced by the condition $N(t) = n$.

2.2. Optimal Bid-Price Trends

In RM, bid price of a unit of capacity is measured by the opportunity cost of selling one unit of capacity. More precisely, at any time t, given a capacity level $n \geq 1$, the bid price is equal to the first-order difference of the value function, that is, $\Delta V(t,n) \triangleq V(t,n) - V(t,n-1)$. According to the literature of *intensity control of point process* (e.g., Brémaud 1981), a sufficient condition for a function $V(t,n)$ to be the optimal value function is that it satisfies the following Hamilton–Jacobi–Bellman (HJB) equation (see also Feng and Xiao (2000)). That is,

$$0 = \frac{\partial}{\partial t}V(t,n) + \max_{d\in[0,1]}\{\Lambda_t d[\psi_t(d) - \Delta V(t,n)]\}. \qquad (1)$$

Clearly, given any inventory level n at time t, if one knows the value of the bid price, $\Delta V(t,n)$, the optimal demand rate can be obtained by solving $\max_{d\in[0,1]}\{r_t(d) - d\Delta V(t,n)\}$. Since $r_t(d)$ is concave, for any $\Delta V(t,n) \geq 0$, the objective function is concave, and thus the optimal decision, if it is an interior point in [0,1], satisfies the first-order condition: $r_t'(d) = \Delta V(t,n)$. Let $d(t,n)$ and $p(t,n) = \psi_t(d(t,n))$ denote the optimal demand rate and pricing decisions for any given t and n.

Zhao and Zheng (2000) show that the optimal bid price $\Delta V(t,n)$ is decreasing in t and n. Their result implies that the elapsed time and remaining capacity are two key factors driving the bid price. When one of these two factors is fixed, an increase in the other factor leads to a decrease in the bid price. For any t, the bid price is decreasing in n, which reflects the *resource scarcity effect*. For any n, the bid price is decreasing in t, which reflects the *resource perishability effect*. However, in a dynamic stochastic system, as time goes by and more of the inventories are sold, the decrease in the inventory level and the increase in the elapsed time have opposite effects on the bid price: the former drives the bid price up, while the latter drives the bid price down. At any point in time, the bid price in the future is determined by the future dynamics and is therefore uncertain. To predict the probabilistic trend of the bid-price process, we ask the following question: which of the two driving factors is the dominant factor for the bid-price trend?

Note that the bid-price process stops when the inventory level falls to zero. When the system runs out of inventory, it may be more natural to virtually let the future bid price be the one right before the stockout, so as not to drive up prices artificially. Thus, given an inventory process $N(t)$, the optimal bid-price process can be defined as

$$\begin{aligned} B(t) &\triangleq \Delta V(t\wedge\tau_0-, N(t\wedge\tau_0-)) \\ &= \begin{cases} \Delta V(t,N(t)) & \text{if } t<\tau_1, \\ V(t,1) & \text{if } \tau_1 \leq t<\tau_0, \\ V(\tau_0-,1) & \text{if } t\geq\tau_0. \end{cases} \end{aligned}$$

THEOREM 1 (OPTIMAL BID-PRICE TRENDS.). *For any $t \in [0,T)$ and $s \in (t,T]$,*

(i) if $N(t) > 1$, then $B(t) \leq E[B(s\wedge\tau_1)|\mathcal{F}_t]$;
(ii) if $N(t) = 1$, then $B(t) \geq E[B(s)|\mathcal{F}_t]$.

Theorem 1 shows that the bid-price process has an upward trend before the inventory falls to one and then it moves downward. It implies that the resource scarcity is the dominant factor in the bid-price process before the inventory falls to one and that resource perishability then becomes the dominant factor in the bid-price process. A stochastic process, $\mathcal{B}(t)$, is said to be a martingale over $[0,T]$ if for all $0 \leq t \leq s \leq T$, $\mathcal{B}(t) = E[\mathcal{B}(s)|\mathcal{F}_t]$. $\mathcal{B}(t)$ is said to be a submartingale (or supermartingale) if the equality above is replaced by $\leq$ (or $\geq$). A martingale is also a submartingale and a supermartingale. Theorem 1 implies that the bid-price process before the inventory level falls to one, $B(t\wedge\tau_1)$, is a submartingale, and the bid-price process after the inventory level falls to one, $B(t\vee\tau_1)$, is a supermartingale.

2.3. Optimal-Price Trends

This section analyzes how the bid-price trends drive the dynamics of optimal prices. For simplicity, in the following analysis of optimal-price trends, we further assume that $r_t(d)$ is strictly concave in d.

Like Xu and Hopp (2009), the optimal-price process can be defined as

$$P(t) \triangleq \begin{cases} p(t,N(t)) & \text{if } t<\tau_0, \\ p(\tau_0-,N(\tau_0-)) & \text{if } t\geq\tau_0. \end{cases}$$

That is, the price process stops immediately after a stockout occurs and it takes the value it had just before the stockout.

For any $t < T$ and $n \geq 1$, the optimal price $p(t,n)$ is obtained by solving the problem

$$\max_{p \in [\underline{p}, \bar{p}]} \{\bar{\Phi}_t(p)[p - \Delta V(t,n)]\}. \tag{2}$$

Since the support of valuation distribution is $[\underline{p}, \bar{p}]$, we know that the objective function is strictly increasing in p as $p \leq \underline{p}$ and is zero when $p \geq \bar{p}$. Hence, the optimal price $p(t,n)$ must be in $[\underline{p}, \bar{p}]$. Differentiating the objective function of (2), we can write the marginal revenue rate function as

$$\bar{\Phi}_t(p) - \phi_t(p)p + \phi_t(p)\Delta V(t,n) = \phi_t(p)[\Delta V(t,n) - H_t(p)],$$

where $H_t(p) \triangleq p - m_t(p)$. Recall that the term $m_t(p) = 1/h_t(p)$ is the Mills ratio.

The following lemma helps characterize the optimal pricing policy.

Lemma 1. *For any t, $H_t(p)$ is strictly increasing in $p \in [\underline{p}, \bar{p}]$.*

Lemma 1 implies that the optimal price satisfies the first-order condition that $\Delta V(t,n) - H_t(p) = 0$ if and only if $H_t(\underline{p}) \leq \Delta V(t,n) \leq H_t(\bar{p})$. Since $\bar{p}$ is the upper bound of the support of the valuation distribution, $\bar{\Phi}(\bar{p}) = 0$, which implies that $H_t(\bar{p}) = \bar{p}$. It is clear that the bid price (or, the opportunity cost of the marginal inventory unit) $\Delta V(t,n)$ must be smaller than $\bar{p}$ for any (t,n), which implies that $\Delta V(t,n) \leq H_t(\bar{p})$. Moreover, if $H_t(\underline{p}) \geq \Delta V(t,n)$, the optimal price is $\underline{p}$. Then the optimal price $p(t,n)$ can be expressed as $\varphi_t(\Delta V(t,n)) = H_t^{-1}(\Delta V(t,n) \vee \underline{B}_t)$, where $\underline{B}_t = H_t(\underline{p})$ and H_t^{-1} is the inverse function of H_t.

Xu and Hopp (2009) identify a set of sufficient conditions under which an optimal-price process has an upward trend or a downward trend. Their discussion focuses on the effect of the customer valuation process. In particular, they show that when customer valuation increases (or declines) rapidly, the optimal price may have an upward (or downward) trend. However, they do not consider the influence of the bid prices on the optimal-price trends. To gain insight into how the two competing forces interact to drive the optimal-price trends, we first consider the following two special cases of valuation distributions.

Example 1 (Exponential Valuation Distribution). Suppose that customer valuation is exponentially distributed with a time-varying mean μ_t and the price range is $p \in (0,\infty)$. For any t and n, by solving the price optimization problem (3), one can obtain the optimal price $p(t,n) = \mu_t + \Delta V(t,n)$. Then the optimal-price process is $P(t) = \mu_{t \wedge \tau_0 -} + B(t)$, and the optimal-price trends depend on which force is dominant. When μ_t is non-decreasing in t, the optimal price has the upward trend before the inventory falls to one. Suppose that μ_t is a differentiable function of t. If, in particular, the increasing rate of μ_t is large enough (e.g., $\frac{d}{dt}\mu_t > -\frac{\partial}{\partial t}V(t,1)$ for all t), then the optimal price has an upward trend over the *entire* selling horizon. Similarly, since the increase in the bid-price trends may be limited, when μ_t falls rapidly enough that the valuation trend is the dominant driver of the optimal-price process, the optimal-price process will have a downward trend. When μ_t is constant, the optimal-price trends are driven entirely by the bid-price trends.

Example 2 (Iso-Elastic Demand). Another commonly used demand model is the iso-elastic (or constant elasticity) demand model $\bar{\Phi}_t(p) = p^{-\gamma_t}$, where $\gamma_t > 1$ and $p \in [1,\infty)$ (see, e.g., McAfee and te Velde 2008). The assumption $\gamma_t > 1$ indicates that the customers are relatively sensitive to price changes. It is clear that the optimal price is $p(t,n) = \frac{\gamma_t}{\gamma_t - 1}\Delta V(t,n)$. Note that $\frac{\gamma_t}{\gamma_t - 1}$ decreases in γ_t. Then when the price elasticity $\{\gamma_t\}$ falls over time, the optimal-price process has an upward trend before the inventory falls to one. When the price elasticity $\{\gamma_t\}$ increases over time, the optimal-price process has a downward trend after the inventory falls to one. When the price elasticity is constant over time, the optimal-price process has the same trend as the bid-price process.

For a more general valuation distribution, the following theorem identifies a sufficient condition under which the optimal-price process may have the same trend as the bid-price process.

Theorem 2 (Optimal-Price Trends.). *Assume that $m_t(p)$ is convex in p for any t. Then, for any $0 \leq t < s \leq T$,*

(i) if $N(t) > 1$ and h_t increases in t, then $P(t) \leq E[P(s \wedge \tau_1)|\mathcal{F}_t]$;
(ii) if $N(t) = 1$ and h_t decreases in t, then $P(t) \geq E[P(s)|\mathcal{F}_t]$.

Theorem 2 demonstrates how the temporal trend and static structure of valuation distribution and the bid-price process drive the trend of the optimal-price process. In particular, if the valuation distribution is constant (independent of t), Theorem 2 shows that the optimal-price process has the same trend as that of the bid-price process: it moves upward until the inventory level falls to one and then moves downward. That is different from the patterns demonstrated by Xu and Hopp (2009), which show that the price trends can have either upward or downward trends under some strong conditions on the customer valuation process.

Xu and Hopp (2009) impose three sufficient conditions to ensure the upward (or downward) price trends: (i) $1/h_t(p)$ is convex in p, (ii) the failure rate $h_t(p)$ is increasing in p, and (iii) price sensitivity decreases (or increases) rapidly. Their condition (i) is the same as our convex Mills ratio condition. As pointed out by Xu and Hopp (2009) and Xu (2013), such a condition is satisfied by many common distributions, including uniform, logistic, normal, Weibull, and gamma with shape parameters no less than one. Moreover, their condition (ii) implies that $H_t(p)$ is also increasing in p. Because $r_t''(d) = -\frac{1}{\phi_t(\psi_t(d))} H_t'(\psi_t(d))$, their condition (ii) further implies the concavity of $r_t(d)$ but not necessarily vice versa. For example, the iso-elastic demand function $\bar{\Phi}_t(p) = p^{-\gamma_t}$ in Example 2 violates condition (ii) while satisfying the concavity of the revenue rate function, because $H_t(p) = p - p/\gamma_t$ is increasing in p while $h_t(p) = \gamma_t/p$ is decreasing in p. This suggests that our assumption that $r_t(d)$ is concave is more general than their condition (ii). Lastly, their condition (iii) is relatively strong: the price sensitivity decreases or increases so rapidly that the optimal-price trend is mainly driven by the trend of customer price sensitivity. Our result complements their study of price trends from a bid-price perspective.

2.4. Numerical Study

In the numerical study, we will demonstrate the difference between the trends of bid price and selling price, and the effects of the valuation trend, of the demand/capacity ratio and of the demand and capacity scaling, on the trends of bid price and selling price.

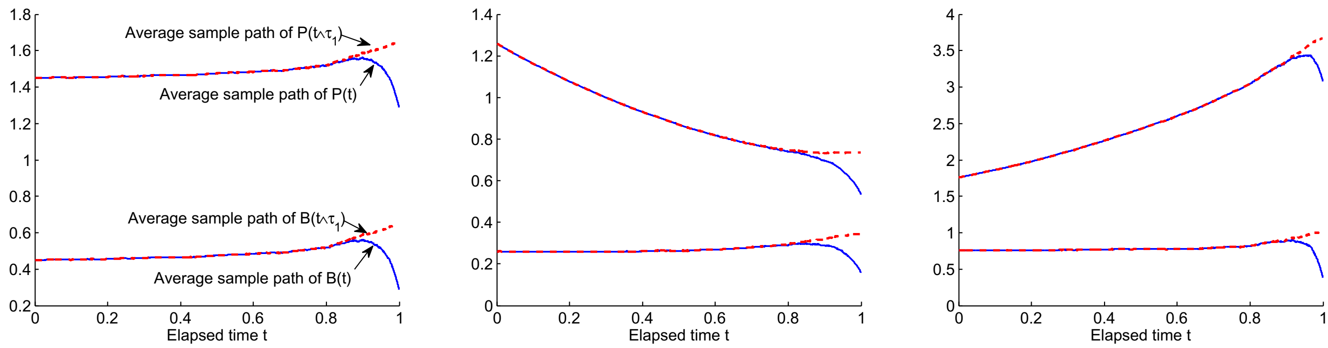
Following Xu and Hopp (2009), we consider the exponential valuation distribution. Firstly, we consider the setting with $T = 1$, $C = 25$, $\lambda_t(p) = \Lambda e^{-p/\mu_t}$, where $\Lambda = 100$ is the arrival rate, and $\mu_t = e^{kt}$ is the mean of the valuation distribution with $k \in \{-1,0,1\}$. Note that when $k = 0$, the mean valuation is constant; when $k = -1$ (or $k = 1$), the mean valuation falls (or increases) exponentially over time. We treat the constant valuation case as the base case. Secondly, we change the demand/capacity ratio by varying the arrival rate in {50,150,200} while keeping the capacity level at 25. Thirdly, we scale the demand and capacity by varying the arrival rate and capacity level (Λ,C) in {(20,5), (200,50), (1000,250)} while keeping the same demand/capacity ratio as in the base case.

In the computation, we discretize the time interval $[0,T]$ into $L = 10{,}000$ small intervals with an equal length $\delta = T/L$. The time interval is indexed by $j = 1, \ldots, L$. Let $t_j = (j-1) \times \delta$ be the beginning of j-th interval. We first compute the optimal selling prices and bid prices. Then, we simulate the demand arrival in each interval $(t_{j-1}, t_j]$, $j = 1, \ldots, L$, by a Bernoulli random variable with mean $\lambda(t_j)\delta$, and the customer valuation upon arrival by an exponential variable with mean μ_t. We record all the trajectories of bid prices and selling prices along the simulated sample paths and then calculate their means.

Figure 1 demonstrates the average sample paths of bid prices and optimal prices under different valuation trends. Recall that with the exponential valuation distribution, the optimal price and bid price satisfy $p(t,n) = \mu_t + b(t,n)$, where $b(t,n) = \Delta V(t,n)$, and hence the optimal-price trends are driven by the valuation trends and bid-price trends simultaneously. Sub-figure (a) shows that when μ_t is constant, the optimal prices and bid prices have the same trends. As predicted, before the inventory level falls to one, the bid price and optimal price both move upward; since the time is close to the end of the horizon after inventory falls to one, the bid price and optimal price both move downward. The bid-price trend plays a dominant role when consumer valuation distribution is time-invariant. It is also notable that as time goes by, the average sample path of $B(t \wedge \tau_1)$ rises from about 0.45 to about 0.65 (an increase of about 50%), which implies that the upward trend of the bid prices is relatively significant. Sub-figure (b) shows that when μ_t decreases rapidly, the optimal price declines, while the bid-price trend still exhibits the up-then-down pattern. Sub-figure (c) shows that when μ_t increases rapidly, the optimal price increases rapidly until the inventory level hits one. As in the second case, the bid-price process exhibits an upward trend, but then falls quickly after the inventory level hits one. This shows that the trend of valuation distribution can be the dominant driver of the optimal-price process when the mean valuation increases or decreases rapidly, which is consistent with Xu and Hopp (2009). These numerical results confirm that the optimal-price trend is determined by the combined effect of the trend of valuation distribution and the up-then-down pattern of the bid-price trend.

Figure 2 demonstrates the bid-price trends and optimal-price trends with different demand/capacity ratios. If the capacity level is 25, a higher demand/capacity ratio corresponds to a scenario with a relatively higher demand. The three sub-figures plus the base case show that the higher the demand/capacity ratio the more significant the up-then-down trends for both bid price and optimal price. Sub-figure (a) shows that when there is ample capacity (e.g., for this case, the average optimal price is around 1.03, the likelihood of a purchase upon a customer's arrival is

Figure 1 Bid Prices Vs. Optimal Prices: Effect of Valuation Trend



(a) μ_t stays constant $(K = 0)$ **(b)** μ_t decreases rapidly $(K = -1)$ **(c)** μ_t increases rapidly $(K = 1)$

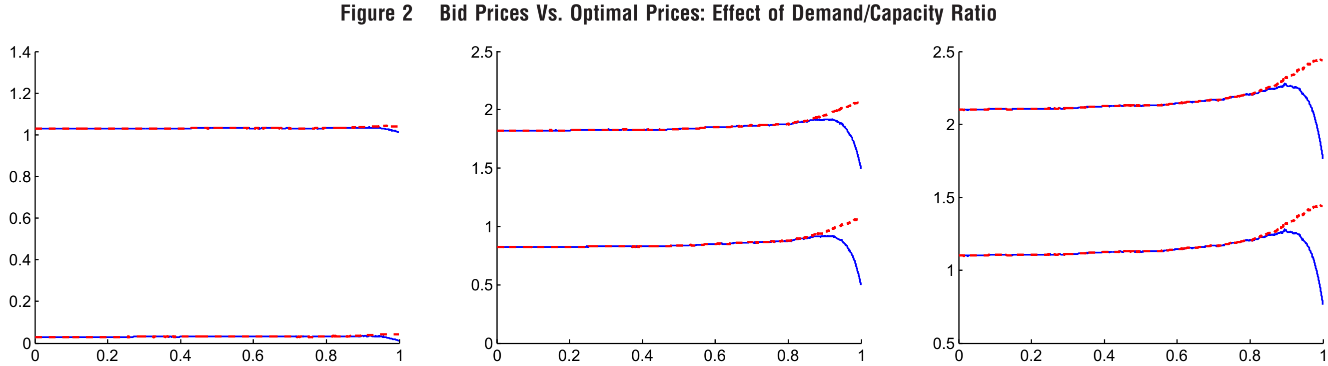
0.357 and it is expected to have a leftover of 7.15 units), the bid-price and optimal-price trends can be quite flat. In other words, they behave like martingales. However, when the capacity is stringent in which case RM is most interesting to be applied to, see sub-figures (b) and (c), the up-then-down pattern is more significant, with a moderate upward trend of the bid-price process in the early selling horizon.

Figure 3 shows the effect of the demand and capacity scaling on the the bid-price and optimal-price trends. It is clear that the arrival rates and capacity levels of the base case and of the cases in sub-figures (b) and (c) are respectively, 5, 10, and 50 times of those of the case in sub-figure (a). We can observe that the higher the arrival rate and the capacity, the flatter the optimal-price and bid-price trends, though the downward trend at the end of the selling horizon is still very significant. Ignoring the downward trends after the inventory levels fall to one, the bid-price processes and optimal-price processes behave like martingales when the arrival rates and capacity levels are high. Our observations are consistent with the finding of Akan and Ata (2009). They argue that the fluid model can be a good approximation of the RM problems when arrival rate and capacity are both very large, and they show that the optimal bid-price processes are martingales in the stochastic fluid models. Note that in the fluid models there is no such "last unit" of inventory; that may explain why there is no such downward trend as we observe at the end of the selling season. See Online Appendix B for more discussions on the fact that the bid-price process in a fluid approximation model becomes a martingale.

3. Network Revenue Management

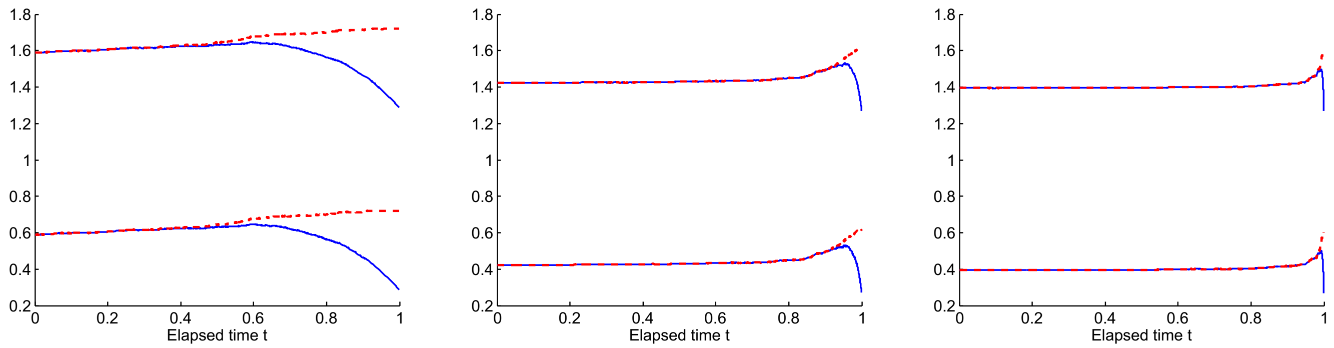
We extend results to the network RM. Consider an airline network that consists of I legs, indexed by $i = 1, \ldots, I$, and J origin–destination itineraries, indexed by $j = 1, \ldots, J$, and that can be treated as a multi-product system where each product corresponds to an itinerary and each resource corresponds to a leg. The initial capacity of the resource i is C_i. Define the vector $C = (C_1, \ldots, C_I)' \in \mathbb{Z}_+^I$, where $\mathbb{Z}_+ \triangleq \{x \in \mathbb{Z} | x \geq 0\}$. Let $Q = [q_{ij}]$, $q_{ij} \in \{0, 1\}$, denote

Figure 2 Bid Prices Vs. Optimal Prices: Effect of Demand/Capacity Ratio



(a) $K = 0, \Lambda = 50, C = 25$ **(b)** $K = 0, \Lambda = 150, C = 25$ **(c)** $K = 0, \Lambda = 200, C = 25$

Figure 3 Bid Prices Vs. Optimal Prices: Effect of Demand and Capacity Scaling



(a) $K=0, \Lambda=20, C=5$ **(b)** $K=0, \Lambda=200, C=50$ **(c)** $K=0, \Lambda=1000, C=250$

the bill of resource matrix, where product j requires q_{ij} units of capacity of resource i. Assume that Q has no zero columns; that is, each product consists of at least one unit from one of I resources. The j-th column of Q, Q_j, is the incidence vector for itinerary j. At any time $t \in [0,T]$, let $N_i(t)$ be the remaining inventory level of resource i. Let $N(t) = (N_1(t), \ldots, N_I(t))'$. The state of the network is denoted by a vector $n = (n_1, \ldots, n_I)' \in \mathbb{Z}^I_+$. Note that the inventory level of a product is equal to the minimum level of the inventories of all the resources that are built into this product.

The demands arrive as a J-variate Poisson process defined on a measurable space $(\Omega, \mathcal{F})$ with the intensity $\lambda^u_j(t)$ for itinerary j, $j = 1, \ldots, J$, where u is an admissible policy in an admissible policy set $\mathcal{U}$. Let $A^u(t) = (A^u_1(t), \ldots, A^u_J(t))'$, where $A^u_j(t)$ is the total number of sales of product j up to time t. Assume that, for any given u, $\lambda^u_j(t)$ is independent of $A^u(t)$. The unit revenue from a sale of product j at time t under policy u is denoted by $p^u_j(t-) \in [\underline{p}_j, \bar{p}_j]$. Let $p^u(t) = (p^u_1(t), \ldots, p^u_J(t))'$. A product is said to be out-of-stock if the inventory level of any resource of the product is out-of-stock. As in the single-product case, we assume that the price and sales process of a product stops when it is out-of-stock, and the price and accumulate sales of this product are equal to the values right before it is out-of-stock. The firm's expected revenue maximization problem is expressed as

$$\max_{u \in \mathcal{U}} E\left[\int_0^T p^u(s-)' dA^u(s)\right],$$

subject to $\int_0^T Q dA^u(s) \le C$.

Let u^* be an optimal policy. Given the inventory levels n at time t, the expected revenue over period $[t,T]$ under the optimal policy is

$$V(t,n) = E\left[\int_t^T p^{u^*}(s-)' dA(s) | \mathcal{F}_t\right].$$

Note that $V(T,n) = 0$ for any n and $V(t,0) = 0$ for any t, where 0 can be a vector with an appropriate dimension and all its elements being zero.

Define $\Delta_i V(t,n) \triangleq V(t,n) - V(t, n-e_i)$ as the bid price (shadow price) of the inventory of i-th resource at time t, where e_i is an I-dimensional unit vector with 1 as the i-th element. Let $\tau^i_m \triangleq \inf\{t \in [0,T] | \max_j K^{ij}(t) = m\}$ be the earliest time that the maximum inventory level of all products that contain resource i falls to $m(\le C_i)$, where

$$K^{ij}(t) = \begin{cases} \min_k\{N_k(t) : q_{kj} = 1\} & \text{if } q_{ij} = 1, \\ 0 & \text{if } q_{ij} = 0 \end{cases}.$$

Note that $K^{ij}(t)$ is the maximum available inventory level of product j that contains resource i. As each product may consist of several resources, the effectiveness of the inventory of a resource depends on the inventories of all other resources that can be combined into a product. Note that $K^{ij}(t) > 1$ implies that $N_i(t) > 1$, but not *vice versa*. The term $\max_j K^{ij}(t)$ represents the effective inventory level of resource i at time t, that is, the maximum amount of inventories at a product level that resource i can contribute to. If $\max_j K^{ij}(t) = 0$ and $N_i(t) > 0$, then $V(t, N(t)) = V(t, N(t) - e_i)$, that is, $\Delta_i V(t, N(t)) = 0$.

Given the inventory process $N(t)$, the bid-price process of the i-th resource is defined as

$$B^i(t) \triangleq \Delta_i V(t \wedge \tau^i_0-, N(t \wedge \tau^i_0-)).$$

THEOREM 3 (BID-PRICE TRENDS OF A RESOURCE.). *Given the inventory vector n at time t, for any $t < s \le T$, the bid-price process for resource i has the following trends:*

(i) if $\max_{j=1}^J K^{ij}(t) > 1$, then $B^i(t) \le E[B^i(s \wedge \tau^i_1) | \mathcal{F}_t]$;

(ii) if $\max_{j=1}^J K^{ij}(t) = 1$, then $B^i(t) \ge E[B^i(s) | \mathcal{F}_t]$.

Like Theorem 1, Theorem 3 shows that the bid price of a resource has an upward trend before the effective inventory level of this resource falls to one and that it then continues on a downward trend. This indicates that the up-then-down pattern of bid-price trends is a general property in RM problems. It is known that the desired structural properties of value functions, such as concavity, supermodularity, or submodularity, do *not* necessarily hold for general network RM models. From this perspective, our results shed new light on the network RM problem.

To see the implications of the bid-price trends for the optimal-price processes, we consider a simple setting where the demand processes and the consumer valuations for all the products are independent of each other. Let Φ_t^j be the consumer valuation distribution for product j and assume that the corresponding revenue rate function $r_t^j(d) = (\Phi_t^j)^{-1}(d)d$ is concave in d. For any $t < T$ and $n \geq Q^j$, the optimal price $p_j(t,n)$ for product j is obtained by solving the problem

$$\max_{p_j \in [\underline{p}_j, \bar{p}_j]} \left\{ \bar{\Phi}_t^j(p_j)[p_j - (V(t,n) - V(t, n - Q^j))] \right\}. \qquad (3)$$

Let $\hat{\tau}_m^j$ be the earliest time that the inventory level of product j (defined as the minimum inventory level of all its resources) falls to m. We extend the bid price concept from a resource to a product. The bid price of a product measures the opportunity cost of selling one unit of the product, which serves as a cost base for optimal product pricing. Define the bid-price process for product j's inventories as

$$\hat{B}^j(t) \triangleq V\left(t \wedge \hat{\tau}_0^j-, N(t \wedge \hat{\tau}_0^j-)\right) - V\left(t \wedge \hat{\tau}_0^j-, N(t \wedge \hat{\tau}_0^j-) - Q^j\right).$$

As in Theorem 3, we can characterize the bid-price trends for a product. The proof replicates that of Theorem 3 and is therefore omitted.

THEOREM 4 (BID-PRICE TRENDS OF A PRODUCT). *Assume that the demand processes for different products are independent of each other. Given the inventory vector n at time t, the product j's optimal bid-price process has the following trends:*

(i) if $t < \hat{\tau}_1^j$, *then for any* $t < s \leq T$, $\hat{B}^j(t) \leq E[\hat{B}^j(s \wedge \hat{\tau}_1^j)|\mathcal{F}_t]$;

(ii) if $t \geq \hat{\tau}_1^j$, *then for any* $t < s \leq T$, $\hat{B}^j(t) \geq E[\hat{B}^j(s)|\mathcal{F}_t]$.

Define the optimal-price process of product j as

$$P_j(t) \triangleq \begin{cases} p_j(t, N(t)) & \text{if } t < \hat{\tau}_0^j, \\ p_j(\hat{\tau}_0^j-, N(\hat{\tau}_0^j-)) & \text{if } t \geq \hat{\tau}_0^j. \end{cases}$$

Then under the same assumption as in Theorem 2 for the valuation distribution of each product, we can have the same characterization for the optimal-price trends as that of Theorem 2. That is, when the hazard rate for product j, denoted by h_t^j, increases in t, the optimal-price process for product j has an upward trend before the inventory level of the product falls to one (i.e., $t < \hat{\tau}_1^j$); when h_t^j decreases in t, the optimal-price process for product j has a downward trend after the inventory level of the product falls to one.

4. Concluding Remarks

Bid price is one of the fundamental driving forces for determining the optimal decisions in dynamic pricing and revenue management models. Our study reveals a general pattern of bid-price processes: an optimal bid-price process has an upward trend before the inventory level falls to one and then has a downward trend. The optimal-price trends are then driven by the bid-price trends and customer valuation trends. In the following, we briefly summarize several extensions. The detailed analysis is provided in the Online Appendix.

Heuristic bid-price trends. Gallego and van Ryzin (1994) analyze a fixed-price heuristic based on the deterministic fluid approximation model and show that the heuristic policy is asymptotically optimal when the volumes of expected sales and capacity are sufficiently large. It is natural to ask whether the bid-price processes under the heuristic also have the same trends as they do under optimal policies. Employing the same approach to the optimal bid-prices, we identify some sufficient conditions under which the bid-price processes and optimal-price processes under the fluid heuristics have either an upward or a downward trend before the inventory level falls to one. That is, the heuristic bid prices may move in either the same direction as the optimal bid prices or the opposite direction. This provides some structural insights into the design of the heuristics: when the heuristic bid price moves in an opposite direction to the optimal bid price and the gap between them becomes larger as time goes by, one may need to double check the heuristic policy to make sure it can indeed perform well.

Fluid approximation from a martingale perspective. Note that both Xu and Hopp (2006) and Akan and Ata (2009) show that the optimal bid-price processes are martingales in stochastic fluid models (which serve as approximations of the standard RM models), while we show that in the standard RM models the optimal bid-price processes may have different trends before and after inventory levels fall to one. We provide further analysis to bridge this gap. We find that it may be the discreteness in the arrival process that drives the non-martingale structure of the bid-price processes. As pointed out by Akan and Ata (2009), when the volume and capacity are sufficiently large, the RM systems can be approximated by fluid models and the bid-price processes can therefore also be viewed approximately as martingales.

Capacity rationing. Our analysis also applies to capacity rationing models, also called quantity-based RM models (see, e.g., Feng and Xiao 2001), in which customers are segmented into several distinct fare classes with predetermined prices. As in the analysis for the dynamic pricing models in network settings, the analysis for the capacity rationing models can also be readily extended to network settings.

Bid-price trends under dynamic price competition. We also try to extend our analysis to competitive dynamic pricing models (see, e.g., Gallego and Hu 2014). We show that when there is only one unit of inventory left for a firm in an oligopolistic market, in equilibrium, the bid-price process of the firm has a downward trend. Furthermore, for a duopoly with linear demand functions, the equilibrium price process of each firm also has a downward trend when both firms have a single unit of inventory. Sweeting (2012) has conducted an interesting empirical study of the dynamic pricing behavior of the secondary markets for Major League Baseball tickets where sellers are small and most sellers offer a single unit. He finds that the sellers cut their prices dramatically, by 40% or more, as an event approaches. He argues that the simple dynamic pricing models (without explicitly addressing the customer's strategic behavior) predict the seller's behavior pretty accurately. Our work provides a rigorous analysis and theoretical justification for his empirical findings.

The above discussions show that the up-then-down bid-price trends are indeed a general property in RM models. Our work is a complement to that of Xu and Hopp (2009), who characterize the optimal-price trends and to Akan and Ata (2009) who show that the bid-price processes in stochastic fluid models are martingales. These findings enhance our understanding on how RM systems work. But we must acknowledge that knowing the bid-price trends or price trends does not directly help compute the optimal policies or design effective heuristics.

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Appendix: Technical Proofs

Proof of Theorem 1. We first prove the first assertion. Note that $d(t,n)$ denotes the optimal demand rate when the inventory level is n at t. From the HJB Equation (1), for any $n \geq 2$ and $0 < t < T$, we have the following equations.

$$0 = \frac{\partial}{\partial t} V(t,n) + \Lambda_t d(t,n)[\psi(d(t,n)) - \Delta V(t,n)], \quad \text{(A1)}$$

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} V(t,n-1) + \Lambda_t d(t,n-1)[\psi(d(t,n-1)) \\ &\quad - \Delta V(t,n-1)] \geq \frac{\partial}{\partial t} V(t,n-1) + \Lambda_t d(t,n)[\psi(d(t,n)) \\ &\quad - \Delta V(t,n-1)], \end{aligned} \quad \text{(A2)}$$

where the inequality is due to the sub-optimality of $d(t,n)$ when the inventory level is $n-1$ at t.

Subtracting Equation (A2) from Equation (A1) yields the following inequality.

$$0 \leq \frac{\partial}{\partial t} \Delta V(t,n) + \Lambda_t d(t,n)[\Delta V(t,n-1) - \Delta V(t,n)]. \quad \text{(A3)}$$

Following the derivation of the proof of Theorem 1 of Chapter VII of Brémaud (1981) (in particular, equation (2.16)), for the inventory process $N(s)$ when $s < \tau_1$, one can decompose the expected value function $V(s,N(s))$ at the jumps of the point process $N(t)$ to derive the following integral expression.

$$\begin{aligned} V(s,N(s)) &= V(0,N(0)) + \int_0^s \mathcal{L}V(\xi,N(\xi))d\xi \\ &\quad - \int_0^s \Delta V(\xi,N(\xi-))(dN(\xi) \\ &\quad - \Lambda_\xi d(\xi,N(\xi))d\xi), \end{aligned} \quad \text{(A4)}$$

where

$$\mathcal{L}V(t,n) \triangleq \frac{\partial}{\partial t} V(t,n) + \Lambda_t d(t,n)[V(t,n-1) - V(t,n)].$$

Note that Equation (A4) is slightly different from the corresponding equation in Brémaud (1981), which is due to the fact that $N(t)$ is an inventory process in our model.

Then for any $0 < s_1 < s_2 < \tau_1$, we have

$$\begin{aligned} &V(s_2,N(s_2)) - V(s_1,N(s_1)) \\ &= \int_{s_1}^{s_2} \mathcal{L}V(\xi,N(\xi))d\xi - \int_{s_1}^{s_2} \Delta V(\xi,N(\xi-))(dN(\xi) \\ &\quad - \Lambda(\xi)d(\xi,N(\xi))d\xi). \end{aligned} \quad \text{(A5)}$$

Similarly, we have

$$V(s_2,N(s_2)-1)-V(s_1,N(s_1)-1) = \int_{s_1}^{s_2} \mathcal{L}V(\xi,N(\xi)-1)d\xi - \int_{s_1}^{s_2} \Delta V(\xi,N(\xi-)-1)(dN(\xi) - \Lambda(\xi)d(\xi,N(\xi)-1)d\xi). \tag{A6}$$

Subtracting Equation (A6) from Equation (A5) yields

$$\begin{aligned}
&\Delta V(s_2,N(s_2)) - \Delta V(s_1,N(s_1)) \\
&= \int_{s_1}^{s_2} \{\mathcal{L}V(\xi,N(\xi)) - \mathcal{L}V(\xi,N(\xi)-1)\}d\xi \\
&- \int_{s_1}^{s_2} \Delta V(\xi,N(\xi-))(dN(\xi) - \Lambda(\xi)d(\xi,N(\xi))d\xi) \\
&+ \int_{s_1}^{s_2} \Delta V(\xi,N(\xi-)-1)(dN(\xi) - \Lambda(\xi)d(\xi,N(\xi)-1)d\xi) \\
&\geq - \int_{s_1}^{s_2} \Delta V(\xi,N(\xi-))(dN(\xi) - \Lambda(\xi)d(\xi,N(\xi))d\xi) \\
&+ \int_{s_1}^{s_2} \Delta V(\xi,N(\xi-)-1)(dN(\xi) - \Lambda(\xi)d(\xi,N(\xi)-1)d\xi),
\end{aligned} \tag{A7}$$

where the inequality is due to inequality (A3).

Note that $\Delta V(\xi,N(\xi-))$ and $\Delta V(\xi,N(\xi-)-1)$ are both bounded $\mathcal{F}_t$-predictable processes, and $N(t)$ is an integral stochastic point process with $\mathcal{F}_t$ intensity $\Lambda_t d(t,N(t))$. It follows from Dynkin's Lemma (see, e.-g., Rogers and Williams 1987) and the optional sampling theorem of martingale (see, e.g., Karatzas and Shreve 1988) that $N(t\wedge\tau_1) - \int_0^{t\wedge\tau_1} \Lambda_u d(\xi,N(\xi-))d\xi$ is a martingale. In addition, it satisfies $E[N(t\wedge\tau_1)] = E[\int_0^{t\wedge\tau_1} \Lambda_t d(\xi,N(\xi-))d\xi]$. Then, following the arguments of Brémaud (1981) (see, in particular, equation (2.3) on page 24 and equation (2.18) on page 204), we have

$$E\left[\int_{s_1}^{s_2} \Delta V(\xi,N(\xi-))(\mathrm{d}N(\xi) - \Lambda(\xi)d(\xi,N(\xi))\mathrm{d}\xi)|\mathcal{F}_t\right] = 0, \tag{A8}$$

and

$$E\left[\int_{s_1}^{s_2} \Delta V(\xi,N(\xi-)-1)(\mathrm{d}N(\xi) - \Lambda(\xi)d(\xi,N(\xi)-1)\mathrm{d}\xi)|\mathcal{F}_t\right] = 0. \tag{A9}$$

By integration on both sides of the inequality (A7) and applying Equations of (A8) and (A9), we have

$$E[\Delta V(s_1,N(s_1))|\mathcal{F}_t] \leq E[\Delta V(s_2,N(s_2))|\mathcal{F}_t],$$

which implies that $E[\Delta V(s\wedge\tau_1,N(s\wedge\tau_1))|\mathcal{F}_t]$ is increasing in s for any $t < s < T$. The first assertion holds.

We next prove the second assertion. When $n = 1$, the HJB Equation (1) implies that for $0 < t < T$,

$$0 = \frac{\partial}{\partial t}V(t,n) + \Lambda_t d(t,n)[\psi(d(t,n)) - V(t,n)] \geq \frac{\partial}{\partial t}V(t,n) - \Lambda_t d(t,n)V(t,n). \tag{A10}$$

Then, similar to the arguments made for the first assertion, for any s_1, s_2 such that $\tau_1 \leq t < s_1 < s_2 < \tau_0$, that is, $N(t) = 1$, we have

$$\begin{aligned}
&V(s_2,N(s_2)) - V(s_1,N(s_1)) = \\
&\int_{s_1}^{s_2} \mathcal{L}V(\xi,N(\xi))d\xi \\
&- \int_{s_1}^{s_2} V(\xi,N(\xi-))(dN(\xi) - \Lambda(\xi)d(\xi,N(\xi))d\xi) \\
&\leq - \int_{s_1}^{s_2} V(\xi,N(\xi-))(dN(\xi) - \Lambda_\xi d(\xi,N(\xi))d\xi),
\end{aligned} \tag{A11}$$

where the inequality is due to the inequality (A10). Similar to the preceding analysis for the first assertion, we have

$$E\left[\int_{s_1}^{s_2} V(\xi,N(\xi-))(dN(\xi) - \Lambda_\xi d(\xi,N(\xi))d\xi)|\mathcal{F}_t\right] = 0. \tag{A12}$$

By integration on both sides of the inequality (A11) and applying Equation (A12), we have $E[V(s_1,N(s_1))|\mathcal{F}_t] \geq E[V(s_2,N(s_2))|\mathcal{F}_t]$. Note that $V(t, N(t)) = 0$ as $t \geq \tau_0$. We have that $E[V(s,N(s))|\mathcal{F}_t]$ decreases in $s \in (t,T]$. The desired result holds.

Remark 1 (An Alternative Proof Using Martingale Arguments). *The bid-price trends can also be proved using Martingale arguments similar to that of Xu and Hopp (2009). Note that* $\int_0^{s\wedge\tau_1} \mathcal{L}V(\xi,N(\xi))d\xi$ *is well defined. It follows from Dynkin's Lemma (Rogers and Williams 1987) and optional sampling theorem of martingale (Karatzas and Shreve 1988) that* $\Delta V(s\wedge\tau_1, N(s\wedge\tau_1)) - \int_0^{s\wedge\tau_1}\{\mathcal{L}V(\xi, N(\xi)) - \mathcal{L}V(\xi, N(\xi)-1)\}d\xi$ *is an* $\mathcal{F}_t$*-martingale. Then by inequality (A3) for any* $t < \tau_1$ *we know that* $E[\Delta V(s\wedge\tau_1, N(s\wedge\tau_1))|\mathcal{F}_t]$ *is increasing in s. The downward trend can also be proved similarly.*

Proof of Lemma 1. We first claim that $H_t(p)$ is strictly increasing in p for any t. In fact, we have $H_t'(p) = 2 + \frac{\bar{\Phi}_t(p)\phi_t'(p)}{\phi_t(p)^2}$ and

$$r_t''(d) = -\frac{2}{\phi_t(\bar{\Phi}_t^{-1}(d))} - \frac{d\phi'(\bar{\Phi}_t^{-1}(d))}{\phi_t^3(\bar{\Phi}_t^{-1}(d))} = -\frac{1}{\phi_t(\psi_t(d))}\left[2 + \frac{\bar{\Phi}_t(\psi_t(d))\phi_t'(\psi_t(d))}{\phi_t(\psi_t(d))^2}\right] = -\frac{1}{\phi_t(\psi_t(d))}H_t'(\psi_t(d)),$$

where $\psi_t(\cdot) = \bar{\Phi}_t^{-1}(\cdot)$. Then, the strict concavity of $r_t(d)$ implies that $r_t''(d) < 0$ and hence $H_t(p)$ is strictly increasing in p. □

Proof of Theorem 2. We first show that the convexity of m_t implies the convexity of H_t^{-1}. In fact, we have

$$(H_t^{-1})''(x) = -\frac{H_t''(H_t^{-1}(x))}{H_t'(H_t^{-1}(x))^3}.$$

Since $m_t(p) = 1/h_t(p)$ is convex in p, $H_t(p)$ is concave and increasing in p, which implies that $H_t^{-1}(x)$ is increasing and convex in x. Clearly, $\varphi_t(x) = H_t^{-1}(x \vee \underline{B}_t)$ must also be increasing and convex in x.

When $h_t(p)$ increases in t for any p, we know that $H_t(p)$ increases in t for any p and so are $H_t^{-1}(x), \underline{B}_t$ and $\varphi_t(x)$ for any x. Then, for any $0 \le t < s \le T$, $N(t) \ge 2$, we have

$$\begin{aligned} E[P(s\wedge\tau_1)|\mathcal{F}_t] &= E[\varphi_s(\Delta V(s\wedge\tau_1, N(s\wedge\tau_1)))|\mathcal{F}_t] \\ &\ge \varphi_s^{-1}(E[\Delta V(s\wedge\tau_1, N(s\wedge\tau_1))|\mathcal{F}_t]) \\ &\ge \varphi_s^{-1}(\Delta V(t, N(t))) \\ &\ge \varphi_t^{-1}(\Delta V(t, N(t))) = P(t). \end{aligned}$$

where the first inequality is by Jensen's inequality, and the second is by Theorem 1 part (i) and the third is by the monotonicity of φ_t^{-1} in t.

When $h_t(p)$ decreases in t for any p, we know that $H_t(p)$ and $H_t^{-1}(p)$ must decrease in t for any p. For $N(t) = 1$, the bid price $\Delta V(s,N(s)) = V(s,N(s))$ is a decreasing process, that is, $\Delta V(t,N(t)) \ge \Delta V(s,N(s))$ for all the realizations of $N(s)$ at $s > t$. Then, $P(t) = \varphi_t^{-1}(\Delta V(t,N(t))) \ge \varphi_s^{-1}(\Delta V(s,N(s))) = P(s)$. Thus, $P(t) \ge E[P(s)|\mathcal{F}_t]$. □

Proof of Theorem 3. First, if $\max_{j=1}^J K^{ij}(t) > 1$, then we have $n_i \ge 2$ and the HJB equations:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}V(t,n) + \sum_{j=1}^J \lambda_j(t,n)[p_j(t,n) \\ &\quad + V(t, n - Q^j) - V(t,n)], \\ 0 &= \frac{\partial}{\partial t}V(t,n-e_i) + \sum_{j=1}^J \lambda_j(t,n-e_i)[p_j(t,n-e_i) \\ &\quad + V(t,n-e_i-Q^j) - V(t,n-e_i)] \\ &\ge \frac{\partial}{\partial t}V(t,n-e_i) + \sum_{j=1}^J \lambda_j(t,n)[p_j(t,n) \\ &\quad + V(t,n-e_i-Q^j) - V(t,n-e_i)], \end{aligned}$$

where $p_j(t,n)$ is the optimal pricing decision at (t,n) and $\lambda_j(t,n)$ is the corresponding demand rate, and in particular, when $n_i = 0$, for j such that $q_{ij} = 1$, $\lambda_j(t,n) = 0$.

A subtraction yields

$$0 \le \frac{\partial}{\partial t}\Delta_i V(t,n) + \sum_{j=1}^J \lambda_j(t,n)[\Delta_i V(t,n-Q^j) - \Delta_i V(t,n)]. \tag{A13}$$

For convenience, define the operator $\mathcal{L}_i$ such that

$$\mathcal{L}_i V(t,n) = \frac{\partial}{\partial t}\Delta_i V(t,n) + \sum_{j=1}^J \lambda_j(t,n)[V(t,n-Q^j) - V(t,n)].$$

Note that $N(t) = (N_1(t), N_2(t), \ldots, N_I(t))'$ is an I-dimensional stochastic process with $\mathcal{F}_t$ intensity $\sum_{j=1}^J \lambda_j(t,N(t))Q^j$. Clearly, $\int_0^{s\wedge\tau_1^i} \mathcal{L}_i V(\xi, N(\xi))\mathrm{d}\xi$ is well defined. Analogous to Remark 1, it follows from Dynkin's Lemma (Rogers and Williams 1987) and optional sampling theorem of martingale (Karatzas and Shreve 1988) that $\Delta_i V(s\wedge\tau_1^i, N(s\wedge\tau_1^i)) - \int_0^{s\wedge\tau_1^i}\sum_{j=1}^J \lambda_j(\xi,N(\xi))[\mathcal{L}_i V(\xi,N(\xi)) - \mathcal{L}_i V(\xi,N(\xi)-Q^j)]d\xi$ is an $\mathcal{F}_t$-martingale. Then, inequality euqation (16) implies that $B^i(s) = \Delta_i V(s\wedge\tau_1^i, N(s\wedge\tau_1^i))$ is a submartingale.

Second, if $\max_{j=1}^J K^{ij}(t) = 1$, then the inventory levels of products involving resource i are at most one. It is clear that for any $t \ge \tau_1^i$, the bid price $B^i(t)$ declines over time and the desired result holds. □

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