8. Introduction to cooperative games[26]

A cooperative game theory generalizes some of the ideas and thinking behind equilibrium theory to other, typically finite social situations. The starting point is to think about the utility levels that can be achieved by group of players. For instance, a group of firms, by choosing different production plans, can jointly achieve certain level of profits that can be shared across firms. A group of municipalities may consider different options for transit systems that cost money, but that also affect positively the welfare of the participating communities. A group of agents in an exchange economy can trade their goods in some way and affect their consumption and utility levels.

There are two fundamental assumptions. First, groups of individuals can choose allocations that only rely on the resources of the coalition members available to members of the coalitions. Thus, it is appropriate to think about utility levels that are available only to members of some coalitions, and that may differ if we change (for example, expand) the coalition. Second, the cooperative theory abstracts from the incentive or problems involved in implementing particular allocations. The assumption is that coalitions can write enforceable contracts. The only question is how a coalition decides which of the available choices to implement.

We are going to define a cooperative game as a mapping that associates a coalition with sets of utility levels that are available to the coalition. Next, we are going to choose about solution concepts that, in some principled way, choose subsets of such utility levels (and, possibly, the formed coalition).

8.1. **NTU-cooperative game.** A basc elements of a cooperative games is the set of players I, and its subsets called coalitions $S \subseteq I$. Each coalition is assigned with a set of utility vectors $V(S) \subseteq \mathbb{R}^S$. Interpretation is that each $u \in V(S)$ is a possible vector of utility for each member of the coalition.

Definition 11. An NTU cooperative game (in a characteristic form) consists of a set of players I and a utility possibility allocation $V(S) \subseteq S$ for each $S \subseteq I$ such that each utility possibility set $V(S)$.

We refer to I as a grand coalition.

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MWG assumes that each utility-possibility set must be also comprehensive: for each $u \in V(S)$, if $u' \in \mathbb{R}^S$ and $u' \leq u$, then $u' \in V(S)$ as well. (In other words, the utility possibility sets must satisfy free disposal.) I am not exactly clear why they make this requirement. We are simply going to say that the NTU game is comprehensive if each utility-possibility set is comprehensive.

8.1.1. *Shapley-Shubik assignment.*

Example 15. Shapley-Shubik assignment game (Example 11). There are I agents. Each agent i has 1 unit of good i. Each agent wants 1 unit of a good, with utility being $u_i(j)$ of agent i for utility of good j. For each $S \subseteq I$, define

$$V(S) = \left\{ (u_i(a(i)))_{i \in S} : a : S \to S \text{ is a bijection } \right\}.$$

The assignment game can be further complicated by more general property rights. For instance, certain goods can be used only if two or more individuals allow (joint ownership). Maybe some individuals do own have anything. Maybe some goods can be allocated only if majority of certain coalition agrees. Etc.

8.1.2. *Voting.*

Example 16. Voting game. There are I agents. There is a set of A of collective decisions. Each agent can withdraw from the group and obtain payoff 0. Otherwise, if a member of the group, the agent i receives utility $u_i(a)$ from some decision $a \in A$. The set of available decisions includes the decision $0 \in A$ that corresponds to the staying with status quo. Status quo can be replaced by some other decisions only only by so-called winning coalitions. The collection of all winning coalitions is denoted as $\mathcal{C}$. We take:

$$V(S) = \{\mathbf{0}_S\} \cup \{(u_i(a)) : a \in A\} \text{ if } S \in \mathcal{C}$$
$$V(S) = \{\mathbf{0}_S\} \text{ if } S \notin \mathcal{C}.$$

(1) For example, majority voting has $\mathcal{C} = \left\{ S : |S| > \frac{1}{2} I \right\}$.
(2) Dictatorship of agent i has $\mathcal{C} = \{S : i \in S\}$.
(3) Majority with a veto-right by individual i. $\mathcal{C} = \left\{ S : |S| > \frac{1}{2} I \text{ and } i \in S \right\}$.

(4) Security Council voting. By majority, but any change from styatus quo must be accepted by all five permanent members of the Council.

This example can be further complicated by allowing different actions available to different winning coalitions. For instance, in the UofT insurance plan, the decision to raise member contributions to the plan is made by majority voting. But the decision to change essential parameters of the plan has an additional veto points.

8.1.3. *Super-additive games.*

Definition 12. A cooperative game (I, V) is *super-additive* if for any two $S, T \subseteq I$ that are disjoint, $S \cap T = \emptyset$, we have

$$V(S) \times V(T) \subseteq V(S \cup T).$$

or, in other words, for each $x^S \in V(S), x^T \in V(T)$, we have $\left(x^S x^T\right) \in V(S \cup T)$.

In other words, the utilities that can be achieved within each coalition separately, can be be also achieved together.

Exercise 13. Show that the assignment game is super-additive .Consider voting game with winning coalitions $\mathcal{C}$. Show that if the winning coalitions is an upper contour set (i.e., if $S \in \mathcal{C}$and $S \subseteq S'$, then $S' \in \mathcal{C}$), then the voting game is super-additive. Is the assumption on $\mathcal{C}$ necessary? Check that all the examples of winning sets from Example above satisfy the condition.

8.1.4. *Core.* For each utility vector $x \in \mathbb{R}^I$, let $x|_S \in \mathbb{R}$ denote vector x restricted to coalition S.

Definition 13. A utility vector $x \in V(I)$ is in *the core* for game (I, V) if for any coalition $S \subseteq I$, there is no $y \in V(S)$ such that $y > x|_S$ (i.e., for each $y \in V(S)$, either $y = x|_S$ or there is $i \in S$ such that $y_i < x_i$). A core of the game is the set of utility vectors that are in the core.

We say that a grand-coalition allocation x is *blocked* by coalition S and allocation $y \in V(S)$ if $y > x|_S$. The core is the set of grand-coalition allocations that cannot be blocked.

Exercise 14. Show that each core allocation is Pareto-optimal in the grand coalition: there is no allocation $y \in V(I)$ that Pareto-dominates x.

The next Example shows that the core might be empty.

Exercise 15. Consider a majority voting game (Example 16) with three players. We assume that each winning coalition can redistributie pie of value 1 between its members. Formally, suppose that for any winning coalition S (i.e., a colaition of 2 or 3 players) $V(S) = \{(x_i) : \sum_{i \in S} x_i = 1\} \cup \{\mathbf{0}_S\}$. (Thus, our game has a transferable utility.) Show that the game has empty core. This fact is also known as the Condorcet paradox.

8.1.5. *Solution of a cooperative game.* A solution of a cooperative game assigns a subset of utility vectors to the grand coalition $\sigma(I, V) \subseteq V(I)$. There are two types of definitions of solutions.

- One is to define the solution according to some properties of how negotiations that choose an allocation may look like. Example is core, or stability in case of matching problems.
- Another one is axiomatic: We define a series of properties of a "good" solution. An example of a typical property is that a good solution should be invariant to certain changes in the game (or at least, the solution should change in a regular way, when we change the game in some regular way). And then, we use them to show that there exists a unique (in some sense) solution that satisfies such properties. Examples: Nash-bargaining solution or Shapley Value.

8.2. **TU-games.** IN a TU- game, the utilities can be transferred between players without a loss. The motivation is that all players have quasi-linear utility over unobouded resource that can lean

In terms of a definition, it means that if $u \in V(S)$, then for any $\tau \in \mathbb{R}^S$ such that $\sum_{i \in S} \tau_i = 0$, we have $u + \tau \in \mathbb{R}^S$.

Definition 14. A TU cooperative game (in a characteristic form) is a tuple (I, v), where I is the set of players and $v : 2^I \to \mathbb{R}$ assigns value to each coalition. Function v is called *characteristic* of the game.

Each TU-game corresponds to a NTU game defined so that $V(S) = \left\{u \in \mathbb{R}^S : \sum_{i\in S} u_i \leq v(S)\right\}$. Conversely, say that NTU game has a transferable utility if and only if for each S, each $u \in V(S)$, and each $\tau \in \mathbb{R}^S$ such that $\sum_{i\in S} \tau_i = 0$, we have $u + \tau \in \mathbb{R}^S$. Then, if the game has bounded payoffs (i.e., $\sup_{i,S:i\in S} \sup_{u\in V(S)} u_i < \infty$), then we can define the characteristic of the associated TU-game as

$$v(S) = \max\left\{\sum_{i\in S} x_i : x \in V(S)\right\}.$$

In the latter case, we say that the TU game is derived from an NTU-game.

For a TU game, the core definition goes like this

Definition 15. A utility vector $x \in \mathbb{R}^I$ is in *core* for a TU-game (I, v) if

$$\sum_{i\in I} x_i = v(I), \text{ and}$$
$$\sum_{i\in S} x_i \geq v(S) \text{ for each coalition } S \subseteq I.$$

The first condition guarantees that the core allocation redistributes the value of the grand coalition, and the second set of conditions ensure that no coalition can block the allocation.

Convince yourself that the two definitions are equivalent if a NTU game has a transferrable utility and the TU game is derived from it.

8.2.1. *Minimal cost network.*

Example 17. There are I cities that want to be connected with an electric network to the power supply. The power supply is in city 0. The cost of connection between cities i and j is equal to $a_{ij} = a_{ji}$. An electric grid for coalition S is a connected symmetric graph $g = (g_{ij})$ on the set of nodes $S \cup \{0\}$, i.e., a graph such that $g_{ij} = g_{ji} = 1$ and for each city $i \in S$ there is a path $i = i_1, ..., i_m = 0$ from i to 0 such that $g_{i_1 i_2} \cdot ... \cdot g_{i_{m-1} i_m} = 1$. The cost of such a graph is

$$c(g) = \frac{1}{2}\sum_{i\neq j} a_{ij} g_{ij}.$$

(The coefficient $\frac{1}{2}$ comes from the fact that we do not want to double-count the connections.) Let $G(S)$ be the set of networks for coalition S. Define

$$v(S) = -\min_{g \in G(S)} c(g).$$

8.2.2. *Market game.*

Example 18. *Market game.* There are L goods (inputs in a production process) and J producers. Each producer has an endowment of inputs ω_j. Each producer has a production function $f_j(x)$ where $x \in \mathbb{R}_+^L$ (that, we can assume, is increasing in each input). For each coalition $S \subseteq J$, we define

$$v(S) = \max_{\sum_{i \in S} x_i \leq \sum_{i \in S} \omega_i} \sum f_i(x_i).$$

8.2.3. *Scheduling game.*

Example 19. *Scheduling game.* There are I tasks. Each task has execution time w_i and a cost function $c(t_i)$ that depends and is increasing in the moment t_i in which the task is executed. No two tasks can be executed at the same time. An allocation in coalition S is a bijection $a : \{1, ..., |S|\} \to S$ that essentially orders the tasks from the first to the last one. We define the value of a coalition as (the negative of) the minimal cost of scheduling task in a coalition:

$$v(S) = -\min_{\text{bijection } a: S \to \{1,...,|S|\}} \sum_{i \leq |S|} c_{a(i)} \left(w_{a(1)} + ... + w_{a(S)}\right)$$

For an example, you can think about a queue of airplanes that wait to land on an airport; packets of information that need to be transferred through a network node; emergency room waiting room with patients that asylum applications that wait on the desk of an CIC agent.

8.2.4. *Super-additive games.* A TU-game is *super -additive* if its associated NTU game is super-additive. Equivalently,m we say that the TU game (I, v) is super-additive if for any two disjoint coalitions S and T, we have

$$v(S \cup T) \geq v(S) + v(T).$$

The interpretation is that there are positive externalities.

8.3. **Core existence for TU games.** The next theorem provides conditions for the non-emptiness of core for TU games. (There is a NTU-version of it, but it is more complicated to state, so we ignore it).

The conditions are not exactly intuitive. We say that a collection of weights $\{\delta_S \in [0,1] : S \subseteq I\}$ is *balanced* if for each player i, we have

$$\sum_{S:i\in S} \delta_S = 1.$$

(I don't know a good interpretation a balanced collection of weights. There is a mathematical interpretation - where teh collection comes from the duality theory and corresponds to a certain set of Lagrange multipliers. But, we are not going to worry about it.)

Theorem 11. *(Bondareva-Shapley). Consider a TU game (I, v). A necessary and sufficient condition for the core of (I, v) to be non-empty is that for each balanced collection of weights $\{\delta_S\}$, we have*

$$\sum \delta_S v(S) \leq v(I).$$

Definition 16. We say that a game is *balanced* if it satisfies the above condition. (Thus, the Bondareva-Shapley Theorem can be restated as saying that a game has a non-empty core if and only if it is balanced.) We also say that a game is *totally balanced*, if each of its subgames is balanced.

The balanced condition is not easy to check for a particular game, but it turns out to be easy to check for classes of games. For example, one shows that Market Games from Example 18 are totally balanced (and that essentially, each TU-game that is totally balanced is equivalent to some market game - see Game Theory Maschler, Zamir, and Solan.

Proof. Necessity. Suppose that x is in the core of (I, v). It means that for any coalition S, $\sum_{i\in S} x_i \geq v(S)$ (otherwise, there would be a vector that could redistribute $v(S)$ and improve the utility for each member of coalition S). It follows that for any balanced weights $\{\delta_S\}$, we have

$$\sum_S \delta_S v(S) \leq \sum_S \delta_S \sum_{i\in S} x_i = \sum_i x_i \sum_{S:i\in S} \delta_S = \sum_i x_i = v(I).$$

Sufficiency. Suppose that the game is balanced. Construct two sets:

$$B_1 = \left\{ \left(\sum_{S:1\in S} \lambda_S, \sum_{S:2\in S} \lambda_S, ..., \sum_{S:I\in S} \lambda_S, \sum_S \lambda_S v(S) \right) : \lambda_S \geq 0 \text{ for each } S \subseteq I \right\},$$
$$B_2 = \{(1, 1, ..., 1, v(I) + \varepsilon) : \varepsilon > 0\}.$$

Both sets are convex and, because the game is balanced, they are disjoint. Hence, by the Separating Hyperplane Theorem, there exists a non-zero vector $(z_1, ..., z_I, z_0)$ such that for each $\lambda_S \geq 0$ for each $S \subseteq I$ and each $\varepsilon > 0$

$$\begin{aligned} \sum z_i + z_0 v(I) + z_0 \varepsilon &\geq \sum_i z_i \sum_{S:i\in S} \lambda_S + z_0 \sum_S \lambda_S v(S) \\ &= \sum_S \lambda_S \left(\sum_{i:i\in S} z_i + z_0 v(S) \right) \end{aligned}$$

Because each $\varepsilon > 0$ can be chosen, it must be that $z_0 \geq 0$ (otherwise, the left-hand side could become arbitrarily small, and smaller than the right-hand side.)

Because arbitrary (but positive) weights can be chosen, we could also choose $\lambda_S = 0$ for each S, which implies

$$\sum z_i + z_0 v(I) \geq 0. \tag{8.1}$$

Moreover, we could have chosen $\lambda_S \to \infty$, which would make the right–hand side arbitrarily large (and bigger than the left hand side) unless

$$\sum_{i:i\in S} z_i + z_0 v(S) \leq 0. \tag{8.2}$$

□

We will show that $z_0 \neq 0$. If not, than the equations imply that $z_i \leq 0$ for each i and $\sum_i z_i \geq 0$, which contradicts the fact that the vector $(z_1, ..., z_I, z_0)$ is non-zero.

Notice that inequalities (8.1) and (8.2) imply that

$$-\frac{1}{z_0} \sum z_i \leq v(I) \text{ and } -\frac{1}{z_0} \sum_{i\in S} z_i \geq v(S).$$

Take $x_i = -\frac{1}{z_0 v(I)} z_i$. The above inequalities mean that $x = (x_1, ..., x_I)$ is in the core.

8.4. **Convex games.**

Definition 17. A TU-game (I, v) is convex if for any two coalition $T, S \subseteq I$, we have

$$v(T) + v(S) \leq v(T \cap S) + v(T \cup S).$$

To understand this condition, let $A = T \cap S, B = T \backslash S, C = S \backslash T$. Notice that all three coalitions are disjoint. The above condition is equivalent to

$$v(A \cup B \cup C) - v(A \cup B) \geq v(A \cup C) - v(A).$$

In other words, the increase of the value caused by adding coalition C to coalition A is higher if also members of coalition B are present.

Exercise 16. Show that any convex game is super-additive.

Exercise 17. Show that the voting game from the Exercise 15 is not convex.

Proposition 4. *Each convex game has a non-empty core.*

Proof. We assume w.l.o.g. that $v(\emptyset) = 0$ (such choice is really a convention and it doesn't affect the definition of convexity - convince yourself!) The proof constructs one of core allocations. Suppose that all players are arranged in some (arbitrary order $i = 1, 2, ..., |I|$). We define $x_1 = v(1)$ and for each $i > 1$

$$x_i = v(1, ..., i) - v(1, ..., i-1).$$

We show that x is a core allocation.

First, notice that such allocation satisfies

$$\sum_{i \in I} x_i = v(1) + v(1,2) - v(1) + v(1,2,3) - v(2) + ... + v(1,2,...,|I|) - v(1,2,...,|I-1|) = v(I).$$

Second, for each coalition $S \subseteq I$, let $i_1 < ... < i_{|S|}$ be the list of names (i.e., elements of the original order) of the members of the coalition. For each s, let $S_s = \{i_1, ... i_s\}$

and let $T_s = \{1, ..., i_s - 1\}$. Then, by convexity, for each $s \leq S$

$$\begin{aligned} v(i_1, ..., i_{s-1}, i_s) - v(i_1, ..., i_{s-1}) &= v(S_s) - v(S_s \cap T_s) \\ &\leq v(S_s \cup T_s) - v(T_s) \\ &= v(1, ..., i_s) - v(1, ..., i_s - 1) \\ &= x_{i_s}, \end{aligned}$$

where we use the convention that $v(\emptyset) = 0$. It follows that

$$\begin{aligned} v(S) &= v\left(i_1, ..., i_{|S|}\right) - v(\emptyset) \\ &= \sum_{1 \leq s \leq |S|} v(i_1, ..., i_{s-1}, i_s) - v(i_1, ..., i_{s-1}) \\ &\leq \sum_{1 \leq s \leq |S|} v(1, ...i_{s-1}, i_s) - v(1, ..., i_{s-1}) = \sum_{i \in S} x_i. \end{aligned}$$

□

Remark 4. Alternatively, one can show that any convex game is totally balanced. Then, the non-emptiness of the core would follow directly from the Bondareva-Shapley Theorem 11.

Comparing to balancedness, convexity is very easy to check (and to develop intuitions). Unfortunately, it is very strong, often too strong for applications.

For a more discussion of the existence of solutions in non-convex games (and many great applications, like matching) see a very nice recent paper "Hyperadditive Games and Applications to Networks or Matching Problems" by Eric Bahel.

8.5. **Nash bargaining solution.** For the sake of reference and completeness, I mention two other important solution concepts: Nash bargaining solution and the Shapley value. You will be talking more about the first one in the first part of the game theory class.

A bargaining problem is a very special type of a cooperative game in which the agreement of all players is necessary for the non-trivial outcome. Formally, a bargaining game is defined as (U, u^*), where $U \subseteq \mathbb{R}^I$ is the convex(!) utility possibility set for the grand coalition, and $u^* \in U$ is the status-quo, i.e., u_i is the solution for single-player coalition $\{i\}$. In other words, (U, u^*) is a NTU-cooperative game with

$I, V(I) = U, V(S) = \left\{(u_i^*)_{i \in S}\right\}$ foer $S \subset I$. Let $\mathcal{U}$ be the collection of all bargaining problems.

A solution to the bargaining problem is a mapping $f : \mathcal{U} \to \mathbb{R}^I$ such that $f(U, u^*) \in U$ for each problem (U, u^*).

We can define some properties of a "good" solution:

(1) **Independence of Affine Transformations** (IAT): for any $\beta \in \mathbb{R}_+^I$, and $\alpha \in \mathbb{R}^I$, we have

$$f(\beta U + \alpha, \beta u^* + \alpha) = \beta f(U, u^*) + \alpha.$$

(Here, $\beta U = \{(\beta_1 u_1, \beta_2 u_2, ..., \beta_I u_I) : (u_1, u_2, ..., u_I) \in U\}$ is the set of transformed utilities.

(2) **Symmetry** (S): If U is symmetric (i.e., does not change after permutation of the axis) and $u_i^* = u_j^*$ for each $i \neq j$, then if $u = f(U, u^*)$, then $u_i = u_j$ for each $i \neq j$.

(3) **Pareto-Optimality** (PO): There is no $u \in U$ that Pareto-dominates $f(U, u)$.

(4) **Individual Rationality** (IR): $f(U, u^*) \geq u^*$. Effectively, PO and IT together mean that the solution is in the core of the NTU-cooperative game.

(5) **Independence of Irrelevant Alternatives** IIA: If $u^* \in U' \subseteq U$ and $f(U, u^*) \in U'$, then $f(U, u^*) = f(U, u^*)$. The last axiom is the most substantial. It says if that the solution to the bargaining problem should not change if some of the alternative allocations that are not the solution are removed.

Proposition 5. *(Nash bargaining solution.) There exists a unique solution to the bargaining problem that satisfies IAT, S, PO, IR, IIA. The solution can be described as*

$$f(U, u^*) = \arg\max_{u \in U} \prod_i (u_i - u_i^*). \tag{8.3}$$

Exercise 18. Show that the solution (8.3) satisfies all the axioms.

Proof. We are left with showing that any solution that satisfies the axioms has the form of (8.3). TBA. □

8.6. **Shapley value.** Shapley value is an attempt to assign a unique solution $f(v) \in \mathbb{R}^I$ to each TU-cooperative game with population I: Shapley proceeds axiomatically: Define some properties of a "good" solution:

(1) **Independence of Affine Transformations** (IAT): For any $b \in \mathbb{R}_+$, and $a \in \mathbb{R}^I$, let $bv + a$ be a new characteristic function defined as: for each $S \subseteq I$

$$(bv + a)(S) = bv(S) + \sum_{i \in S} a_i$$

The IAT axiom says that

$$f(bv + a) = bf(v) + a.$$

(2) **Symmetry** (S): Suppose that $\pi : I \to I$ is a permutation of name of the players. Let $\pi(v)$ be a characteristic function of a permuted game defined as: $(\pi(v))(S) = v(\pi(S))$. Then, $f_{i(v)}(\pi(v)) = f_{\pi(i)}(v)$.

(3) **Pareto-Optimality** (PO): $\sum_i f_i(v) = v(I)$.

(4) **Dummy axiom** (DA): Individual i is a dummy in game v if for each coalition $S \subseteq I$, we have $v(S \cup \{i\}) = v(S)$, i.e., if i does not affect the value. Then, the solution satisifes the dummy axiom if $f_i(v) = 0$ for each dummy i in game v.

(5) **Additivity** (A): For any two TU-games v, u with the same grand-coalition I, we have

$$f(v + u) = f(u) + f(u).$$

To characerize the solution, we need the following result. An ordering (a "permutation") of a grand coalition is a bijection $o : I \to \{1, ... |I|\}$. Let $O(I)$be the set of orderings. Notice that there is $|I|!$ of them.

For each ordering π and each agent i, let

$$v\{i' : o(i') \leq o(i)\} - v\{i' : o(i') < o(i)\}$$

is a contribution of agent i to the coalition of all agents that are before him under π.

We define the Shapley value as

$$\text{Sh}_i(v) = \frac{1}{|I|!} \sum_{\pi \in O(I)} (v\{i' : o(i') \leq o(i)\} - v\{i' : o(i') < o(i)\}).$$

The Shapley value of player i. Suppose that we order the players at random (each permutation o has an equal probability $\frac{1}{|I|!}$). Each player enters the room in the order, and each player receuives, as a payoff, his or her contribution to the current coalition in the room. The Shapley value of player i is the average (i.e., the expected) payoff of player i obtained in such a way.

Proposition 6. *(Shapley value) The Shapley value $Sh(.)$ is the unique solution that satisfies IAT, S, PO,DA and A.*

Exercise 19. Show that the Shapley value satisfies all axioms IAT, S, PO,DA and A.

Proof. Given the Exercise, it is enough to show that if f satisfies all the axioms, then it is unique and equal to Sh. The proof is quite straightforward given the powerful axioms.

First, notice that axiom A and axiom IAT (specifically, being able to scale by factor β) implies that Sh is a linear mapping from the space $\mathbb{R}^{2^I}$ (i.e., the space of characteristic functions) to $\mathbb{R}^I$ (the space of payoff vectors). Hence, there must exists coefficients α_{iS} for each coalition $S \subseteq I$ and each agent i such that

$$f_i(v) = \sum_S \alpha_{i,S} v(S) = \sum_{k<n} \sum_{S \subseteq I \setminus \{i\} : |S|=k} \alpha_{i,S\cup\{i\}} v(S \cup \{i\}) + \alpha_{i,S} v(S) + .$$

The second equality turns out to be a useful representation.

Second, notice that the Symmetry implies that for each permutation $\pi : I \to I$ we have

$$\alpha_{i,S} = \alpha_{\pi(i),\pi(S)}.$$

For each k and two coalitions S, S' such that $|S| = |S'| = k$ and $i \notin S, S'$, we can find a permutation $\pi_{S,S'}$ such that $\pi_{S,S'}(S) = S'$ and $\pi_{S,S'}(i) = i$. It follows that for any such two coalitions, we have

$$\alpha_{i,S\cup\{i\}} = \alpha_{i,S'\cup\{i\}} =: \beta_{i,k}$$

$$\beta_{i,S} = \beta_{i,S'} =: \beta_{i,k},$$

It follows that

$$f_i(v) = \sum_{k<n} \sum_{S \subseteq I \setminus \{i\}: |S|=k} \alpha_{i,k} v(S \cup \{i\}) + \beta_{i,k} v(S).$$

By symmetry, the coefficients $\alpha_{i,k}$ and β_{ik}do not depend on i.

Third, the Dummy axiom implies that $\beta_{i,k} = -\alpha_{i,k}$. To see it, consider a class of characteristic functions such that $v(S \cup \{i\}) = v(S) = v$ for some coaltion $|S| = k$ and $v(S') = 0$ for any other coalition. If $t\beta_{i,k} \neq -\alpha_{i,k}$, then the f would have to depend on i, but it should not. Hence,

$$\mathrm{Sh}_i(v) = \sum_{k<n} \alpha_k \sum_{S \subseteq I \setminus \{i\}: |S|=k} (v(S \cup \{i\}) - v(S)).$$

Fourth, the Pareto-Optimality implies that

$$\begin{aligned} &\sum_i \alpha_k \sum_{S \subseteq I \setminus \{i\}: |S|=k} (v(S \cup \{i\}) - v(S)) \\ &= \sum_{S \subseteq I} \Big(|S| \, \alpha_{|S|-1} - (n - |S|) \, \alpha_{|S|} \Big) v(S). \end{aligned}$$

It follows that for each $k \leq n$, we have

$$\alpha_k = \frac{k}{n-k} \alpha_{k-1},$$

or

$$\alpha_k = \alpha_0 \frac{1}{n-1} \cdot \frac{2}{n-2} \cdot \ldots \cdot \frac{k}{n-k} = \frac{(n-k-1)!k!}{(n-1)!} \alpha_0 = \frac{1}{\begin{pmatrix} n-1 \\ k \end{pmatrix}} \alpha_0.$$

Finally, the additive part of the Independence to Affine Transformation means that for each a_i, $f_i(v + (0, ...0, \alpha_i 0, ..., 0)) = f_i(v) + a_i$, which implies that

$$\sum_{k<n} \alpha_k \sum_{S \subseteq I \setminus \{i\}: |S|=k} a_i = a_i \sum_{k<n} \alpha_k \begin{pmatrix} n-1 \\ k \end{pmatrix} = a_i,$$

which implies that

$$1 = \sum_{k<n} \alpha_k \begin{pmatrix} n-1 \\ k \end{pmatrix} = \alpha_0 \sum_{k<n} \frac{1}{\begin{pmatrix} n-1 \\ k \end{pmatrix}} \begin{pmatrix} n-1 \\ k \end{pmatrix} = \alpha_0 n = 1,$$

or that $\alpha_0 = 1/n$. Hence, we get

$$f_i(v) = \sum_{k<n} \frac{1}{\begin{pmatrix} n \\ k \end{pmatrix}} \sum_{S \subseteq I \setminus \{i\} : |S| = k} \left(v\left(S \cup \{i\}\right) - v\left(S\right) \right).$$

In particular, there is a unique f that satisfies all the axiom. Some algebra shows that it is indeed the same as the Shapley value. (Or it follows from the exercise.) $\square$