1. Feasible allocations[12]

We define exchange economy in steps.

1.1. **Two examples.** This lecture is about allocation problems. There are some goods that can be given to some agents in the economy (or, posibly, first used as inputs in a production, and then given to somebody). The agents have opinions over the allocations, i.e., they have preferences.

Example 1. Alice and Bob like cookies. There are two cookies, on with chocolate chips, and one with strawberry jelly. There are 4 possible allocations that do not involve throwing away cookies $\left\{s^A c^A, s^A c^B, s^B c^A, s^B c^B\right\}$. Agent $i = A, B$ utility depends only on what she gets; her utility from no cookie, strawberry, chocolate or both is respectively, $u_i(0), u_i(s), u_i(c), u_i(sc) \in \mathbb{R}$.

Very often, we are going to assume that the space of possible allocations are continuous rather than discrete.

Example 2. Alice and Bob divide a cake with two parts, chocolate and strawberry. A division of the cake (i.e., a feasible allocation in the economy) is a tuple $(x_{A,c}, x_{A,s}, x_{B,c}, x_{B,s}) \in [0,1]^4$ such that $\sum_i x_{i,l} = 1$ for each l. Both consumers have linear preferences over their consumption bundles: $u_i(x_i) = \sum_l u_i^l x_{i,l}$, where we assume that $u_i^l > 0$ for each i, l, i.e., each consumer likes each part of the cake. PICTURE.

The last example can be drawn in the Edgeworth box (allocations, preferences - indifference curves, endowments). Many intricacies of the general equilibrium theory, can be explained using the simplest of example: the exchange economy with two consumers and two goods. The Edgeworth box represents all *non-wasteful allocations*:

$$x = (x_A^s, x_A^c, x_B^s, x_B^c) = E = \left\{(x_A^s, x_A^c, x_B^s, x_B^c) : x_A^l + x_B^l = \omega^l \text{ for } l = s, c\right\}.$$

It turns out that the Edgeworth box of non-wasteful allocations is sufficient if both consumers preferences are increasing in their consumption, as we will assume here.

[12] $October 22, 2019$.

(But, this is not without loss of generality - maybe Alice has allergy to strawberries, or Bob is lactose intolerant.

1.2. **Mathematical notation.**

(1) Some mathematical notations and definitions for consumption space $\mathbb{R}^L$
   (a) when we write x, we always think about it as a vertical vector, i.e.,
   $$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_L \end{bmatrix}.$$
   (b) Comparison of vectors:
      (i) for each $x, x' \in X$, $x \leq x'$ if and only if $x_i \leq x'_i$ for each i, and
      (ii) for each $x, x' \in X$, $x < x'$ if and only if $x_i \leq x'_i$ for each i with at least one strict inequality,
      (iii) for each $x, x' \in X$, $x \ll x'$ if and only if $x_i < x'_i$ for each i,
   (c) Scalar product. For any two vectors $x, y \in \mathbb{R}^L$, we have $x \cdot y = \sum_l x_l y_l$
   (d) Topology on X:
      (i) "Maximum" distance: $\|x - x'\| = \max_i |x_i - x_i|$.
      (ii) For each $x \in X$, set U is a *neighborhood* of x if $x \in U$ and there exists $\epsilon > 0$ such that for each $x' \in X$, if $\|x' - x\| \leq \epsilon$, then $x' \in U$.
      (iii) Set $A \subseteq X$ is closed if for each $x_n \to x$, if $x_n \in A$ for each n and $x_n \to x$ (x_n is a convergent sequence with limit x), then $x \in A$.
   (e) Operation on sets:
      (i) (Minkowski) sum: for any two sets $X, Y \in \mathbb{R}^L$, we define
      $$X + Y = \{x + y : x \in X, y \in Y\}.$$
      This operation generalizes: if $X_i \subseteq \mathbb{R}^L$ for each i., we define
      $$\sum_i X_i = \left\{ \sum_i x_i : x_i \in X_i \text{ for each } i \right\}.$$
      (ii) Multiplication by scalar: for each $a \in \mathbb{R}$ and each $X \subseteq \mathbb{R}^L$, we take
      $$aX = \{ax : x \in X\}.$$

Example 3. We have:

- $\{0\} \times \{0, 1\} + [0, 1] \times \{0\} = [0, 1] \times \{0, 1\}$,
- $[0, 1] \times \{0\} + \{0\} \times [0, 1] = [0, 1]^2$.
- $[0, 1]^2 + [0, 1]^2 = [0, 2]^2 = 2\, [0, 1]^2$.

Exercise 1. Show that the (Minkowski) sum of two convex sets is convex.

1.3. **Main model.** Production economy = goods, consumers, producers, ownership structure

commodities: goods or bads (factors)	consumers	firms	ownership structure
$l = 1, .., L$,	$i = 1, .., I$,	$j = 1, .., J$,	
Commodity space $Z = \prod Z_l$	$X_i \subseteq X_i \subseteq Z$ consumption space,	$Y_i \subseteq Z$ technology	$\omega_i \in Z$ endowment of consumer i
$\omega \in \mathbb{R}^L$ total endowment of economy	$\preceq_i$ preferences,		θ_{ij} share of firm j owned by consumer i

1.3.1. *Commodities.*

- A good is a category that contains possible multiplicities of objects, including fractional quantities.
  - Very often, we assume that the commodities are *perfect divisible*. But, sometimes we also talk about non-divisible goods. They become more important later in the lecture, when we talk about matching, assignment problems, etc.
  - To make it clear, whenever I say “goods”, I assume that they are divisible. When I say “objects”, they are not divisible.
  - Sometimes, it is possible to incorporate objects into continuous consumption space by simply saying that agents utility does not increase over fractional parts of a good. But, it may lead to difficulties, as it requires the utility functions to be discontinuous.
  - Alternatively, one can consider lotteries over objects. But, there are different ways of doing so. We discuss some of them below.
- Each good belongs to its commodity set $x_l \in Z_l \subseteq \mathbb{R}$;

- A typical restriction is that each good in the commodity set has to be positive: $Z_l = \mathbb{R}_+$. This reflects basic physical constraint that we do not know what are negative goods.
- A common exception to this rule are financial products, like money (or insurance, derivatives, etc.) for which the negative quantity has an interpretation. In such a case, we take $Z_l = \mathbb{R}$.
- if either $Z_l = \mathbb{R}$ or $Z_l = \mathbb{R}_+$for each l, we say taht the commodity sets are standard.

- Commodity space $Z = \prod_l Z_l \subseteq \mathbb{R}^L$. An element of commodity space is called either *consumption bundle*, or *production plan*, depending on whether we talk about firms or consumers.
  - We assume that the commodity space is equipped with the topology induced by the "maximum" distance.
- A total endowment in the economy $\omega \in Z$.

1.3.2. *Consumer preferences.*

- Consumption space $X_i \subseteq Z = \mathbb{R}^L$.
- Preference $\preceq_i$ on X_i: binary relation that is rational (complete, transitive). Derived relation of *strict preferences* "$\prec$" and indifference "$\sim$".
  - typically, we assume that preference relation $\preceq_i$ is *represented* by utility function $u_i : X_i \to \mathbb{R}$: for each $x, y \in X$,

$$x \preceq y \text{ if and only if } u(x) \leq u(y).$$

Properties of the preferences:

(1) *Monotonicity:*
   (a) Preference relation is monotone, if for each $x, y \in X$, if $x \ll y$, then $x \prec y$. Thus, if bundle y contains strictly more each of the good than x, then it is prefered to x.
   (b) It is strongly monotone if for each $x, y \in X$, if $x \leq y$ and $x \neq y$, then $x \prec y$.

(2) *Local non-satiation*: In any neighbohood $U \subseteq X$ of $x \in U$, there is $x' \in U$ such that $x' \succ x$. For each x, there is always something that is better and arbitrarily close to x.

(3) *Convexity:*

   (a) preferences are convex (or u is *quasi-concave*), if for each $x \in$, the upper-contour set $\{x' : x \preceq x'\}$ is convex,

   (b) preferences are strictly convex (or u is *strictly quasi-concave*) if for each x, the upper-contour set $\{x' : x \preceq x'\}$ is strictly convex (i.e., $\alpha x' + (1-\alpha) x'' \succ x$ for each $\alpha \in (0,1)$ and $x', x'' \succeq x$)

(4) *Continuity*:

   (a) Preference relation is *continuous* if it is preserved under limits: for all convergent sequences $x^n \to x$ and $y^n \to y$, if $x^n \preceq y^n$ for all n, then $x \preceq y$, or if

   (b) for each $x \in X$, the upper-contour set $\{x' : x \preceq x'\}$ and the lower-contour set $\{x' : x' \preceq x\}$. Preferences are continuous iff both upper- and lower-contour sets are closed.

Example 4. *Quasi-linear preferences*: We say that consumer i has quasi-linear preferences if one of the goods, say the last, has unlimited consumption, and the utility has form

$$u_i(x_i) = u_i^0(x_{i,-L}) + x_{i,L},$$

where $X_i = X_i^0 \times \mathbb{R}$ and $u_i^0 : X_i^0 \to \mathbb{R}$ is the "utility" of the non-linear component. We typically refer to the last good as a *numeraire*, or even better, money, and typically assume that the initial endowment of money is 0.

1.3.3. *Firms.*

- Interpretation of production vector $y \in Y$: negative coordinate is an input and positive coordinate is an output.
- Technology:
  - nonempty, closed (for each $y_n \in Y$, if $\lim_n y_n$ exists, then $\lim_n y_n \in Y$).
  - possibility of inaction: $\mathbf{0} \in Y$.

  - free disposal: For each $y \leq y'$ if $y' \in Y$, then $y \in Y$. (Hewre, $y \leq y'$ means $y_l \leq y'_l$ for each l).
  - convexity.
- Aggregate technology $Y = \sum_j Y_j$.

Exercise 2. Show that if for some firm i,Y_i has free disposal, then the aggregate technology Y has free disposal as well.

1.4. **Feasible allocations.** An allocation in the production economy is a tuple $(x_1, ..., x_I, y_1, ..., y_L) \in Z^{I \cup J}$.

Definition 1. An allocation is *(weakly) feasible* if it satisfies:

- *individual feasibility*: $x_i \in X_i$ for each i, $y_j \in Y_j$ for each j, and
- *(weak) aggregate feasibility*:

$$\sum_i x_i \leq \omega + \sum_j y_j. \tag{1.1}$$

The space of all feasible allocations is denoted with X. We take a convention that for each $x \in X$, each agent i, $u_i(x) := u_i(x_i)$.

The above inequality is called a *resource constraint.* We require the resource constraint to be satisfied with inequality. An alternative is called *strict aggregate feasibility*:

$$\sum_i x_i = \omega + \sum_j y_j.$$

We have the following observation:

Lemma 1. *Suppose that the aggregate technology satisfies free disposal. Then, for each feasible allocation $x = (x_1, ..., x_I, y_1, ..., y_L)$ that satisfies weak aggregate feasibility, there is a feasible allocation $x' = (x_1, ..., x_I, y'_1, ..., y'_L)$, with the same consumption bundles and possibly different production plans, that satisfies strict aggregate feasibility:*

$$\sum_i x_i = \omega + \sum_j y_j.$$

Exercise 3. Prove Lemma 1.

Remark 1. The "weak" vs "strict" feasibility difference has an important economic meaning. Notice that the "weak" feasibility parallels the budget constraint of a consumer (we will talk about it later). The assumption sort of captures the idea that consumers may not necessarily want to buy all the goods that are available to them. Perhaps, because they became satiated, or because some of the goods are actually "bads".

There is no one uniformly accepted way of dealing with "bads" in equilibrium. So, we follow Kreps's textbook and we sort of ignore an issue by working with the weak notion of the aggregate feasibility. Alternatively, we can assume free disposal. Notice that MWG assumes strong monotonicity in their existence proof Propositions 17.B.2 and 17.C.2, which eliminates "bads" as well, in a different way. See 2 below.

1.5. **Examples.**

1.5.1. *Exchange economy.* See above Example 2.

1.5.2. *Robinson Cruzoe economy.* There are $L = 2$ goods: coconuts and leisure time.

There is $I = 1$ agent, Robinson. Robinson has three roles: (a) owner of the firm, (b) worker who owns L_0 unit of leisure time, and (c) consumer, who consumes coconuts and leisure

There is $J = 1$ firm with technology that converts leisure time to coconuts $q = f(l)$.

1.5.3. *Geographical goods and transportation.* So far, one good split between consumers. What if consumers live in different locations? And it is costly to transport goods between locations?

Our model is able to incorporate different locations with a simple trick. We can represent a good as different "goods" in different locations, and use the "transportation technology" to convert goods between the locations. The transportation technology can be as complicated or as simple as one wants.

Example 5. Transportation technology uses good called "fuel" to convert good x_A to good x_B.

1.5.4. *Temporal goods and storage.* The same trick with "multiplying" commodities, or, more precisely, being careful about the right definition of commodity, can be used in many other contexts.

For instance, suppose that we have agents that live through different periods and they derive utility from consumption in different points in time. Then, a good in period 1 is not the same as the "good" consumed in the next period. A storage technology serves to convert one good into another.

1.5.5. *State-contingent goods and insurance (uncertainty).* We can also incorporate uncertainty about some utility- or production-relevant event. To do so, we can think about the "same" good consumed in different states of the world as different goods. Think about consumption of "umbrella if it is sunny", and consumption of"umbrella if it rains". The utilities should be defined on these different, state-contingent, goods (and, depending on other assumptions, they may have a general form of utilities from acts, or something more special, like expected utility).

We talk more about the trading under uncertainty in Section 6.

1.5.6. *Public good.* Exclusions and tricks. MWG Example 16.G.3.

1.6. **Discrete allocations and lotteries over discrete allocations.** For both good and bad reasons, most of the general equilibrium theory developed initially with commodities being perfectly divisible. Nevertheless, many questions are naturally modeled as discrete allocation problems (for instance, Example 1). Sometimes, we want to convert such problems into continuous problems. There are both methodological and conceptual reasons to do it. For the first, it is often easier to work with continuous problems, as we can use techniques like differentiability. For the second, often, even if the original problem has a natural discrete description, it is possible that a natural solution involves a continuous object. Like a lottery.

In many situations, lotteries over objects are natural extensions of goods. The problem is that there are multiple ways of converting a discrete problem to a continuous one using lotteries. The most appropriate one is to take the space of all allocations and replace it by the space of probability distributions over it.

Example 6. (Continuation of Example 1). In the cookie example, the space of discrete allocations is equal to $X_A = \left\{s^A c^A, s^A c^B, s^B c^A, s^B c^B\right\}$. Let $X_L = \Delta X_A$ be the space of probability distributions over discrete allocations. Each element $x \in X_L$ assigns probability $x(a)$ to each element $a \in X_A$. The space of probability distributions has $2^2 - 1 = 3$ dimensions.

One problem with the above method is that it may lead to very large spaces of allocations. To see it, suppose that we have I agents and L objects. Then, there are $|X_A| = L^I$-possible (discrete) allocations (that are non-wasteful, i.e., each object gets allocated to somebody). The space of all probability distributions $X_L = \Delta X_A$ is a $\left(L^I - 1\right)$-dimensional simplex. It gets large very fast with I.

There is another problem that becomes more apparent when we study the general equilibrium theory. That theory needs allocations to have a product form, in which goods are clearly separated across agents. Like in the space of allocations defined above is a product of the consumption spaces of individual agents. The space of all discrete allocations has such a product form. The space of all distributions over allocations does not. Think about why. (Hint: the issue is in multiple ways that one can capture correlations between agents)

An alternative that is available sometimes is to consider individual lotteries over consumption bundles.

Example 7. (Continuation of Examples 1 and 6). An individual lottery for agent i is a pair $(x_{is}, x_{ic}) \in [0,1]^2$, where x_{il} is the probability that agent i gets cookie l. To create a non-wasteful allocation, we need to make sure the probabilities add-up. Let $X_{IL} = \left\{(x_{il})_{i=A,B}^{l=s,c} : \forall_l \sum_i x_{il} = 1\right\}$. Such a space has $4 - 2 = 2$ dimensions.

More generally, we can create the space of individual lotteries $X_{i,IL} = [0,1]^L$ and then the space of all feasible allocations as the subset of the product of individual spaces$X_{IL} \subseteq \prod_i X_{i,IL}$ that satisfies feasibility restrictions. This leads to $L \times I - L = (L-1) \times I$ dimensions.

This is not a perfect solution and there are difficulties. One of them is that, in the end, to figure out who gets what, we still need to determine the allocation of objects. (People cannot eat lotteries, or live in them.) In the former case, it is easy: we use the

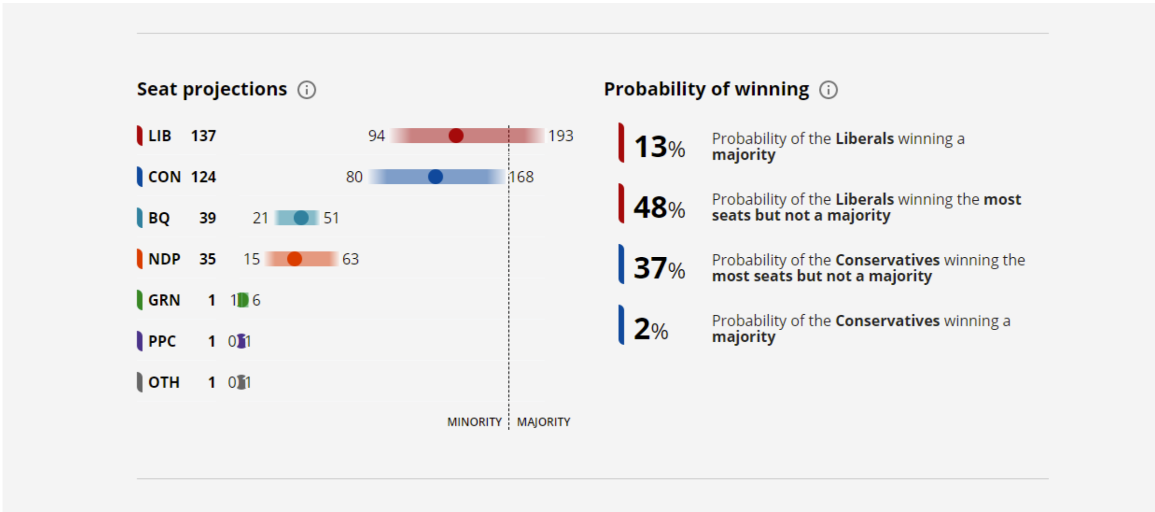

FIGURE 1.1. Canada Poll Tracker prediction of the outcome of 2019 Federal Election in Canada.

probability distribution x to draw an allocation a and give each agent i the good(s) a_i.

In the latter case, the final allocation is not obvious and can be drawn in multiple ways. For example, we can use the feasibility restrictions for each object to construct a probability distribution $x_l = (x_{1,l}, ..., x_{I,l}) \in \Delta I$ to draw an agent for each object. But this is only one way - it assumes that objects are allocated independently. It is not the only way, and sometimes, it is also the wrong way, when, for example there are complementarities in preferences across objects. (Think about allocation of left and right shoes using lotteries, where each shoe is allocated independently.)

Example 8. Figure 1.1 shows the last prediction of the outcome of 2019 federal election in Canada from the CBC website Canada Poll Tracker. You can think about the election as an allocation problem with 338 objects ("seats") and parties who want to get them. The election day itself is a randomized allocation. The projection above (the left hand side of it) shows the mean and the range of the distribution (roughly, the support) of the number of seats. Notice that the randomization is done in a individual way, like in Example 7 (and not in the aggregate way, like in Example 6). Of course, the allocations cannot be chosen independently. Although it is possible according to the above prediction that Liberals take 193 seats and the Conservatives take 168 seats, because the total number of seats in the House of Commons is 338>193+168, these two outcomes cannot happen simultaneously. Hence the prediction is only a projection of the marginal distribution of some “true” aggregate distribution. In the “true” aggregate distribution, the seats of the Liberals and the Conservatives are negatively correlated.

There are other ways of resolving individual lotteries, that may involve correlations, etc. that may avoid some of the difficulties. But, in the end, which extension is good and works well depends on the problem at hand.